 Name ____________Solutions_______________

Electrodynamics, Physics 323 **Second midterm** **8.20 am, Monday May 21, 2007**
Spring 2007 Instructor: David Cobden

Do not turn this page until the buzzer goes at 8.20. You must hand your exam to me before I leave the room at 9.25.

Attempt all the questions.

Please write your name on every page and your SID on the first page.

Write all your working on these question sheets. Use this front page for extra working. It is important to show your calculation or derivation. Some of the marks are given for showing clear and accurate working and reasoning.

Watch the blackboard for corrections or clarifications during the exam.

This is a closed book exam. ***No books, notes or calculators allowed.***

"'Relativity,' you say? — Well, it can't be any worse than your 'speed bump' theory."

1. [12] State the relations between the electromagnetic fields and the scalar and vector potentials, and show that a gauge transformation of the appropriate form leaves the fields unchanged.

$\vec{B} = \vec{\nabla} \times \vec{A}$

$\vec{E} = -\vec{\nabla} V - \dfrac{\partial \vec{A}}{\partial t}$

If $\vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}\lambda \quad V \to V' - \dfrac{\partial \lambda}{\partial t}$ gauge transformation

Then $\vec{B} \to \vec{B}' = \vec{\nabla} \times \vec{A}' = \vec{\nabla} \times \vec{A} + \underbrace{\vec{\nabla} \times (\vec{\nabla}\lambda)}_{0} = \vec{B}$

$\vec{E} \to \vec{E}' = -\vec{\nabla} V' - \dfrac{\partial \vec{A}'}{\partial t}$

$= -\vec{\nabla}\left(V - \dfrac{\partial \lambda}{\partial t}\right) - \dfrac{\partial}{\partial t}\left(\vec{A} + \vec{\nabla}\lambda\right) = -\vec{\nabla} V - \dfrac{\partial \vec{A}}{\partial t} = \vec{E}$

2. [4] Why is the Coulomb gauge usually a poor choice in electrodynamics?

In this gauge, V responds instantaneously to a change in ρ, which is not manifestly consistent with special relativity.

3. [6] Jefimenko's equation for the magnetic field due to a time dependent charge and current density confined to a region of size L is

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int\left[\frac{\mathbf{J}(\mathbf{r}',t_r)\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} + \frac{\dot{\mathbf{J}}(\mathbf{r}',t_r)\times(\mathbf{r}-\mathbf{r}')}{c\,|\mathbf{r}-\mathbf{r}'|^2}\right]d^3r'. \qquad \text{(i)}$$

Under what conditions (neglecting quantum effects) is Equation (i) valid?

It's exact. (but may diverge if $\vec{J} \neq 0$ at $\vec{r}$)

So make sure $r > L$

4. [10] What is the idea of the quasistatic approximation? Using Equation (i), deduce the condition for the quasistatic approximation to be applicable at frequency ω, and give the expression for **B** in this limit.

For small ω and r, the time-derivative terms become negligible relative to the static ones → quasistatic.

In this case that means $\dfrac{\dot{J}}{cr} \ll \dfrac{J}{r^2} \quad \therefore \dfrac{\omega J}{c} \ll \dfrac{J}{r} \quad \therefore \omega \ll \dfrac{c}{r}$ for all r

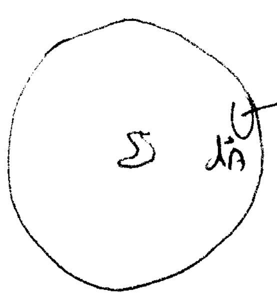
$\therefore \omega \ll \dfrac{c}{L}$, or $L \ll \dfrac{c}{\omega} = \lambda 2\pi$ Then $\vec{B}(\vec{r},t) \doteq \dfrac{\mu_0}{4\pi}\displaystyle\int \dfrac{\vec{J}(\vec{r}',t)\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\,d^3r'$ NOT RETARDED

5. [4] By the same token, what is the radiation zone, and what is the condition to be in the radiation zone?

for LARGE r, the time-dependent (radiation) terms dominate.

Then $\omega \gg \dfrac{c}{r}$, $r \gg \dfrac{c}{\omega} = 2\pi\lambda$

6. [8] Explain why the power flowing out through a large sphere centered on the source is independent of the size of the sphere, in terms of Jefimenko's equations, the Poynting vector, and the basic notion of radiation.



$\vec{S} = \dfrac{1}{\mu_0}\vec{E}\times\vec{B}$

Power $= \oint \vec{S}\cdot d\vec{A} \propto \int E.B.dA$

$\langle \text{Power} \rangle \propto \dfrac{1}{r}\cdot\dfrac{1}{r}\cdot 4\pi r^2 = \text{const}$

$B \propto \dfrac{1}{r}$ for radiation term. $E \doteq cB \propto \dfrac{1}{r}$ too

$\langle \text{Power} \rangle$ = energy rate lost for good by the source.

 Name Solutions

7. [10] A uniform current density $\mathbf{K}(t) = \mathbf{K}_0\Theta(t)$ is switched on at time $t = 0$ simultaneously everywhere on the x-y plane, which is neutral. Write down the general integral giving the scalar potential in the Lorentz gauge, involving the retarded time t_r, and use it to deduce the (retarded) scalar potential $V(\mathbf{r},t)$ at any point out of the plane. Be sure to define t_r.

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,',t_r)}{|\vec{r}-\vec{r}\,'|}\,d^3r' \qquad t_r = t - \frac{|\vec{r}-\vec{r}\,'|}{c} \qquad \text{(Lorentz gauge)}$$

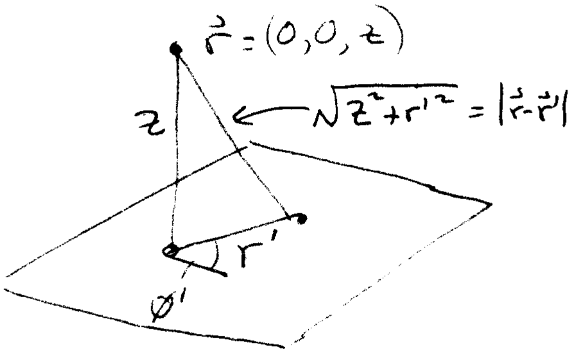
In this problem $\rho = 0$ everywhere and always

$\rightarrow V(\vec{r},t) = 0.$

8. [20] Show that the (retarded) vector potential is $\mathbf{A}(\mathbf{r},t) = \begin{cases} 0, & t < |z|/c \\ \dfrac{\mu_0\mathbf{K}_0}{2}(ct - |z|), & t \ge |z|/c \end{cases}$.

(Hint: you can put the measurement point on the z-axis and use cylindrical coordinates to do the integral. This is probably the hardest question on the exam.)

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,',t_r)}{|\vec{r}-\vec{r}\,'|}\,d^3r' = \frac{\mu_0}{4\pi}\int_0^{2\pi} d\phi' \int_0^{\infty} r\,dr' \int_{-\infty}^{\infty} dz'\,\frac{\vec{J}(\vec{r}\,',t_r)}{|\vec{r}-\vec{r}\,'|}$$



$$\vec{J}(\vec{r}\,',t_r) = \vec{K}_0\,\delta(z)\,\Theta\left(t - \frac{\sqrt{z^2+r'^2}}{c}\right)$$

$$\therefore \vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int_{\phi'=0}^{2\pi}\int_{r'=0}^{\sqrt{c^2t^2-z^2}} \frac{\vec{K}_0\,r'\,d\phi'\,dr'}{\sqrt{z^2+r'^2}}$$

for $ct \ge |z|$

or $= 0$ for $ct < z$.

$$\therefore \text{For } ct \ge z,\quad \vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\cdot 2\pi\cdot\vec{K}_0\cdot\int_0^{\sqrt{c^2t^2-z^2}} \frac{r'\,dr'}{\sqrt{z^2+r'^2}}$$

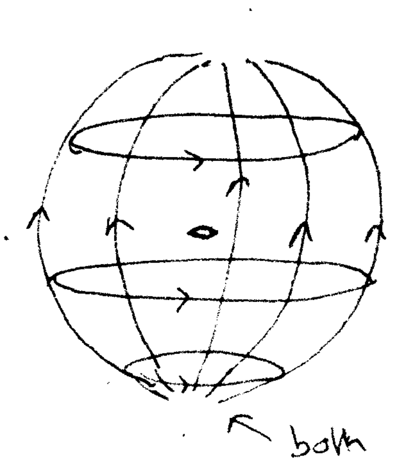
$$= \frac{\mu_0\vec{K}_0}{2}\left[\sqrt{z^2+r'^2}\right]_{r'=0}^{\sqrt{c^2t^2-z^2}}$$

$$= \frac{\mu_0\vec{K}_0}{2}\left(ct - |z|\right)$$

obviously need positive root for continuity at $ct = z$, $t > 0$

 Name solutions

9. [10] Sketch the lines of **E** and **B** on a large sphere in the radiation zone centered on a small wire ring carrying an oscillating current. Indicate where the fields are weakest and strongest.



$\vec{B} \to$ lines of longitude
$\vec{E} \to$ " " latitude

$\leftarrow$ both max at equator $\propto \sin\theta$

$\nwarrow$ both vanish at poles

The instantaneous Poynting vector at point (**r**,*t*) for a source of size *L* located at the origin is given approximately by the following, where $\mathbf{p}(t_r)$ is the dipole moment of the source at the retarded time:

$$\mathbf{S}(\mathbf{r},t) = \frac{\mu_0}{16\pi^2 c r^2}\left[\hat{\mathbf{r}}\times\ddot{\mathbf{p}}(t_r)\right]^2\hat{\mathbf{r}}. \qquad \text{(ii)}$$

10. [6] Under what approximations is Equation (ii) valid? You will lose marks for including approximations which are *not* necessary.

Radiation zone : $r \gg \lambda\,(2\pi) = \frac{c}{\omega}$

Small source : $L \ll \lambda$

These imply that dipole dominates multipole expansion. If $\vec{p} = 0$, another multipole may win.

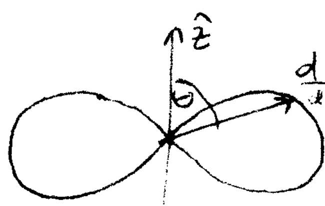
11. [14] An electron (of charge *e*) oscillates about the origin with position $b\hat{\mathbf{z}}\cos\omega t$. Using Equation (ii), find the time-averaged power radiated per unit solid angle, $dP/d\Omega$, as a function of angle θ to the z-axis, and sketch it on a polar plot.

$\vec{p} = eb\hat{z}\cos\omega t \quad \therefore \ddot{\vec{p}} = -\omega^2 eb\hat{z}\cos\omega t$

$\langle S\rangle = \frac{1}{r^2}\frac{dP}{d\Omega}$ if $\vec{S}$ is radial.

$$\therefore \frac{dP}{d\Omega} = r^2\Big\langle \frac{\mu_0}{16\pi^2 c r^2}\Big[\underbrace{\hat{r}\times}_{\substack{\downarrow(\hat{r}\times\hat{z})\\ \sin\theta}} \omega^2 eb\,\hat{z}\underbrace{\cos\omega t}_{\substack{\downarrow\\ \frac{1}{2}}}\Big]^2\Big\rangle$$

$$= \frac{\mu_0}{16\pi^2 c}\cdot\frac{1}{2}\cdot\sin^2\theta\cdot(\omega^2 eb)^2 = \frac{\mu_0 e^2 b^2\omega^4\sin^2\theta}{32\pi^2 c}$$



$\frac{dP}{d\Omega}(\theta) \propto \sin^2\theta$