Name Solutions

Electrodynamics, Physics 323 **First midterm** **8.20 am, Wednesday May 6, 2008**
Spring 2008 Instructor: David Cobden

Do not turn this page until the buzzer goes at 8.20. You must hand your exam to me before I leave the room at 9.25.

Attempt all the questions.

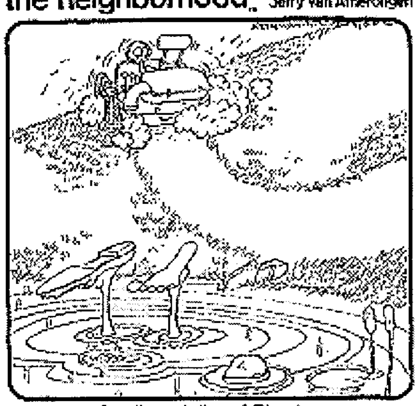
Please write your name on every page and your SID on the first page.

Write all your working on these question sheets. Use this front page for extra working. It is important to show your calculation or derivation. Some of the marks are given for showing clear and accurate working and reasoning.

Watch the blackboard for corrections or clarifications during the exam.

This is a closed book exam. ***No books, notes or calculators allowed.***

the neighborhood. Jerry Van Amerongen



Another victim of Physics

 Name solutions

1. [10] Complete Maxwell's equations in matter:

(i) $\nabla . \mathbf{D} = \rho_{free}$ (ii) $\nabla \times \mathbf{E} = -\frac{\partial \vec{B}}{\partial t}$

(iii) $\nabla . \mathbf{B} = 0$ (iv) $\nabla \times \mathbf{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}_{free}$

2. [5] Is **B** *ever not* perpendicular to the wavevector **k** for a monochromatic plane wave (MPW) in matter, and why or why not?

No. $\vec{\nabla} \to i\vec{k}$ for any MPW, so $\vec{\nabla} . \vec{B} = 0 \Rightarrow \vec{k} . \vec{B} = 0$

3. [10] Starting with these equations, show that in a linear dielectric the electric field obeys a wave equation, and deduce the relation between the refractive index n and the constants ε_r and μ_r.
$[\mathbf{\nabla} \times (\mathbf{\nabla} \times \mathbf{E}) = \mathbf{\nabla}(\mathbf{\nabla} . \mathbf{E}) - \nabla^2 \mathbf{E}]$

(i) $\to \varepsilon_r \varepsilon_0 \vec{\nabla} . \vec{E} = \rho_{free} = 0$ if $\vec{J}_{free} = 0$ (ac only) $\therefore \vec{\nabla} . \vec{E} = 0$

(iv) $\to \frac{1}{\mu_r \mu_0} \vec{\nabla} \times \vec{B} = \varepsilon_r \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

$\vec{\nabla} \times$ (ii) $\to \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\vec{\nabla} \times \left(\frac{\partial \vec{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B}) =$ by (iv)

$= \vec{\nabla}(\vec{\nabla} . \vec{E}) - \nabla^2 \vec{E}$ $= \mu_r \mu_0 \varepsilon_r \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$

$= -\nabla^2 \vec{E}$

$\therefore \nabla^2 \vec{E} = -\mu_r \mu_0 \varepsilon_r \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} - \frac{-1}{v^2} \frac{\partial^2 \vec{E}}{\partial t^2}$ $\therefore n = \frac{c}{v} = \sqrt{\varepsilon_r \mu_r}$

$= -\frac{\mu_r \varepsilon_r}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$

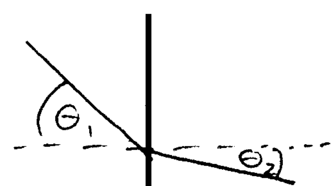
4. [10] Consider a boundary between two dielectrics with refractive indices $n_1 = 2$ and $n_2 = \sqrt{3}$. Using Snell's law, show that there is a range of angles of incidence for which an incident electromagnetic wave is completely reflected from the boundary. Indicate that range of angles on a polar plot, covering the whole 360 ° range of possible incident directions.

Snell: $n_1 \sin\theta_1 = n_2 \sin\theta_2$ $\frac{n_2}{n_1} = \frac{\sqrt{3}}{2}$

Incident from n_2 side,

$\sin\theta_1 = \frac{n_2}{n_1} \sin\theta_2 < 1$ always

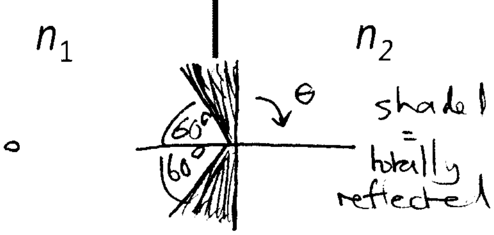
so there's always a transmitted wave.



Incident from n_1 side

$\sin\theta_2 = \frac{n_1}{n_2} \sin\theta_1 > 1$ if $\sin\theta_1 > \frac{n_2}{n_1}$

$\therefore$ no transmission for $|\theta_1| > \sin^{-1}\frac{\sqrt{3}}{2} = 60°$



 Name solutions

Graphene is a conducting sheet of pure carbon that is only one atom thick. Earlier this year, it was discovered that the sheet conductivity of graphene at optical frequencies has the quantized value $\sigma = e^2/(4\hbar)$ (a resistance of about 16 kΩ). By definition, an in-plane field **E** produces a surface current $\mathbf{K} = \sigma\mathbf{E}$ within the sheet.

5. [6] What is the average rate of Joule heat dissipation per unit area in the graphene for an ac in-plane field of magnitude E_0 (ie, $E = E_0 \cos \omega t$), in terms of σ and E_0?

Joule power/unit area $= P_J = \langle \vec{K}.\vec{E} \rangle = \frac{1}{2}\sigma E_0^2$

$\uparrow \langle \cos^2 \rangle$

6. [15] When a laser beam is incident normally on a graphene sheet, it is found that the transmission coefficient (ie, the fraction of incident energy transmitted) is $T = 1 - \pi\alpha$, where $\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c}$ is the 'fine structure constant' ($\pi\alpha$ is about 2.3 %). Show that this is as expected, given σ as above and assuming that the difference between the incident and transmitted intensity is entirely due to Joule heating in the graphene by the incident radiation field.

incident intensity

$I_i = \frac{1}{2}c\varepsilon_0 E_i^2 \longrightarrow$

power/unit area

graphene sheet

transmitted

$I_t = \frac{1}{2}c\varepsilon_0 E_t^2 \longrightarrow$

$\downarrow P_J = \frac{1}{2}\sigma E_i^2$

Cons. energy/unit area:

$I_i = I_t + P_J$

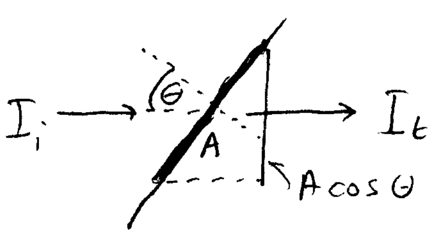
$\therefore \frac{1}{2}\varepsilon_0 c E_i^2 = \frac{1}{2}c\varepsilon_0 E_t^2 + P_J$

Transmission coeff $T = \frac{I_t}{I_i}$

$= 1 - \frac{P_J}{I_i} = 1 - \frac{\sigma}{\varepsilon_0 c}$

$= 1 - \frac{1}{\varepsilon_0 c}\frac{e^2}{4\hbar} = 1 - \frac{\pi e^2}{4\pi\varepsilon_0 \hbar c} = 1 - \pi\alpha$

7. [6] If the graphene is rotated so that the laser is at an angle θ to the normal, how would you expect T to vary with θ if the laser light is polarized perpendicular to the plane of incidence (ie parallel to the axis about which the graphene was rotated)?



E is in plane of graphene still,

so $P_J = \frac{1}{2}\sigma E_i^2$ per unit area of graphene

$= \frac{1}{2}\frac{\sigma E_i^2}{\cos\theta}$ per unit area perpendicular to beam

$\rightarrow \frac{P_J}{I_i} = \frac{\pi\alpha}{\cos\theta} \quad \therefore T = 1 - \frac{\pi\alpha}{\cos\theta}$

9. [6] What is the radiation pressure exerted on the graphene by the laser beam if the incident intensity is I?

Pressure if beam completely absorbed is $p = \frac{I}{c}$

Since a fraction $\pi\alpha$ is absorbed here, $P_{rad} = \frac{\pi\alpha I}{c}$

Name solutions

10. [8] Show how the fields **B** and $\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$ are invariant under a transformation of the potentials **A** and V involving an arbitrary scalar field $\lambda(\mathbf{r},t)$.

$$\vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}\lambda \qquad \text{then } \vec{B}' = \vec{\nabla}\times\vec{A}' = \vec{\nabla}\times\vec{A},\ \underbrace{\vec{\nabla}\times\vec{\nabla}\lambda}_{0} = \vec{\nabla}\times\vec{A} = \vec{B}$$

$$V \to V' = V - \frac{\partial\lambda}{\partial t} \qquad \vec{E}' = -\vec{\nabla}V' - \frac{\partial\vec{A}'}{\partial t} = -\vec{\nabla}V + \vec{\nabla}\left(\frac{\partial\lambda}{\partial t}\right) - \frac{\partial\vec{A}}{\partial t} - \frac{\partial}{\partial t}(\vec{\nabla}\lambda)$$

$$= -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t} = \vec{E}$$

11. [8] The following are two possible expressions for the scalar (electric) potential:

(i) $V(\mathbf{r},t) = \int \frac{\rho(\mathbf{r}',t)d^3r'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|}$, and (ii) $V(\mathbf{r},t) = \int \frac{\rho(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/c)d^3r'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|}$.

How can it be that both of these are valid? Specify the relevant gauges by name and in terms of $\nabla.\mathbf{A}$.

They are in different gauges. (i) is in the Coulomb gauge, where $\vec{\nabla}.\vec{A} = 0$. (ii) is Lorentz gauge, $\vec{\nabla}.\vec{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t}$. $\vec{A}$ must be such that when you calculate $\vec{B}$ or $\vec{E}$ you get the same answer in each case.

12. [6] In what way does one of these expressions appear to violate the principles of special relativity (causality), and why does it actually not do so?

(i) appears to have instantaneous response at finite distance; $V(\vec{r}_1,t)$ depends on $\rho(\vec{r}_2,t)$, $\vec{r}_1 \neq \vec{r}_2$. The badly behaved part, corresponding to information moving faster than c, must be cancelled by a corresponding part of $\vec{A}$.

13. [10] This is a *cross-section* through a strip transmission line (perpendicuar to the signal direction). Sketch the lines of **E** and **B** *both between and around* the plates for the TEM mode propagating in the direction into the page. Write down the relationship between k and ω for this mode. (The transmission line consists of conductors in vacuum.)

$\vec{E}.\vec{B} = 0$

$B_\perp = 0$ at conductor

$E_\parallel = 0$

E has same form as parallel plate capacitor