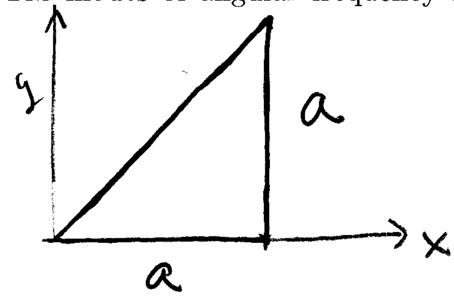
323 Midterm2 Solutions

1. Consider a wave guide with a hollow cross sectional area in the form of a right isosoles triangle as shown in the figure. The direction of propagation is the z direction, with z axis perpendicular to the page. This entire problem is concerned with TM modes of angular frequency ω. Hint: **if** $x = y$, $e^{i(\alpha x+\beta y)} - e^{i(\alpha y+\beta x)} = 0$.



(a) What is B_z? TM means mag. field is transverse $B_z = 0$

(b) For waves propagating in the positive z direction, we may write the z component of the electric field as $E_z(x,y)e^{i(kz-\omega t)}$. State the boundary conditions that $E_z(x,y)$ must obey.

E_z vanishes on conductor surfaces

$E_z(x=a, y) = 0$, $E_z(x, y=0) = 0$, $E_z(x,x) = 0$ if $0 \le x \le a$

(c) Derive a wave equation for $E_z(x,y)$. Maxwell's eqns give

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}(x,y,z,t) = 0$$

Take z-component + using $E_z(x,y)\, e^{i(kz-\omega t)}$

to get

$$\boxed{\left(\nabla_\perp^2 - k^2 + \frac{\omega^2}{c^2}\right) E_z(x,y) = 0}$$

Problem 1 continued

Determine

(d) Determine the set of functions $E_z(x, y)$ that satisfy the wave equation and boundary conditions. k.

$E_z(x,y)$ must vanish as in part (b)

$$E_z(x,y) = \psi_{mn}(x,y) = \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{a} - \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}$$

This is a solution if $\Big\} = -2\left[\cos\frac{\pi}{a}(nx+my) - \cos\frac{\pi}{a}(ny+mx)\right]$

$$k^2 = k_{mn}^2 = \frac{\omega^2}{c^2} - \frac{\pi^2}{a^2}(n^2+m^2)$$

(e) Determine the lowest cutoff frequency.

Need $k_{mn} \geq 0$ $\qquad \omega > \omega_{mn} = \frac{\pi}{a}\sqrt{n^2+m^2}$

Lowest ω_{mn} has $m=1, n=2$

$$\omega_{12} = \frac{\pi}{a}\sqrt{5}$$

$\left(\begin{matrix}\text{if } m=1=n \\ \psi_{11}=0\end{matrix}\right)$

2. This problem concerns radiation from a system of charges and currents that is localized near an origin, surrounded by empty space. Suppose that, for positions $\mathbf{r}$ in the radiation zone, the vector potential can be represented by the expression $\mathbf{A}(\mathbf{r},t) = e^{i(kr-\omega t)}\ \frac{\mathbf{a}(k,\widehat{r})}{r}$, with $\mathbf{a}(k,\widehat{r})$ a given but unspecified function. This problem is concerned with fields in the radiation zone. ($\widehat{r} = \frac{\mathbf{r}}{\mathbf{r}}$).

(a) Determine $\mathbf{B}(\mathbf{r},t)$.

$$\vec{B} = \vec{\nabla} \times \vec{A} = i k\, \hat{r} \times \frac{\vec{a}}{r}\, e^{i(kr-\omega t)}$$

(b) Determine $\mathbf{E}(\mathbf{r},t)$. $\vec{E} = -c\hat{r} \times \vec{B}$ (from $\mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t} = \vec{\nabla} \times \vec{B}$)

(c) Show that the angular distribution of (time averaged) radiated power, $\frac{dP}{d\Omega}$ is given by $\frac{dP}{d\Omega} = \frac{k^2 c}{2\mu_0}|\widehat{r} \times \mathbf{a}|^2$.

$$\frac{dP}{d\Omega} = \frac{1}{2} r^2 \operatorname{Re}\left(\hat{r} \cdot \left(\vec{E} \times \frac{\vec{B}^*}{\mu_0}\right)\right) = \frac{1}{2}\frac{k^2 c}{\mu_0}\, \hat{r} \cdot \left((\hat{r} \times \vec{v}) \times \vec{v}^*\right)$$

with $\vec{v} \equiv \hat{r} \times \vec{a}$, $\hat{r} \cdot \left((\hat{r} \times \vec{v}) \times \vec{v}^*\right) = |\vec{v}|^2$ so

$$\frac{dP}{d\Omega} = \frac{1}{2}\frac{k^2 c}{\mu_0} |\hat{r} \times \vec{a}|^2$$

Problem 2 continued

(d) A system consists of a particle of charge Q moving along a trajectory given by $\mathbf{r}_Q(t) = \hat{\mathbf{x}} L\cos(\omega_0 t)$ and a wire loop of radius $R > L$. This loop is located in the xy plane and is centered at the origin with current $I(t) = I_0\cos(\omega_0 t)$. Suppose $\omega R/c \ll 1$. Determine the vector function $\mathbf{a}(k,\hat{r})$. We may use the long wave length approximation. From text with both electric and magnetic dipole:

$$\vec{a} = -i\omega \frac{\vec{p}\mu_0}{4\pi} - \frac{\mu_0}{4\pi}\frac{i\omega}{c}(\vec{m}\times\hat{r})$$

$$\vec{p} = QL\hat{x} \qquad p = QL$$

$$\vec{m} = I\pi R^2\hat{z} \qquad m = I\pi R^2$$

(e) For the situation of part (d) determine $\frac{dP}{d\Omega}$.

$$\frac{dP}{d\Omega} = \frac{1}{2}\frac{k^2 c}{\mu_0}\left|\hat{r}\times\left(\vec{p}+\frac{\vec{m}}{c}\times\hat{r}\right)\right|^2\left(\frac{\omega\mu_0}{4\pi}\right)^2$$

$$\left|\hat{r}\times\left(\vec{p}+\frac{\vec{m}}{c}\times\hat{r}\right)\right|^2 = |\hat{r}\times\vec{p}|^2 + \frac{m^2}{c^2}\sin^2\theta + 2(\hat{r}\times\vec{p})\cdot\left(\frac{\vec{m}}{c} - \frac{\vec{m}\cdot\hat{r}\,\hat{r}}{c}\right)$$

$$|\hat{r}\times\vec{p}|^2 = \vec{p}^2 - (\hat{r}\cdot\vec{p})^2 = p^2(1-\sin^2\theta\cos^2\phi)$$

$$(\hat{r}\times\vec{p})\cdot\vec{m} = -\hat{r}_y pm = -\sin\theta\sin\phi\, pm$$

$$\boxed{\frac{dP}{d\Omega} = \frac{1}{2}\frac{k^2 c}{\mu_0}\left(\frac{\omega\mu_0}{4\pi}\right)^2\left[p^2(1-\sin^2\theta\cos^2\phi) + \frac{m^2}{c^2}\sin^2\theta + 2pm(-)\sin\theta\sin\phi\right]}$$