2003 323 Final exam Solutions

Prof Miller

1. Suppose light is propagated from air ($n = 1$) into a medium of index of refraction $n = 4/3$, with $\mu = \mu_0$.

(a) Determine ϵ/ϵ_0 for the medium.

$$n^2 = \frac{\varepsilon}{\varepsilon_0} = \frac{16}{9}$$

(b) In most situations, there will be a reflected wave and a transmitted wave. Explain the physical origin of the reflected wave.

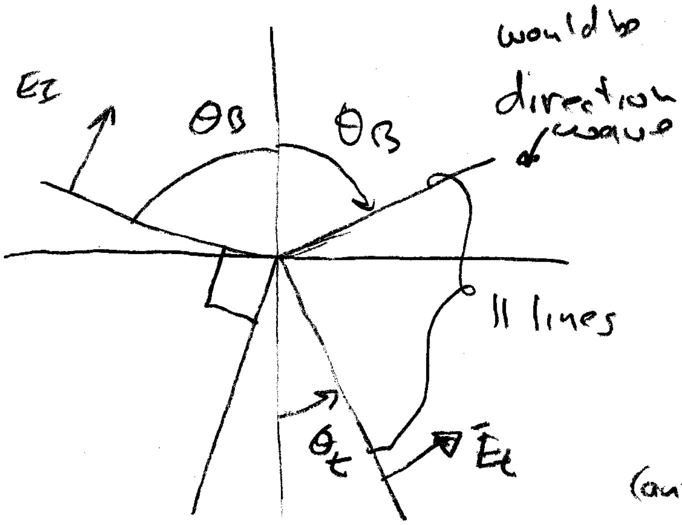
The incident wave causes electric dipoles in the medium (n) to oscillate. This oscillation causes radiation that is the reflected wave

(c) Suppose the incident light is polarized parallel to the plane of incidence and has an amplitude E_{0I} and a frequency ω. Determine the tangent of the angle of incidence θ_I for which the reflection coefficient vanishes.

$$\tan\theta_I = \frac{n_2}{n_1} \qquad \overset{\text{eq}}{(9.112)}$$

$$= 4/3$$

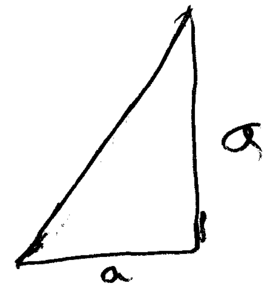
(d) Provide an explanation for the vanishing of part (c). Your explanation should involve geometry and the radiation fields produced by electric dipoles.



$\vec{E}_t$ cause $\vec{p} \parallel \vec{E}_t$

Oscillations of $\vec{p}$ cause radiation. But $\vec{E}_{rad} = 0$ in directions (anti) parallel to the dipole moment.

2. Consider a wave guide with a hollow cross sectional area in the form of a right isosceles triangle as shown in the figure. The direction of propagation is the z direction, with z axis perpendicular to the page. This entire problem is concerned with **TE** modes of angular frequency ω.



(a) What is E_z? 0

(b) For waves propagating in the positive z direction, we may write the z component of the magnetic field as $B_z(x, y)e^{i(kz-\omega t)}$. State the specific boundary conditions that $B_z(x, y)$ must obey.

$\vec{B}\cdot\hat{n} = 0$. This translates (see 9.181 for example) to $\hat{n}\cdot\vec{\nabla} B_z = 0$ on surfaces. So

$$0 = \frac{\partial B_z}{\partial y}(x, y=0) = \frac{\partial B_z}{\partial x}(x=a, y) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) B_z(x,y)\Big|_{x=y}$$

(c) Determine the set of functions $B_z(x, y)$ that satisfy the wave equation (9.181) and the boundary conditions.

$$B_z = \cos\frac{n\pi x}{a}\cos\frac{m\pi y}{a} - \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{a}$$

$n = $ integer

(d) Determine the lowest cutoff frequency.

$$\omega_{1,2} = \frac{\pi}{a}\sqrt{5}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{-i\mu_0\omega}{4\pi r}\mathbf{p}\, e^{i(kr-\omega t)}, \quad \mathbf{A}(\mathbf{r},t) = \frac{-i\mu_0\omega}{4\pi r}\frac{1}{c}\mathbf{m}\times\hat{\mathbf{r}}\, e^{i(kr-\omega t)}$$

3. A system consists of a particle of charge Q moving along a trajectory given by $\mathbf{r}_Q(t) = \hat{\mathbf{z}}L\sin(\omega_0 t)$ and a wire loop of radius $R > L$. This loop is located in the xy plane and is centered at the origin with current $I(t) = I_0\cos(\omega_0 t)$. Suppose $\omega R/c \ll 1$.

(a) Determine the vector potential $\mathbf{A}(\mathbf{r},t)$ for positions in the radiation zone.

Here $\vec{p} = \hat{z}\, QL\,(-i)$ since $\sin\omega_0 t = \cos(\omega_0 t - \frac{\pi}{2})$

$\vec{m} = I_0 \pi R^2 \hat{z}$

$\omega = \omega_0$ $\qquad k = \frac{\omega_0}{c}$

So

$$\vec{A}(\vec{r},t) = \frac{-i\mu_0}{4\pi r}\omega_0\, e^{i(kr-\omega_0 t)}\left[\hat{z}(-i\,QL) + \frac{I_0\pi R^2}{c}(\hat{z}\times\hat{r})\right]$$

(b) For the situation of part (a) determine the electric field.

$\vec{B} = \vec{\nabla}\times\vec{A} = ik\hat{r}\times\vec{A}$, $\quad \vec{E} = -c\hat{r}\times\vec{B}$

$$\vec{E} = -(c\,ik)\,\hat{r}\times(\hat{r}\times\vec{A}) = -(c\,ik)[\hat{r}\,\hat{r}\cdot\vec{A} - \vec{A}]$$

$$\vec{E} = -(c\,k)\left(\frac{-i\mu_0}{4\pi r}\omega_0\right)e^{i(kr-\omega_0 t)}$$

$$\left[\hat{r}\,\hat{r}\cdot\hat{z}\,(-iQL) - \left(\hat{z}(-iQL) + \frac{I_0\pi R^2}{c}\right)(\hat{z}\times\hat{r})\right]$$

$\hat{r}\cdot\hat{z} = \cos\theta$

Problem 3 continued

(c) Measurements at positions in the radiation zone along the y axis determine that the light is circularly polarized. Determine R in terms of L and other given quantities.

On y-axis $\cos\theta = 0$ $(\theta = \pi/2)$

and $\vec{m} \times \hat{r} = \hat{z} \times \hat{y} = -\hat{x}$ Thus

$\vec{E}$ has two orthogonal components

out of phase by $\pi/2$. Circular pol. $\Rightarrow$

Need to have $QL = \frac{m}{c}$

$$\boxed{QL = \frac{I_0 \pi R^2}{c}}$$

4. In a reference frame K two almost evenly matched sprinters are lined up a distance d apart on the x axis for a race parallel to the y axis. Two starters, one beside each sprinter, will fire their starting pistols at slightly different times, giving a handicap to the better of the two sprinters: the slower runner will start at a time T before the faster runner.

(a) For what range of values of T is there a reference frame $\bar{K}$ for which there is no handicap.

No handicap if event has spacelike separation. event = start of race

$$c^2 \Delta\tau^2 = c^2 T^2 - d^2 < 0$$

if $\boxed{T < d/c}$ real no handicap exists

(b) Suppose T is such that there is no handicap. Find the frame $\bar{K}$.

Find frame in which $\Delta \bar{T} = 0$

$$\bar{t}_1 = \gamma(cT - \beta d) \quad \text{slower runner}$$

$$\bar{t}_2 = \gamma(0) = 0$$

$\Delta \bar{T} = 0$ if $cT - \beta d = 0$

$V = d/T$ if slower runner at $x = d$

$= -\frac{d}{T}$ if faster " " "

$\bar{K}$ moves with V down x axis

5. The threshold kinetic energy T_{th} in the laboratory (target particle at rest) for a given reaction is the kinetic energy of the incident particle on a stationary target just sufficient to make the center of mass energy W equal to the sum of the rest energies of the particles in the final state. (W is the sum of the energies of the particles in the frame in which the sum of the particle's three momentum vanishes, for two particles $-W^2/c^2 = (p_1+p_2)_\mu(p_1+p_2)^\mu$. (Griffiths's convention)

(a) Calculate the threshold kinetic energy for the process of neutral pion production in a collision between a proton beam and a target nucleus, A: $pA \to pA\pi^0$. Take the proton mass to be M, the mass of a nucleus to be MA and the pion mass to be m_π.

Compute total momentum^2 in Lab initial and cm. final

$$(E_L + MA)^2 - P_L^2 = \left(M(A+1) + m_\pi\right)^2$$

P_L, E_L = Lab momentum and energy $E_L = M + T_{th}$

$$M^2 + 2(M+T_{th})(MA) + M^2A^2 = M^2(A^2+2A+1) + m_\pi^2 + 2m_\pi M(A+1)$$

$$2MA\,T_{th} = 2M(A+1)m_\pi + m_\pi^2$$

$$T_{th} = \frac{A+1}{A} m_\pi + \frac{m_\pi^2}{2MA}$$

(b) Suppose $Mc^2 = 1000$ MeV and $m_\pi c^2 = 150MeV$ and you have a beam of protons of kinetic energy 180 MeV. Can this beam be used to produce pions in proton proton collisions?

No. This corresponds to $A=1$ (pp)

$$T_{th} = 2m_\pi + \frac{m_\pi^2}{2M} > 300 \text{ MeV}$$

(c) For the situation of (b), how large must A be for pion production to be allowed?

$$\frac{T_{th}}{m_\pi} - 1 = \frac{1}{A} + \frac{1}{A}\left(\frac{m_\pi}{2MA}\right) = 0.2$$

$$A = \frac{1 + \frac{m_\pi}{2MA}}{0.2} = 5\left(1 + \frac{3}{20}\right)$$

$$\boxed{A = 6}$$

$\hat{z}$

6. In a frame $\mathcal{S}$, the vector potential $\mathbf{A}$ is given by $\mathbf{A} = A_0 \hat{z} \exp(-r^2/R^2)$, where R is a given length and A_0 is a given potential strength. The electric potential $V(\vec{r}, t) = 0$.
(a) What is the charge density ρ in the frame $\mathcal{S}$?

$$\rho = 0 \qquad \left(\Box V = \rho(-\mu_0 c) \quad \text{if } V=0 \quad \rho = 0 \right)$$

(b) Determine $\mathbf{E}(\mathbf{r}, \mathbf{t})$ and $\mathbf{B}(\mathbf{r}, \mathbf{t})$ in $\mathcal{S}$.

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} = 0$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$B_x = A_0 \frac{\partial}{\partial y} e^{-r^2/R^2} = A_0 \frac{(-2y)}{R^2} e^{-r^2/R^2}$$

$$B_y = -A_0 \frac{\partial}{\partial x} e^{-r^2/R^2} = A_0 \frac{2x}{R^2} e^{-r^2/R^2}$$

$$B_z = 0$$

Problem 6 continued

(c) The frame $\overline{\mathcal{S}}$ moves relative to $\mathcal{S}$ at a velocity $\mathbf{v} = v\hat{x}$, and its origin overlaps that of $\mathcal{S}$ at $t = 0$. Determine $\overline{\mathbf{E}}(\bar{\mathbf{r}}, \bar{t})$ and $\overline{\mathbf{B}}(\bar{\mathbf{r}}, \bar{t})$ in the frame $\overline{\mathcal{S}}$.

Use 12.108 $\quad \bar{E}_x = 0 \quad \bar{E}_y = -\gamma v B_z = 0 \;, \quad \bar{E}_z = \gamma v B_y$

$$r^2 \text{ in } \bar{S} = \gamma^2 (\bar{x} + v\bar{t})^2 + y^2 + z^2 \equiv \bar{r}^2(t)$$

$$\bar{E}_z = \gamma v \frac{(2A_0)}{R^2} \gamma(\bar{x} + v\bar{t}) e^{-\bar{r}^2/R^2} \qquad \bar{y} = y, \quad \bar{z} = z$$

$$\bar{B}_x = B_x(\bar{r})$$

$$\bar{B}_y = \gamma B_y = \frac{\gamma 2A_0}{R^2} \gamma(\bar{x} + v\bar{t}) e^{-\bar{r}^2/R^2}$$

$$\bar{B}_z = 0$$

(d) Determine the charge density $\bar{\rho}(\bar{\mathbf{r}}, \bar{t})$ in the frame $\overline{\mathcal{S}}$.

$$\frac{\bar{\rho}(\bar{r}, \bar{t})}{\varepsilon_0} = \vec{\bar{\nabla}} \cdot E = \frac{\partial E_z}{\partial z}$$

$$= \gamma v \frac{2A_0}{R^2} \gamma(\bar{x} + v\bar{t}) \left(\frac{-2z}{R^2}\right) e^{-\bar{r}^2/R^2}$$

Can also get $\vec{J}$ from $\Box \vec{A} = -\mu_0 \vec{J}$, then use Lorentz transformation to get $\bar{\rho}, \vec{\bar{J}}$.