Do not turn this page until the buzzer goes at 8.20. You must hand your exam to me before I leave the room at 9.25.

Attempt all the questions.

Please write your name on every page and your SID on the first page.

Write all your working on these question sheets. Use this front page for extra working. It is important to show your calculation or derivation. Some of the marks are given for showing clear and accurate working and reasoning.

Watch the blackboard for corrections or clarifications during the exam.

This is a closed book exam. *No books, notes or calculators allowed.*
1. [12] State the relations between the electromagnetic fields and the scalar and vector potentials, and show that a gauge transformation of the appropriate form leaves the fields unchanged.

\[ \mathbf{B} = \mathbf{\hat{E}} \times \mathbf{A} \]

If \( \mathbf{A} \rightarrow \mathbf{A'} = \mathbf{A} + \mathbf{\hat{A}} \lambda \)

\[ \mathbf{V} \rightarrow \mathbf{V'} = \mathbf{V} - \frac{\partial \lambda}{\partial t} \]

\[ \mathbf{E} = -\mathbf{\hat{E}} \mathbf{V} - \frac{\partial \mathbf{A}}{\partial t} \]

Then \( \mathbf{E} \rightarrow \mathbf{E'} = \mathbf{\hat{E}} \mathbf{\hat{A}}' + \frac{\partial \mathbf{\hat{A}}'}{\partial t} \mathbf{\hat{E}} \mathbf{\hat{A}} + \frac{\partial \mathbf{\hat{A}}'}{\partial t} \mathbf{V} = \mathbf{E} \]

\[ \mathbf{E} \rightarrow \mathbf{E'} = -\mathbf{\hat{E}} \mathbf{\hat{A}}' \]

\[ = -\mathbf{\hat{E}} (\mathbf{\hat{E}} \mathbf{\hat{A}}) = \mathbf{E} \]

2. [4] Why is the Coulomb gauge usually a poor choice in electrodynamics?

In this gauge, \( \mathbf{V} \) responds instantaneously to a charge \( \rho \), which is not manifestly consistent with special relativity.

3. [6] Jefimenko's equation for the magnetic field due to a time dependent charge and current density confined to a region of size \( L \) is

\[ \mathbf{B}(r,t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(r',t) \times (r-r')}{|r-r'|^3} + \frac{\mathbf{J}(r',t) \times (r-r')}{c |r-r'|^2} \, d^3r'. \] (i)

Under what conditions (neglecting quantum effects) is Equation (i) valid?

It's exact (but may diverge if \( \omega \neq 0 \) or \( \mathbf{r} \))

so make sure \( r > L \)

4. [10] What is the idea of the quasistatic approximation? Using Equation (i), deduce the condition for the quasistatic approximation to be applicable at frequency \( \omega \), and give the expression for \( \mathbf{B} \) in this limit.

For small \( \omega \) the time-derivative terms become negligible relative to the static ones - quasistatic.

In this case they mean \( \frac{\delta \mathbf{J}}{\delta t} \ll \frac{\mathbf{J}}{c^2} \) : \( \omega \ll \frac{c}{L} \) for all \( r \),

\( \omega \ll \frac{L}{c} \), or \( L \ll \frac{c}{\omega} = \lambda / \omega \).

Then \( \mathbf{B}(r,t) \approx \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(r',t) \times (r-r')}{|r-r'|^3} \, d^3r' \)

5. [4] By the same token, what is the radiation zone, and what is the condition to be in the radiation zone?

For large \( r \), the time-dependent (radiation) terms dominate.

Then \( \omega \ll \frac{c}{L} \), \( r \ll \frac{L}{\omega} = 2\pi \lambda \)

6. [8] Explain why the power flowing out through a large sphere centered on the source is independent of the size of the sphere, in terms of Jefimenko's equations, the Poynting vector, and the basic notion of radiation.

\[ \text{Power} = \oint \mathbf{S} \cdot d\mathbf{A} \propto \oint \mathbf{E} \cdot d\mathbf{A} \]

\[ \langle \text{Power} \rangle \propto \frac{1}{c} \cdot \frac{1}{c} \cdot 4 \pi r^2 \cdot \text{const} \]

\( \propto \frac{1}{c} \) for radiation term. \( \mathbf{E} \ll \mathbf{B} \gg \frac{1}{c} \to \)

\[ \langle \text{Power} \rangle \] energy rate lost for good by the source.
7. [10] A uniform current density \( \mathbf{K}(t) = K_0 \Theta(t) \) is switched on at time \( t = 0 \) simultaneously everywhere on the \( x-y \) plane, which is neutral. Write down the general integral giving the scalar potential in the Lorentz gauge, involving the retarded time \( t_r \), and use it to deduce the (retarded) scalar potential \( V(\mathbf{r}, t) \) at any point out of the plane. Be sure to define \( t_r \).

\[
V(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} \, d^3\mathbf{r}' \\
t_r = t - \frac{|\mathbf{r} - \mathbf{r}'|}{c} \quad \text{(Lorentz gauge)}
\]

In this problem \( \rho = 0 \) everywhere and always

\[
\Rightarrow V(\mathbf{r}, t) = 0.
\]

8. [20] Show that the (retarded) vector potential is

\[
\mathbf{A}(\mathbf{r}, t) = \begin{cases} 
0, & t < |z|/c \\
\frac{\mu_0 K_0}{2} \left( ct - |z| \right), & t \geq |z|/c 
\end{cases}
\]

(Hint: you can put the measurement point on the \( z \)-axis and use cylindrical coordinates to do the integral. This is probably the hardest question on the exam.)

\[
\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0 K_0}{2} \int_0^{2\pi} d\phi' \int_0^{\infty} r' dr' \int_{\sqrt{r'^2 + z'^2}}^{\infty} \frac{\mathbf{J}(\mathbf{r}', t_r) \, d\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}
\]

\[
\mathbf{J}(\mathbf{r}', t_r) = K_0 \delta(z') \Theta \left( t - \frac{\sqrt{z'^2 + r'^2}}{c} \right)
\]

\[
\Rightarrow \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0 K_0}{2} \int_0^{2\pi} d\phi' \int_0^{\infty} r' dr' \int_{\sqrt{r'^2 + z'^2}}^{\infty} \frac{K_0 \mathbf{r}' \, d\mathbf{r}'}{\sqrt{z'^2 + r'^2}}
\]

for \( ct > |z| \)

or \( = 0 \) for \( ct < |z| \).

\[
\therefore \text{For } ct > |z|, \quad \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0 K_0}{2 \pi} \int_0^{\sqrt{z^2 + r^2}} r' dr' \int_{\sqrt{r'^2 + z'^2}}^{\infty} \frac{K_0 \mathbf{r}' \, d\mathbf{r}'}{\sqrt{z'^2 + r'^2}}
\]

\[
= \frac{\mu_0 K_0}{2 \pi} \left[ \sqrt{z^2 + r^2} \right]_{r'=0}^{r=\sqrt{z^2 + r^2}} \left[ \sqrt{z^2 + r^2} \right]_{r'=0}^{r=\sqrt{z^2 + r^2}}
\]

\[
= \frac{\mu_0 K_0}{2} \left( ct - |z| \right) \text{ obviously need positive root for continuity at } ct = |z|, t > 0
\]
9. [10] Sketch the lines of \( \mathbf{E} \) and \( \mathbf{B} \) on a large sphere in the radiation zone centered on a small wire ring carrying an oscillating current. Indicate where the fields are weakest and strongest.

![Image of Earth with lines and labels indicating maxima and minima at equator and poles.]

The instantaneous Poynting vector at point \((r,t)\) for a source of size \(L\) located at the origin is given approximately by the following, where \(\mathbf{p}(t)\) is the dipole moment of the source at the retarded time:

\[
S(r,t) = \frac{\mu_0}{16\pi c^3 r^3} \left[ \mathbf{r} \times \mathbf{p}(t) \right]^2 \mathbf{r}.
\]  

(ii)

10. [6] Under what approximations is Equation (ii) valid? You will lose marks for including approximations which are not necessary.

- **Radiation zone**: \(r \gg \lambda\) \(\Rightarrow \theta \ll 1\)
- **Small source**: \(L \ll \lambda\)

These imply that dipole dominates multipole expansion. If \(\mathbf{p} = 0\), other multipole may win.

11. [14] An electron (of charge \(e\)) oscillates about the origin with position \(\mathbf{b} \cos \omega t\). Using Equation (ii), find the time-averaged power radiated per unit solid angle, \(dP/d\Omega\), as a function of angle \(\theta\) to the \(z\)-axis, and sketch it on a polar plot.

\[
\mathbf{p} = e\mathbf{b} \cos \omega t \quad \Rightarrow \quad \mathbf{S} = -\mathbf{b} e \mathbf{b} \hat{z} \cos \omega t
\]

\[
\frac{d^2 S}{d\Omega} = \frac{1}{r^2} \frac{dP}{d\Omega} \quad \because \mathbf{S} \text{ is radial.}
\]

\[
\frac{dP}{d\Omega} = r^2 \left( \frac{2}{16\pi c^3} \left[ \mathbf{r} \times (\mathbf{b} e \mathbf{b} \hat{z} \cos \omega t)^2 \right] \right) \sin \theta \frac{1}{2}
\]

\[
= \mu_0 \frac{1}{16\pi c^2} \frac{1}{2} \sin \theta (\omega e b)^2 = \frac{\mu_0 e b^2 \omega^2 \sin \theta}{32 \pi c}
\]