 Name ______________solutions_________

Electrodynamics, Physics 323 **Final exam** **8.20 am, June 8, 2004**
Spring 2004 Instructor: David Cobden

You have 120 minutes. End on the buzzer at 10.20. Attempt all the questions if you can.

Write your name on every page and your ID on the first page.

Write all your working on these question sheets. Use this cover page for extra working (you might get credit for it.)

It is important to show your calculation or derivation. You won't get full marks just for stating the correct answer if you don't show how you get it.

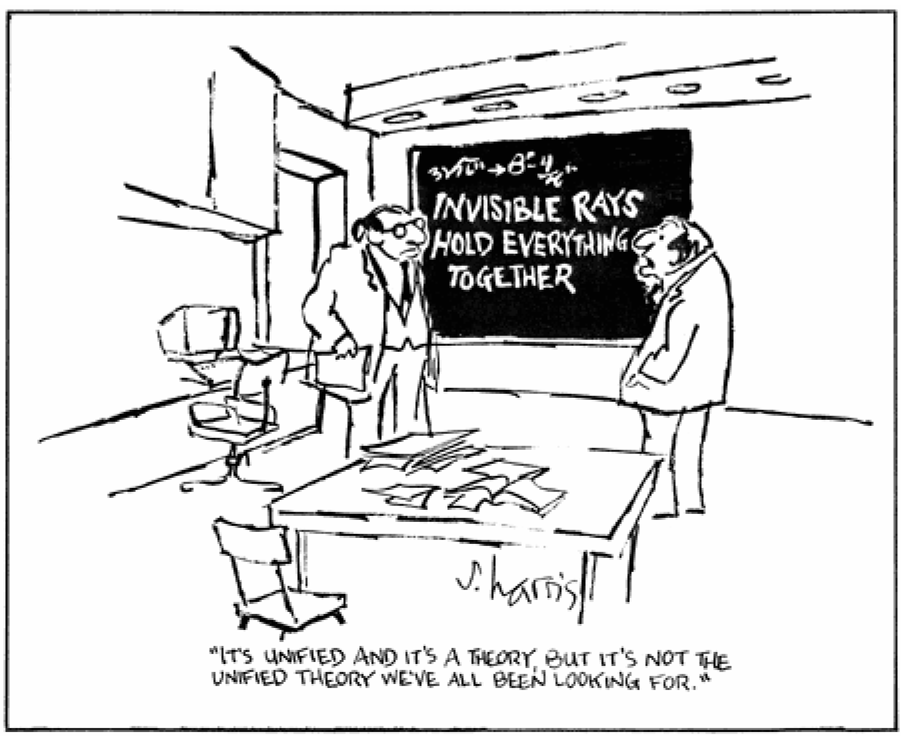
Watch the blackboard for corrections or clarifications during the exam.

This is a **closed book** exam. **No notes allowed. No calculators.**

Do not turn this page until the buzzer goes at 8.20.

$$\Lambda^{m}{}_{n} = \begin{pmatrix} g & -g\,v/c & 0 & 0 \\ -g\,v/c & g & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad F^{mn} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_y/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

$$A^{m} = (V/c, \mathbf{A}) \qquad J^{m} = (c r, \mathbf{J}) \qquad A^{m} = \sum_{n=0}^{3} \Lambda^{m}{}_{n} A^{n} \qquad F'^{mn} = \sum_{a=0}^{3}\sum_{b=0}^{3} \Lambda^{m}{}_{a} \Lambda^{n}{}_{b} F^{ab}$$



"IT'S UNIFIED AND IT'S A THEORY, BUT IT'S NOT THE UNIFIED THEORY WE'VE ALL BEEN LOOKING FOR."

 Name solutions

I. Say whether each of the following is true or false. [5 pts for the right answer, -1 pt for the wrong answer, 0 pts for no answer.]

1. [5] Maxwell's equations are not Lorentz invariant unless they are written solely in terms of Lorentz tensors (of various ranks). FALSE

2. [5] A sphere of uniform charge density ρ which is stationary in inertial frame S will appear as an ellipsoid of charge density $(1-u^2/c^2)^{-1/2}\rho$ in frame S′ traveling with velocity u relative to S. TRUE

3. [5] If event X_2 occurs later than event X_1 in frame S, it is never possible to find a frame S′ in which X_2 occurs earlier than X_1. FALSE

II. The electric field in a medium of conductivity σ obeys the equation $\nabla^2\mathbf{E} = \mu\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2} + \mu\sigma\frac{\partial\mathbf{E}}{\partial t}$. We are concerned with the propagation of electromagnetic waves at frequency ω.

4. [5] Why is $\omega \ll \sigma/\varepsilon$ said to be the criterion for a 'good' conductor?

$\frac{\varepsilon}{\sigma}$ = timescale for charge relaxation. OR: Real current >> displacement current.

5. [15] Show that in a good conductor, the wave decays exponentially with length scale $\delta = (2/\mu\sigma\omega)^{1/2}$.

Try $\vec{E} = \vec{E}_0 e^{i(kx-\omega t)}$ (4) $\rightarrow -k^2\vec{E} = -\mu\varepsilon\omega^2\vec{E} - i\omega\mu\sigma\vec{E}$

$\therefore k^2 = \mu\varepsilon\omega^2 + i\mu\sigma\omega$ (4)

For $\omega \ll \sigma/\varepsilon$, $k^2 \simeq i\mu\sigma\omega$ (2) $\therefore k \simeq (1+i)\sqrt{\frac{\mu\sigma\omega}{2}} = \frac{1+i}{\delta}$ (2)

$\therefore \vec{E} \simeq \vec{E}_0 e^{-\frac{x}{\delta}} e^{i(\frac{x}{\delta}-\omega t)}$ (3)

↑ decays on length $\delta = \sqrt{\frac{2}{\mu\sigma\omega}}$

6. [10] Show that the amplitude of the wave decreases by a factor $e^{-2\pi}$ over a distance of one wavelength in the propagation direction.

$k = \frac{2\pi}{\lambda} = \frac{1}{\delta}$ (3) $\therefore \lambda = 2\pi\delta$ (3)

$\therefore$ at $x = \lambda$, $\vec{E} = \vec{E}_0 e^{-2\pi} e^{i(\frac{x}{\delta}-\omega t)}$ (4)

↑ as required

 Name solutions

III. The power radiated by an electric dipole **p** is $P = \mu_0 \ddot{p}^2 / 6\pi c$.

7. [5] Obtain from this the Larmor formula for the power radiated by a moving charged particle.

$\vec{p} = q\vec{x}^{(2)}$ ∴ $P = \frac{\mu_0 (q\ddot{x})^2}{6\pi c} = \frac{\mu_0 q^2 a^2}{6\pi c}$ (3) a = acceleration of particle

8. [5] Under what conditions does the Larmor fomula break down?

$\frac{v}{c}$ not $\ll 1$ 5 (relativistic limit)

9. [5] Give a simple reason for why a particle moving at constant velocity does not radiate.

5 In its rest frame, which is inertial, it clearly doesn't radiate, so it can't radiate in any frame.
(2pt for a=0)

10. [20] A particle of charge *q* and mass *m* is released from rest and falls under gravity for a time *T*. What fraction of the potential energy lost is radiated away? (Make a reasonable approximation).

q • m ↓ F=mg a=g

Radiated energy $E_{rad} = \int_0^T \frac{\mu_0 g^2 q^2}{6\pi c} dt$ (5)

$= \frac{\mu_0 g^2 q^2 T}{6\pi c}$ (5)

Loss of gravitational energy: $U = mg \cdot \frac{1}{2} g T^2$

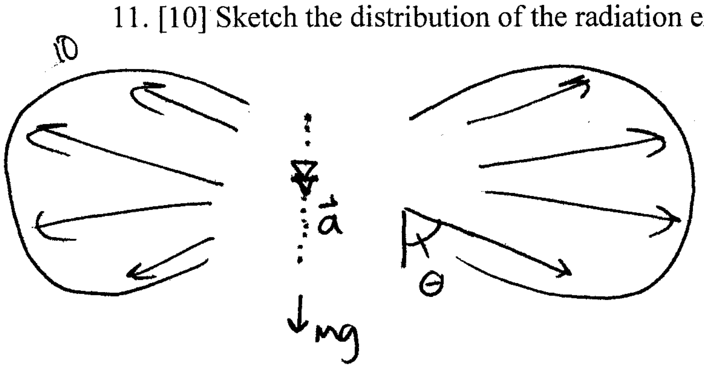
$= \frac{mg^2 T^2}{2}$ (5)

∴ $\frac{E_{rad}}{U} = \frac{\left(\frac{\mu_0 q^2 g^2 T}{6\pi c}\right)}{\left(\frac{mg^2 T^2}{2}\right)} = \frac{\mu_0 q^2}{3\pi m c T}$ (5)

we assumed that $E_{rad} \ll U$, which is true.
(otherwise motion would be more complicated)

11. [10] Sketch the distribution of the radiation emitted during this process.

10



radiation is emitted with $\frac{dP}{d\Omega} \sim \sin^2\theta$
↑ angle to vertical

(4 compensation points for time distribution)

 Name solutions

IV. When a point charge q moves along a trajectory $\mathbf{w}(t)$, the resulting scalar potential is given by the Lienard-Wiechert formula, $V(\mathbf{x},t) = \dfrac{q}{4\pi\varepsilon_0(X - \mathbf{X}.\mathbf{v}/c)}$.

5 12. [5] Is this equation correct at relativistic velocities? YES

13. [10] What *precisely* is the displacement vector **X** in this equation, and how precisely is the velocity vector **v** specified?

$\vec{X} = \vec{x} - \vec{w}(t_r)$ 3 where t_r is the solution of $c(t-t_r) = |\vec{x} - \vec{w}(t_r)|$ 2

5 $\vec{v} = \vec{v}(t_r)$ (at the same point on the trajectory)

14. [5] Write down the standard solution for the retarded potential $V(\mathbf{x},t)$ resulting from a charge density $\rho(\mathbf{x},t)$ in the Lorentz gauge.

5
$$V(\vec{x},t) = \int \frac{\rho\left(\vec{x}', t - \frac{|\vec{x}-\vec{x}'|}{c}\right)}{4\pi\varepsilon_0|\vec{x}-\vec{x}'|} d^3x'$$

15. [5] Briefly, why is it *wrong* just to put the retarded position in this equation, which would lead to the *incorrect* result $V(\mathbf{x},t) = \dfrac{q}{4\pi\varepsilon_0 X}$?

5 As the integral sweeps over the charge, the charge moves. (This changes the range of x' over which ρ is nonzero, for instance, if the particle is taken to be a ball of charge)

16. [10] Evaluate the Lienard-Wiechert formula above for the case of a charge at rest at position $\mathbf{w}_0$.

$\vec{w}(t) = \vec{w}_0 \quad \vec{v}(t) = 0$

$\vec{X} = \vec{x} - \vec{w}_0$

$X = |\vec{X}| = |\vec{x} - \vec{w}_0|$ 5

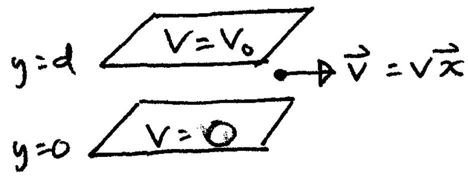
$$\therefore V(\vec{x},t) = \frac{q}{4\pi\varepsilon_0[|\vec{x}-\vec{w}_0| - 0]} = \frac{q}{4\pi\varepsilon_0|\vec{x}-\vec{w}_0|}$$ 5

(static Coulomb potential)

Name solutions

V. In the laboratory frame S there exist two stationary infinite conducting surfaces at $y = 0$ and $y = d$, separated by vacuum. The lower plate is grounded while the upper is held at potential V_0. A particle of charge q moves along with velocity $\mathbf{v} = v\hat{\mathbf{x}}$ in between the plates. The rest frame of the particle is S′.

17. [5] What is the force **F** on the particle in the lab frame S?



5 $\vec{F} = q\vec{E} = -q\dfrac{V_0}{d}\hat{y}$

18. [6] Write a manifestly covariant equation which relates A^μ and J^μ, and state the Lorentz gauge condition in corresponding covariant form.

$\Box^2 A^\mu = -\mu_0 J^\mu$ 3 $\quad$ 3 $\dfrac{\partial A^\mu}{\partial x^\mu} = 0$ (Lorentz gauge)

(or $\dfrac{\partial}{\partial x^\nu}\dfrac{\partial}{\partial x_\nu} A^\mu = -\mu_0 J^\mu$)

19. [14] Find the components in S of:

(a) the 4-potential A^μ,

(b) the corresponding 4-current J^μ (use a Dirac delta function for the charge density), and

(c) the electromagnetic field tensor $F^{\mu\nu}$.

In S, $\rho = \varepsilon_0 \dfrac{V_0}{d}\left[\delta(y-d) - \delta(y)\right]$ (2)

$\vec{J} = 0$ (1)

$V = V_0\, y/d$ (2)

$\vec{A} = 0$ (1)

$\vec{E} = -\dfrac{V_0}{d}\hat{y}$ (2) $\quad \vec{B} = 0$ (1)

$\therefore\ A^\mu = \left(\dfrac{V_0 y}{dc}, 0, 0, 0\right)$ (1) 4T

$$J^\mu = \begin{pmatrix} c\varepsilon_0 \frac{V_0}{d}\left[\delta(y-d) - \delta(y)\right] \\ 0 \\ 0 \\ 0 \end{pmatrix}$$ (2) 5T

$$F^{\mu\nu} = \begin{pmatrix} 0 & 0 & -\frac{V_0}{dc} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{V_0}{dc} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ (2) 5T

−1 not AS

−1 wrong sign for $\vec{E}$

Name solutions

20. [20] By Lorentz transformation (or otherwise), find the components of the three tensors, A'^{μ}, J'^{μ}, and $F'^{\mu\nu}$, in the particle's rest frame S′.

(6)
$$A'^{\mu} = \Lambda^{\mu}{}_{\nu} A^{\nu} = \begin{pmatrix} \gamma & -\frac{v}{c}\gamma & 0 & 0 \\ -\frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{V_0 y}{dc} \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{\gamma V_0 y}{dc} \begin{pmatrix} 1 \\ -\frac{v}{c} \\ 0 \\ 0 \end{pmatrix}$$

(6)
$$J'^{\mu} = \Lambda^{\mu}{}_{\nu} J^{\nu} = \begin{pmatrix} \gamma & -\frac{v}{c}\gamma & 0 & 0 \\ -\frac{v\gamma}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \rho c \\ 0 \\ 0 \\ 0 \end{pmatrix} = \gamma c \varepsilon_0 \frac{V_0}{d}\left[\delta(y-d) - \delta(y)\right] \begin{pmatrix} 1 \\ -\frac{v}{c} \\ 0 \\ 0 \end{pmatrix}$$

(8)
$$F'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} F^{\alpha\beta} = \Lambda^{\mu}{}_{0} \Lambda^{\nu}{}_{2} F^{02} + \Lambda^{\mu}{}_{2} \Lambda^{\nu}{}_{0} F^{20}$$

(all other $F^{\alpha\beta}$ are zero)

$$= \frac{V_0}{dc}\left(-\Lambda^{\mu}{}_{0} \Lambda^{\nu}{}_{2} + \Lambda^{\mu}{}_{2} \Lambda^{\nu}{}_{0}\right)$$

$$= \frac{\gamma V_0}{cd} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & \frac{v}{c} & 0 \\ 1 & -\frac{v}{c} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

 Name solutions

21. [10] Hence, or otherwise, find the potentials V' and $\mathbf{A}'$ and the fields $\mathbf{E}'$ and $\mathbf{B}'$ experienced by the particle in S′.

$$V' = cA'^0 = \frac{\gamma V_0 y}{d} \quad (3) \qquad \vec{A}' = (A'^1, A'^2, A'^3) = \left(-\frac{\gamma v}{c^2}\frac{V_0 y}{d}, 0, 0\right) \quad (3)$$

$$\vec{E}' = (F'^{01}, F'^{02}, F'^{03})c = \left(0, -\frac{\gamma V_0}{cd}, 0\right)c = -\frac{\gamma V_0}{d}\hat{y} \quad (2)$$

$$\vec{B}' = (F'^{23}, -F'^{13}, F'^{12}) = \left(0, 0, \frac{\gamma v}{c^2}\frac{V_0}{d}\right) = \frac{\gamma v}{c^2}\frac{V_0}{d}\hat{z} \quad (2)$$

22. [10] Show that $\mathbf{E}'$ and $\mathbf{B}'$ are generated by the current and charge density on the plates in S′.

$$\rho' = \frac{J'^0}{c} = \gamma \varepsilon_0 \frac{V_0}{d}\left[\delta(y-d) - \delta(y)\right] = \gamma\rho$$

$$\vec{J}' = (J'^1, J'^2, J'^3) = \left(\gamma v \varepsilon_0 \frac{V_0}{d}\left[\delta(y-d) - \delta(y)\right], 0, 0\right)$$

$$= \text{surface current } \vec{K} = \gamma v \varepsilon_0 \frac{V}{d}\left[\delta(y-d) - \delta(y)\right]\hat{x}$$

Boundary conditions on $\vec{E}$ at surface of plate $\rightarrow E'_y = \gamma \frac{V_0}{d} = \frac{\sigma'}{\varepsilon_0}$

Boundary conds on $\vec{B}$ at surface of plates $\longrightarrow B'_z = B'_{\parallel} = \mu_0 K$

23. [5] Find the (Lorentz) force $\mathbf{F}'$ on the particle in S′.

$$\vec{F}' = q(\vec{E}' + \vec{v}' \times \vec{B}') = q\vec{E}' = \frac{-\gamma q V_0}{d}\hat{y} \quad 3$$

(2 above $\vec{v}'$)

24. [5] Why does $\mathbf{F}'$ differ from $\mathbf{F}$?

Time dilation → acceleration is "slower" in frame S than in rest frame S′