 Name Solutions

Electrodynamics, Physics 323 **Final exam** **8.20 am, Tuesday June 8, 2005**
Spring 2005 Instructor: David Cobden

You have 120 minutes. End on the buzzer at 10.20. Answer all questions.
You are strongly recommended to read through all the questions before you begin.
Write your name on every page and your ID on the first page.
Watch the blackboard for corrections or clarifications during the exam.
In this exam you are allowed **no books, no note sheets, and no calculators!**

Write all your working on these question sheets. You will get credit for it. It is important to show your calculation or derivation, and where appropriate to write a few words to indicate the reasoning. You usually won't get full marks just for stating the correct answer.

Do not turn this page until the buzzer goes.

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma v/c & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \quad F^{\mu\nu} = \begin{pmatrix} * & E_x/c & E_y/c & E_z/c \\ * & * & B_z & -B_y \\ * & * & * & B_x \\ * & * & * & * \end{pmatrix}$$

$c = 3 \times 10^8$ ms^{-1} (hope you don't need to be reminded of that.)

A NATION OF EINSTEINS

(possible bonus mark: spot the errors)

 Name solutions

Warm up:

1. [10] Write down the continuity equation for charge (a) in normal form and (b) in manifestly Lorentz-covariant form.

$\vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho}{\partial t}$ $\partial_\mu J^\mu = 0$ $J^\mu = (\rho c, \vec{J})$

2. [10] Write down the differential relationships between the scalar and vector potentials and the charge density and current density, (a) in normal form (2 equations) and (b) in covariant form (one equation).

$$\left.\begin{aligned}\Box^2 \vec{A} &= -\mu_0 \vec{J} \\ \Box^2 V &= -\rho/\varepsilon_0\end{aligned}\right\} \rightarrow \partial_\mu \partial^\mu A^\nu = -\mu_0 J^\nu \quad \text{or} \quad \Box^2 A^\mu = -\mu_0 J^\mu$$

where $A^\mu = (V/c, \vec{A})$

I. Radiation. The power radiated by a changing charge distribution is aproximately $P(t) = \frac{\mu_0 \ddot{p}(t)^2}{6\pi c}$, where $p(t)$ is the instantaneous net electric dipole moment at time t.

3. [6] Derive from this a formula for the power radiated by an oscillating electric dipole with angular frequency ω. Put $p(t) = p_0 \cos\omega t$ $\therefore \ddot{p} = -\omega^2 p_0 \cos\omega t$

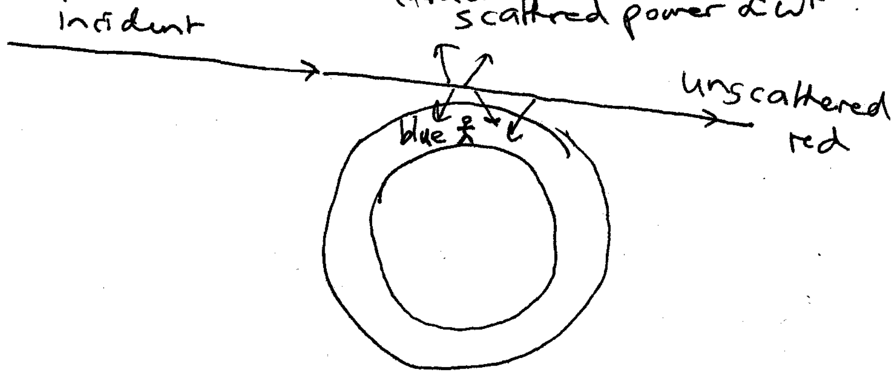
$$\therefore P_{av} = \frac{\mu_0}{6\pi c}(\omega^2 p_0)^2 \langle \cos^2\omega t\rangle = \frac{\mu_0 p_0^2 \omega^4}{6\pi c}\cdot\frac{1}{2} = \frac{\mu_0 p_0^2 \omega^4}{12\pi c}$$

4. [12] Hence, with the help of a diagram, explain why the daytime sky is blue (allegedly).

Oscillating E-field of incident light induces dipoles in molecules (or dust) $\propto E_0$ (incident amplitude)

These radiate at a rate $\propto \omega^4$ (see above)

$\rightarrow$ scattering cross-section $\sigma \propto \omega^4$ (Rayleigh scattering)



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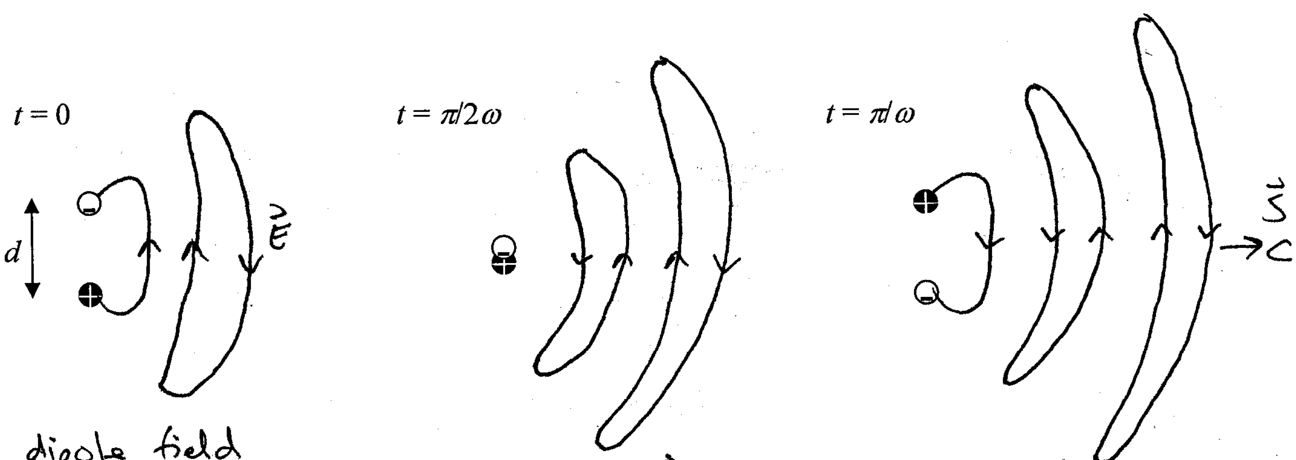
5. [10] Estimate the typical size of a simple broadcasting antenna for 100 MHz radio waves.

Typically use a half-wave antenna:

$$\text{Size } L = \frac{\lambda}{2} = \frac{c}{2f} = \frac{3\times10^8\,\mathrm{ms^{-1}}}{2\times10^8\,\mathrm{Hz}} = 1.5\,\mathrm{m}$$

6. [10] An oscillating dipole consists of two oppositely charged particles which oscillate in their relative coordinate at angular frequency ω as indicated below. Sketch below the electric field patterns at the three different phases shown if the amplitude d of the oscillation is similar to c/ω, indicating how the pattern evolves towards the spherical dipole radiation pattern at larger distances.



dipole field close-up.

$\rho = 0$ $\vec{\nabla}\cdot\vec{E} = 0$
$\to$ closed loops only

blowing bubbles

7. [6] A solid sphere spinning about an axis through its center has a net charge Q spread evenly over its surface. Does it radiate electromagnetic energy? If so, what kind(s) of radiation, and which is dominant? (ie, magnetic dipole, electric quadrupole, etc?) If not, why not?

No. It's a charge monopole and constant magnetic dipole. You need an oscillating dipole (or higher order) moment for radiation.

8. [6] Answer the same question for the case that the spin axis is offset from the center of the sphere but is close to it.

Yes.

Net oscillating electric dipole ✓
" " magnetic dipole ✓
" " electric quadrupole ✓

It emits all orders of radiation. The electric dipole dominates.

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II. Relativity concepts. The 'worldline' of a particle of charge q, rest mass m_0 is specified by $x^\mu(\tau)$.

9. [5] Define the 4-momentum p^μ of the particle in terms of the parameters given.

$$p^\mu = m_0 \frac{dx^\mu}{d\tau}$$

10. [5] What is the definition and meaning of the proper time τ?

$$d\tau = \frac{dt}{\gamma} = dx_\mu dx^\mu$$ = element of time measured in particle's rest frame

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

11. [5] Write x^μ in component form in terms of the trajectory of the particle $\mathbf{w}(t)$ observed in a frame S.

$$x^\mu = (ct, \vec{w}(t))$$

12. [15] If in frame S the 4-momentum has components $p^\mu = (E/c, \mathbf{p})$, show that $E = \gamma m_0 c^2$ and that $E^2 = p^2c^2 + m_0^2c^4$.

$$p^\mu = \left(\frac{E}{c}, \vec{p}\right) = m_0 \frac{dx^\mu}{d\tau} = \gamma m_0 \frac{d}{dt}(ct, \vec{r}) = \left(\gamma m_0 c, \gamma m_0 \frac{d\vec{r}}{dt}\right)$$

$$\therefore \frac{E}{c} = \gamma m_0 c \qquad \therefore E = \gamma m_0 c^2$$

$p_\mu p^\mu$ is a scalar invariant

$$= -\left(\frac{E}{c}\right)^2 + p^2 \quad \text{in } S.$$

In rest frame, $p^\mu = (m_0 c, 0)$ $\quad \therefore p_\mu p^\mu = -m_0^2 c^2$

$$\therefore -\left(\frac{E}{c}\right)^2 + p^2 = -m_0^2 c^2$$

$$\therefore E^2 = p^2 c^2 + m_0^2 c^4$$

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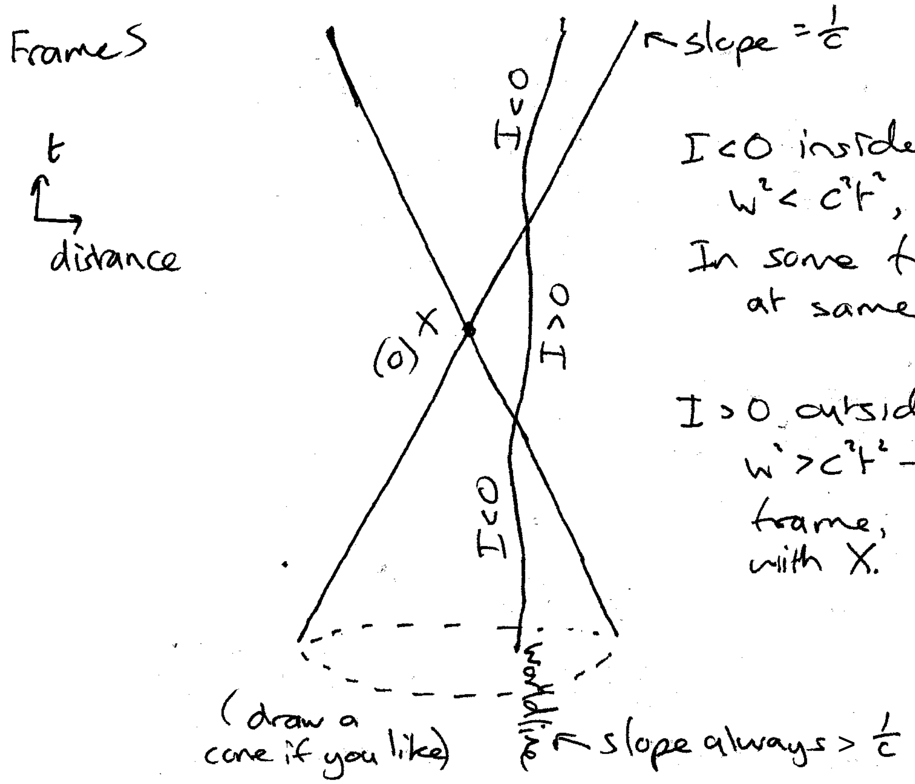
13. [8] An event X occurs at the origin at $t = 0$ in frame S, at which time the particle is *not* at the origin. Sketch the wordline of the particle on a space-time diagram in frame S. Leave room for answers to Q15.



$I < 0$ inside light cone. Here $w^2 < c^2t^2$, → timelike. Here, In some frame, $\vec{w}$ occurs at same position as X

$I > 0$ outside light cone. Here $w^2 > c^2t^2$ → spacelike. In some frame, $\vec{w}(t)$ is simultaneous with X.

14. [4] Define the invariant interval between a point on the trajectory and X.

$$I = \Delta x_\mu \Delta x^\mu = x_\mu x^\mu = -c^2t^2 + w^2$$

(because here $\Delta x^\mu = x^\mu - 0$)

15. [8] With reference to the invariant interval, indicate on the above diagram which parts of the trajectory are timelike and which are spacelike with respect to X, and say what the consequences are for causality and simultaneity.

III. Transforming the potentials. Let us find the potentials of a charge in uniform motion, by considering a charge which is stationary in inertial frame S and moving at $\mathbf{v} = v\hat{\mathbf{x}}$ in frame S'.

16. [6] Show that $\frac{\partial}{\partial x^\mu} A^\mu = 0$ written in component form specifies the Lorentz gauge.

$$\frac{\partial}{\partial x^\mu} A^\mu = \left(\frac{\partial}{\partial(-ct)}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(\frac{V}{c}, \vec{A}\right) = -\frac{1}{c^2}\frac{\partial V}{\partial t} + \vec{\nabla} \cdot \vec{A}$$

$= 0$ ✓ Lorentz gauge

17. [8] Write down the components of A^μ for a charge q which is stationary at the origin in frame S.

In S, $V = \frac{q}{4\pi\varepsilon_0 r}$ and $\vec{A} = 0$ $\therefore A^\mu = \left(\frac{q}{4\pi\varepsilon_0\sqrt{x^2+y^2+z^2}}, 0, 0, 0\right)$

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18. [12] Apply the Lorentz transformation to find the components A'^{μ} of the 4-potential in a frame S'. At this point your answer should be in terms of the coordinates in S.

$$A'^{\mu} = \Lambda^{\mu}{}_{\nu} A^{\nu} = \begin{pmatrix} \gamma & -\frac{\gamma v}{c} & 0 & 0 \\ -\frac{\gamma v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{q}{4\pi\varepsilon_0 \sqrt{x^2+y^2+z^2}} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore \begin{pmatrix} V'/c \\ A_x' \\ A_y' \\ A_z' \end{pmatrix} = \frac{\gamma q}{4\pi\varepsilon_0 \sqrt{x^2+y^2+z^2}} \begin{pmatrix} 1 \\ -\frac{v}{c} \\ 0 \\ 0 \end{pmatrix}$$

19. [12] Now find the coordinates (x,y,z,t) in frame S in terms of the coordinates (x',y',z',t') in S'.

We need the inverse L.T., $x^{\mu} = \Lambda^{\mu}{}_{\nu}(-v)\, x'^{\nu}$

$$t = \gamma\left(t' + \frac{vx'}{c^2}\right)$$

$$\rightarrow \quad x = \gamma(x' + vt)$$

$$y = y'$$

$$z = z'$$

20. [12] Hence deduce the scalar potential $V'(x',y',z',t')$ and vector potential $\mathbf{A}'(x',y',z',t')$ of a charge moving at constant velocity $\mathbf{v} = v\hat{\mathbf{x}}$ in frame S'.

$$V' = \frac{\gamma q}{4\pi\varepsilon_0 \sqrt{\gamma^2(x'+vt)^2 + y'^2 + z'^2}}$$

$$A_x' = \frac{-\frac{\gamma q v}{c}}{4\pi\varepsilon_0 \sqrt{\gamma^2(x'+vt)^2 + y'^2 + z'^2}} \qquad A_y' = 0 = A_z'$$

$$\therefore \vec{A} = \frac{-\gamma q \vec{v}}{4\pi\varepsilon_0 c \sqrt{\gamma^2(x'+vt)^2 + y'^2 + z'^2}}$$

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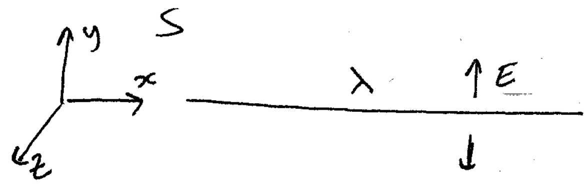
III. Transforming the fields. An infinite line charge is at rest with charge density λ along the x-axis in inertial frame S.

21. [8] Find the electric and magnetic fields in S, or just state them.



$\vec{B}=0 \quad (\vec{J}=0)$

Gauss : $\int \vec{E}\cdot d\vec{S} = \frac{q}{\varepsilon_0} \rightarrow 2\pi r L E(r) = \frac{\lambda L}{\varepsilon_0}$

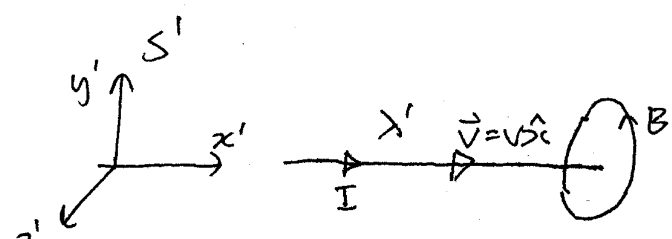
$\rightarrow \vec{E}(r) = \frac{\lambda \hat{r}}{2\pi\varepsilon_0 r} = \frac{\lambda}{2\pi\varepsilon_0 r^2}\left(y\hat{y} + z\hat{z}\right)$

where $r^2 = y^2 + z^2$

22. [10] Construct the electromagnetic field tensor $F^{\mu\nu}$ in Cartesian coordinates in frame S.

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_x & -B_y \\ -E_y/c & -B_x & 0 & B_z \\ -E_z/c & B_y & -B_z & 0 \end{pmatrix} = \frac{\lambda}{2\pi\varepsilon_0 r^2 c}\begin{pmatrix} 0 & 0 & y & z \\ 0 & 0 & 0 & 0 \\ -y & 0 & 0 & 0 \\ -z & 0 & 0 & 0 \end{pmatrix}$$

23. [12] In another frame S' the line charge moves at constant velocity $\mathbf{v} = v\hat{\mathbf{x}}$ along the x-axis. Find the electric field (using Gauss) and the magnetic field (using Ampere) in S'. Don't forget to take account of length contraction.



Charge is invariant

$\lambda' \Delta x' = \lambda \Delta x \quad \therefore \lambda' = \lambda \frac{\Delta x}{\Delta x'} = \gamma \lambda$

(due to length contraction)

now we have current, $I = \lambda' v$ along x-axis

$= \gamma \lambda v$

Now $\vec{E}' = \frac{\lambda' \hat{r}}{2\pi\varepsilon_0\sqrt{y'^2+z'^2}} = \frac{\gamma\lambda(y'\hat{y} + z'\hat{z})}{2\pi\varepsilon_0(y'^2+z'^2)} \qquad r'^2 = y'^2 + z'^2$

$\vec{B}' = \frac{\mu_0 I}{2\pi r'}\hat{\phi} = \frac{\mu_0 \gamma \lambda v (y'\hat{z} - z'\hat{y})}{2\pi(y'^2+z'^2)}$

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24. [10] Construct the field tensor $F^{\mu\nu}$ in frame S'.

$$F'^{\mu\nu} = \frac{\gamma\lambda}{2\pi\varepsilon_0 r'^2 c}\begin{pmatrix} 0 & 0 & y' & z' \\ 0 & 0 & 0 & \frac{v}{c}z' \\ -y' & 0 & 0 & \frac{v}{c}y' \\ -z' & -\frac{v}{c}z' & -\frac{v}{c}y' & 0 \end{pmatrix}$$

25. [10] Show that the B_y' component of $F'^{\mu\nu}$ is related by the appropriate Lorentz transformation to the components of $F^{\mu\nu}$.

$$F'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}F^{\alpha\beta}$$

$$B_y' = F'^{31} \quad (\text{or } -F'^{13}) = \frac{-\gamma\lambda v z'}{2\pi\varepsilon_0 r'^2 c^2} = \frac{-\mu_0\gamma\lambda v z'}{2\pi r'^2}$$

$$= \sum_{\alpha,\beta}\Lambda^3{}_\alpha\Lambda^1{}_\beta F^{\alpha\beta} \quad \text{where } \Lambda = \begin{pmatrix} \gamma & +\frac{\gamma v}{c} & 0 & 0 \\ +\frac{\gamma v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(s' moves at $-v\hat{x}$ relative to S)

$\alpha = \beta \rightarrow F^{\alpha\beta} = 0$

$\alpha \neq 3 \rightarrow \Lambda^3{}_\alpha = 0$

$\beta = 2 \text{ or } 3 \rightarrow \Lambda^1{}_\beta = 0$

$$\therefore F'^{31} = \sum_{\substack{\alpha=3 \\ \beta=0,1}}\Lambda^3{}_\alpha\Lambda^1{}_\beta F^{\alpha\beta}$$

$$= \Lambda^3{}_3\left(\Lambda^1{}_0 F^{30} + \Lambda^1{}_1 F^{31}\right)$$

$$= 1\cdot\left[\left(+\frac{\gamma v}{c}\right)\left(\frac{-z\lambda}{2\pi\varepsilon_0 r^2 c}\right) + \gamma\cdot(0)\right] = \frac{-\mu_0\gamma\lambda v z}{2\pi r^2}$$

$$= \frac{-\mu_0\gamma\lambda v z'}{2\pi r'^2} \quad \text{as } z' = z,\ r' = r$$