Name Solutions

Electrodynamics, Physics 323
Spring 2005

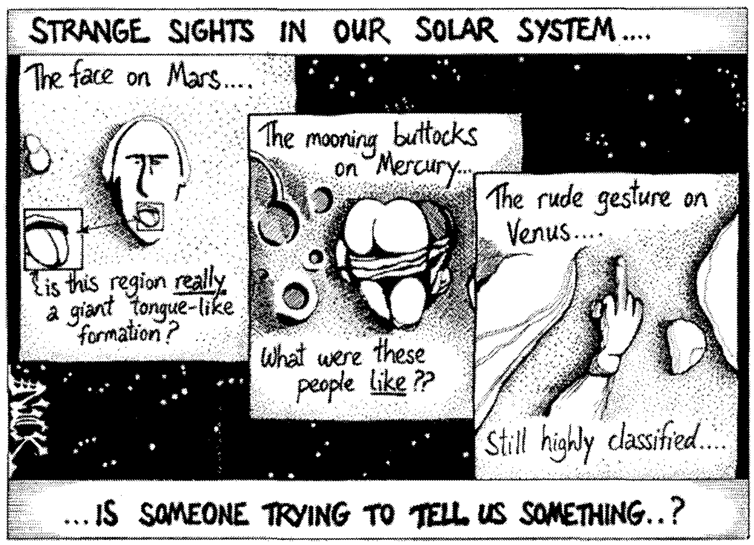
Second midterm
Instructor: David Cobden

8.20 am, May 16, 2005

You have 60 minutes. End on the buzzer at 9.20. Answer all questions.
You are strongly recommended to read through all the questions before you begin.
Write your name on every page and your ID on the first page.
Watch the blackboard for corrections or clarifications during the exam.
In this exam you are allowed **no books, no note sheets, and no calculators!**

Write all your working on these question sheets. You will get credit for it. It is important to show your calculation or derivation, and where appropriate to write a few words to indicate the reasoning. You usually won't get full marks just for stating the correct answer.

Do not turn this page until the buzzer goes.



Name solutions

I. A straight, neutral wire along the z-axis carries a current $I(t) = \begin{cases} 0 & t<0 \\ I_0 t & t \geq 0 \end{cases}$ in the $+\hat{\mathbf{z}}$ direction.

1. [8] State the relations for getting the fields **B** and **E** from the potentials V and **A** in electrodynamics.

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \qquad \vec{B} = \vec{\nabla} \times \vec{A}$$

2. [10] Show that there exists a gauge transformation parameterized by a scalar field $f(\mathbf{r})$ such that neither **E** nor **B** is affected by choice of gauge.

Consider $\vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla} f$ $\quad \therefore \vec{B}' = \vec{\nabla} \times \vec{A}' = \vec{\nabla} \times (\vec{A} + \vec{\nabla} f) = \vec{\nabla} \times \vec{A} = \vec{B}$

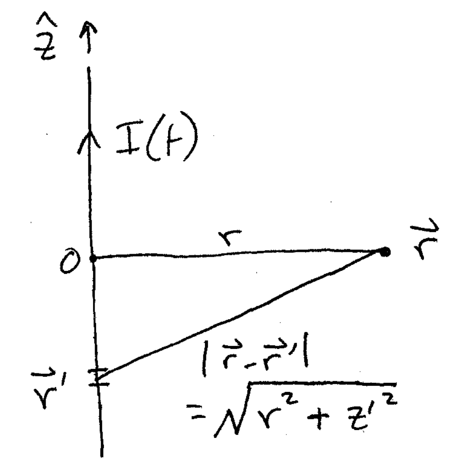
$V \to V' = V - \frac{\partial f}{\partial t}$

$$\vec{E}' = -\vec{\nabla} V' - \frac{\partial \vec{A}'}{\partial t} = -\vec{\nabla}\left(V - \frac{\partial f}{\partial t}\right) - \frac{\partial}{\partial t}\left(\vec{A} + \vec{\nabla} f\right) = \underbrace{-\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}}_{= \vec{E}} + \underbrace{\vec{\nabla}\left(\frac{\partial f}{\partial t}\right) - \frac{\partial}{\partial t} \vec{\nabla} f}_{0}$$

3. [5] State the Lorentz gauge condition. What is the key physical advantage of the Lorentz gauge over the Coulomb gauge?

$$\vec{\nabla} \cdot \vec{A} = -\frac{1}{c^2} \frac{\partial V}{\partial t}$$

4. [16] Find the scalar potential $V(\mathbf{r})$ and vector potential $\mathbf{A(r)}$ for this wire in the Lorentz gauge. Your answer should include a fully explicit integral over the distance z' along the z axis. Do not attempt to evaluate the integral.



$$V(\vec{r}, t) = \int \frac{\rho(\vec{r}'; t_r)}{4\pi\varepsilon_0 |\vec{r} - \vec{r}'|} d^3r' \; ; \; t_r = t - \frac{|\vec{r} - \vec{r}'|}{c}$$

$= 0$ because $\rho = 0$ at all t.

$$\vec{A}(\vec{r}, t) = \int \frac{\mu_0 \vec{J}(\vec{r}', t_r)}{4\pi |\vec{r} - \vec{r}'|} d^3r'$$

$$= \int_{-\infty}^{\infty} \frac{\mu_0 I(t_r) \hat{z} \, dz'}{4\pi |\vec{r} - \vec{r}'|}$$

$t_r = t - \frac{\sqrt{r^2 + z'^2}}{c}$

$$= \frac{\mu_0 \hat{z}}{4\pi} \cdot 2 \int_0 \frac{I_0 t_r \, dz'}{\sqrt{r^2 + z'^2}}$$

$I_0(t_r) = 0$ for $t_r < 0$

ie $r^2 + z'^2 > c^2 t^2$

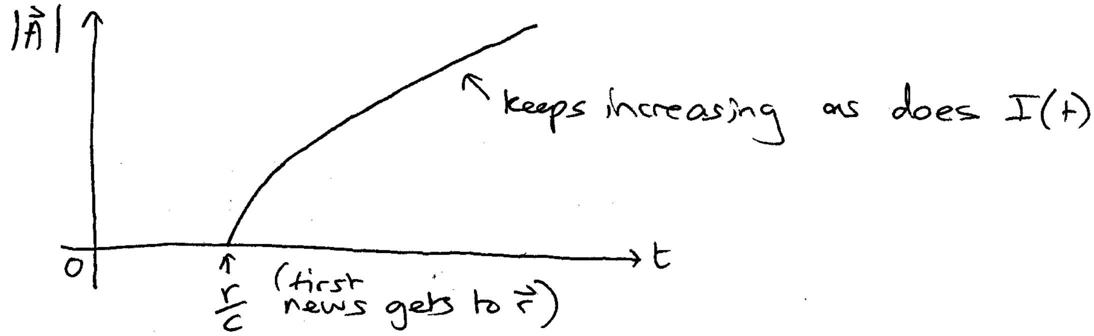
ie $z' > +\sqrt{c^2 t^2 - r^2}$

$$= \frac{\mu_0 I_0 \hat{z}}{2\pi} \int_0^{\sqrt{c^2t^2 - r^2}} \frac{\left(t - \frac{\sqrt{r^2 + z'^2}}{c}\right)}{\sqrt{r^2 + z'^2}} dz'$$

$$= \frac{\mu_0 I_0 \hat{z}}{2\pi} \left[t \int_0^{\sqrt{c^2t^2 - r^2}} \frac{dz'}{\sqrt{r^2 + z'^2}} - \frac{\sqrt{c^2 t^2 - r^2}}{c} \right]$$

Name Solutions

5. [6] Sketch the variation of the magnitude of **A** with time t at a distance r from the wire.



6. [3] Would the result of the previous question be different in the Coulomb gauge?

Yes. (Change in $\vec{A}$ would be instantaneous.)

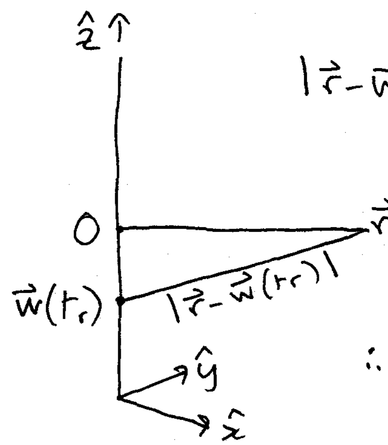
7. [3] Is power radiated during this process?

Yes. ($\vec{B} = \vec{\nabla} \times \vec{A}$ is $/\!/ \hat{z}$, $\vec{E} \sim \int \vec{\nabla} \times \vec{B}\, dt$ is $/\!/ \hat{\phi}$, so $\vec{S}$ is outward).

II. A point charge q moves along the z-axis at constant velocity u, reaching the origin at time $t = 0$.

8. [10] Find the retarded position $\mathbf{w}(t_r)$ on the trajectory which determines the potentials and fields at point $\mathbf{r}$ and time t in the Lorentz gauge.

$$\vec{w}(t) = ut\,\hat{z}.$$



$$|\vec{r} - \vec{w}(t_r)| = c(t - t_r)$$

$$\therefore\ |(x, y, z) - (0, 0, ut_r)| = c(t - t_r)$$

$$\therefore\ x^2 + y^2 + (z - ut_r)^2 = c^2(t - t_r)^2$$

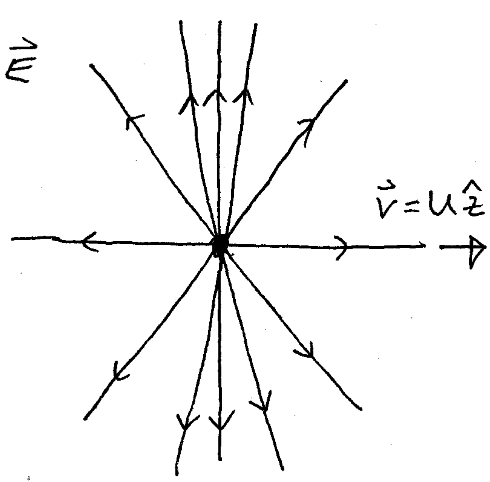
$$\therefore\ x^2 + y^2 + z^2 + u^2 t_r^2 - 2zut_r = c^2 t^2 + c^2 t_r^2 - 2c^2 t t_r$$

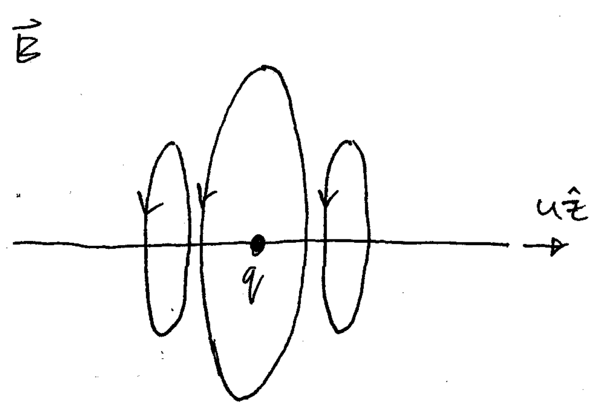
$$\therefore\ (c^2 - u^2) t_r^2 + 2(zu - c^2 t) + c^2 t^2 - r^2 = 0$$

$$\therefore\ t_r = \frac{c^2 t - uz - \sqrt{(zu - c^2 t)^2 - (c^2 - u^2)(c^2 t^2 - r^2)}}{c^2 - u^2}$$

check: must be minus so $t_r = t - r/c$ when $u = 0$

9. [10] Sketch the **E** and **B** fields of the point charge if u is close to c (say, $u = 0.8c$).



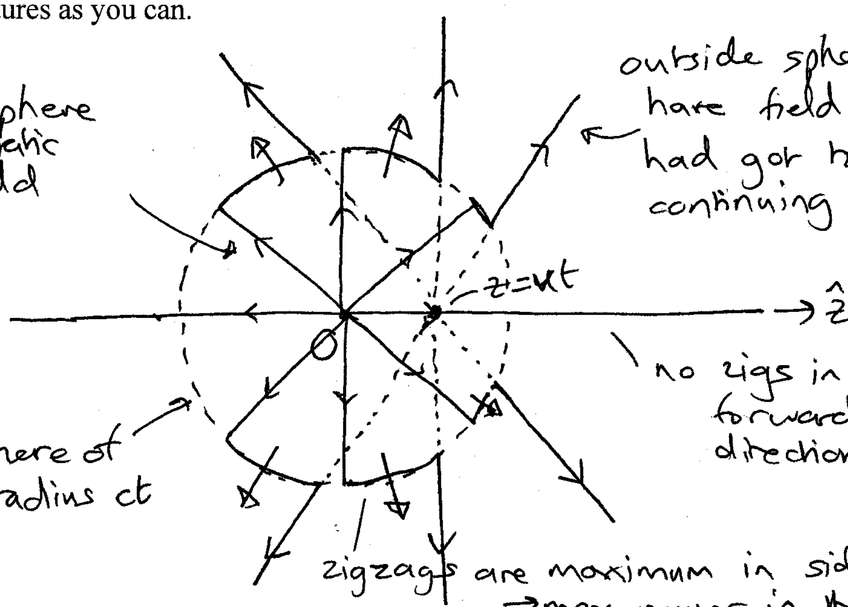


 Name solutions

10. [10] When the charge reaches the origin it suddenly stops. Sketch the lines of electric field at a later time t, showing the effect of this sharp deceleration. Indicate the features in the field lines which tell you that radiation is emitted, and in what direction the power is concentrated, and point out as many other features as you can.

within sphere we have static Coulomb field

outside sphere we have field *as if* charge had got to posit $z=vt$ continuing at speed v.



no zigs in E lines in forward & backward directions → no radiation

sphere of radius ct

zigzags are maximum in sideways directions → max power in these directions.

At a later time, the charge is made to oscillate about the origin, with position given by $z = d\cos\omega t$.

11. [9] What are the conditions under which the electric field at a point **r** due to this oscillating charge is given by the following equation? $\mathbf{E} \simeq \frac{\mu_0\omega^2 d\cos[\omega(t-r/c)]}{4\pi r}\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\hat{\mathbf{z}})$

$d \ll r$ dipole contribution dominates multipole expansion

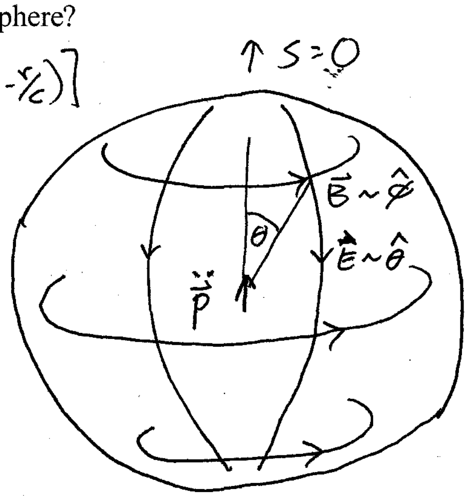
$d \ll \lambda$ "long wavelength" $\lambda = \frac{\omega}{2\pi c}$; phase varies little over source

$r \gg \lambda = \frac{\omega}{2\pi c}$ "radiation zone", radiative terms ≫ static terms.

12. [10] Under these conditions, draw a sketch to indicate the lines of **E** and **B** at a fixed time t over a large sphere centered at the origin. What is the direction and angular dependence of the radiative energy flow through the sphere?

$\ddot{\vec{p}} = -\omega^2 d\,\hat{z}\cos[\omega(t-r/c)]$

This is electric dipole radiation.



→ $S \propto EB \propto \sin^2\theta$

max at $\theta = \frac{\pi}{2}$, equator

zero at poles.