 Name solutions

Electrodynamics, Physics 323 **Final exam** **8.20 am, Tuesday June 5, 2007**
Spring 2007 Instructor: David Cobden

Do not turn this page until the buzzer goes at 8.20. You must hand your exam to me before I leave the room at 10.25.

Attempt all the questions.

Please write your name on every page and your SID on the first page.

Write all your working on these question sheets. Use this front page for extra working. It is important to show your calculation or derivation. Some of the marks are given for showing clear and accurate working and reasoning.

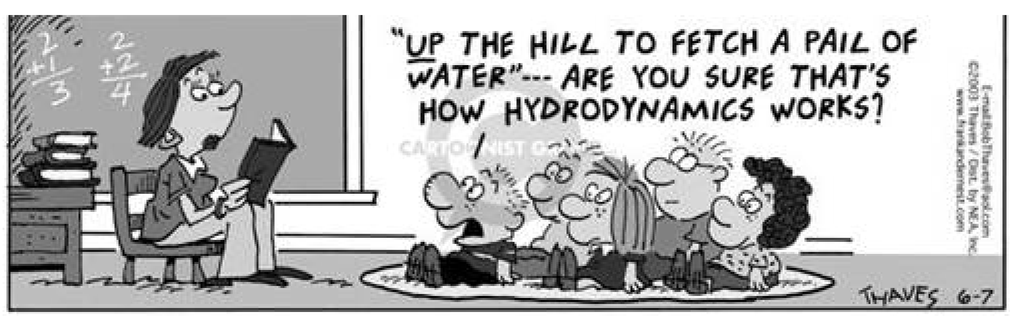
Watch the blackboard for corrections or clarifications during the exam.

This is a closed book exam. ***No books, notes or calculators allowed.***

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

$$x'^{\mu} = \sum_{\nu=0}^{3} \Lambda^{\mu}{}_{\nu} x^{\nu} \qquad F'^{\mu\nu} = \sum_{\alpha=0}^{3}\sum_{\beta=0}^{3} \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} F^{\alpha\beta} \qquad \text{Larmor: } P = \frac{\mu_0 q^2 a^2}{6\pi c}$$

$$E'_{//} = E_{//} \quad B'_{//} = B_{//} \qquad \mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \mathbf{v}\times\mathbf{B}_{\perp}) \qquad \mathbf{B}'_{\perp} = \gamma(\mathbf{B}_{\perp} - \mathbf{v}\times\mathbf{E}_{\perp}/c^2)$$



 Name <u>Solutions</u>

1. [20] For electromagnetic fields in a material, one of Maxwell's equations is

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \qquad \text{(i).}$$

Give the other three Maxwell's equations also in the form appropriate inside materials.
Show by substituting into Eq. (i) the constitutive relations (hint: one of them gives **D** in terms of **E** and **P**) that the curl of **B** is the sum of four current terms, and say what gives rise to each term.

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\cdot\vec{D} = \rho_f \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M} \qquad \vec{D} = \varepsilon_0\vec{E} + \vec{P}$$

$$\therefore (i) \rightarrow \vec{\nabla}\times\left(\frac{1}{\mu_0}\vec{B} - \vec{M}\right) = \vec{J}_f + \frac{\partial}{\partial t}(\varepsilon_0\vec{E} + \vec{P})$$

$$\therefore \vec{\nabla}\times\vec{B} = \mu_0\left[\vec{J}_f + \vec{\nabla}\times\vec{M} + \frac{\partial \vec{P}}{\partial t} + \varepsilon_0\frac{\partial \vec{E}}{\partial t}\right]$$

motion of free charges — "net "current" from nonuniform circulating (magnetic) currents — motion of bound charges — Maxwell's displacement current

2. [8] Now instead rewrite Eq. (i) for the case of a linear medium, as a relation between **B** and **E**, and hence deduce the condition on the conductivity σ and dielectric constant ε_r under which the material can be taken as a good conductor at frequency ω.

linear medium: $\vec{B} = \mu_r\mu_0\vec{H} \qquad \vec{D} = \varepsilon_r\varepsilon_0\vec{E} \qquad \vec{J}_f = \sigma\vec{E}$

$$\therefore \frac{1}{\mu_r\mu_0}\vec{\nabla}\times\vec{B} = \sigma\vec{E} + \varepsilon_r\varepsilon_0\frac{\partial\vec{E}}{\partial t} \qquad \text{Put } \vec{E} \propto e^{-i\omega t}$$

$$= [\sigma + \varepsilon_r\varepsilon_0(-i\omega)]\vec{E} \doteq \sigma\vec{E} \text{ if } \underline{\sigma \gg \varepsilon_r\varepsilon_0\omega}$$

"good conductor"

3. [8] By thinking of a monochromatic plane wave as a beam of photons of frequency ω, deduce the relationship between the energy density, the intensity, and the pressure exerted at normal incidence on a perfectly absorbing flat surface.

Energy density $\underline{u_{em}} = n\hbar\omega \qquad n =$ photon number density

Intensity = energy flow/unit area/unit time

= energy density × velocity = $\underline{c u_{em}}$ ($= c n\hbar\omega$)

Pressure = momentum flow/unit time/unit area

= momentum density × velocity $= n\cdot\left(\frac{\hbar\omega}{c}\right)\times c = n\hbar\omega = \underline{u_{em}}$

4. [8] In what commonly occurring situation is the speed of light in a dielectric greater than c, and why does this not violate special relativity?

$v = \frac{c}{n}$

Just above a resonance n can dip below 1 → phase velocity $v > c$.
The group velocity $v_g = \frac{\partial\omega}{\partial k}$ remains $< c$. This is speed of information.

Name Solutions

5. [8] Define the 4-potential A^μ and the 4-current J^μ in terms of V, **A**, ρ, and **J** in some frame S.

$$A^\mu = \left(\frac{V}{c}, \vec{A}\right) \qquad J^\mu = (\rho c, \vec{J})$$

6. [8] Construct the scalar $A^\mu J_\mu$ and interpret the result.

$$A^\mu J_\mu = \left(\frac{V}{c}, \vec{A}\right).(-\rho c, \vec{J}) = \underbrace{\vec{A}.\vec{J}}_{\text{magnetic energy density}} - \underbrace{\rho V}_{\text{electric energy density}} = \text{scalar, invariant on Lorentz transform}$$

$\therefore$ (magnetic − electric) energy is invariant.

7. [10] Show that the covariant equation $\partial_\nu \partial^\nu A^\mu = -\mu_0 J^\mu$ gives the correct relations between V, **A**, ρ, and **J** in the Lorentz gauge (to satisfy Maxwell's equations).

$$\partial^\nu = \left(-\frac{\partial}{\partial(ct)}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \qquad \partial_\nu \partial^\nu = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

$$\therefore \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\left(\frac{V}{c}, A\right) = -\mu_0(\rho c, \vec{J}) \qquad \therefore \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)V = -\mu_0 c^2 \rho = \frac{\rho}{\varepsilon_0}$$

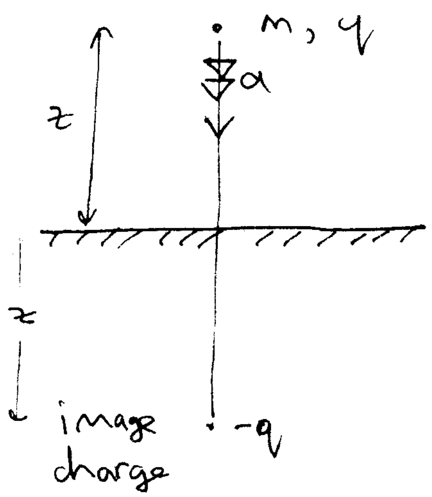
$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\mu_0 \vec{J}$$

8. [10] Write down (i) the continuity equation and (ii) the Lorentz gauge condition, first in terms of V, **A**, ρ, and **J**, and then in covariant form using 4-vectors.

$$\vec{\nabla}.\vec{J} + \frac{\partial \rho}{\partial t} = 0 \quad \text{or} \quad \partial_\mu J^\mu = 0$$

$$\vec{\nabla}.\vec{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0 \quad \text{or} \quad \partial_\mu A^\mu = 0$$

9. [16] A particle of mass m and charge q is released from rest some distance from a flat metal surface. Find the power radiated as a function of distance z from the surface.



Force F due to image charge is

$$F = ma = \frac{q^2}{4\pi\varepsilon_0 (2z)^2}$$

Larmor formula: power radiated is

$$P = \frac{\mu_0 q^2 a^2}{6\pi c} = \frac{\mu_0 q^2}{6\pi c}\left(\frac{q^2/m}{4\pi\varepsilon_0 (2z)^2}\right)^2$$

$$= \frac{\mu_0 q^6}{3.2^9.\pi^3 c\, \varepsilon_0^2 z^4 m^2} = \frac{q^6}{3.2^9.\pi^3 c^3 \varepsilon_0^3 z^4 m^2}$$

$$= \frac{1}{3z^4 m^2}\left(\frac{q^2}{8\pi\varepsilon_0 c}\right)^3$$

Name solutions

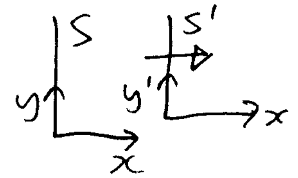
10. [8] An electromagnetic wave of frequency ω travels in the x direction through the vacuum. It is polarized in the y direction, and the amplitude of the electric field is E_0. Write down the electric and magnetic fields, $\mathbf{E}(x,y,z,t)$ and $\mathbf{B}(x,y,z,t)$.

$$\vec{E} = E_0 \hat{y} \cos(kx - \omega t) \qquad \vec{B} = \frac{E_0}{c} \hat{z} \cos(kx - \omega t) \qquad k = \frac{\omega}{c}$$

11. [6] Construct the electromagnetic field tensor $F^{\mu\nu}$ for the wave.

$$F^{\mu\nu} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \frac{E_0}{c} \cos(kx - \omega t)$$

12. [10] This same wave is observed from an inertial system S′ moving in the $+x$ direction with speed v relative to the original system S. Write down the equations giving the coordinates in S in terms of the coordinates x', y', z', and t' in S′.



$$t = \gamma\left(t' + \frac{vx'}{c^2}\right)$$
$$x = \gamma(x' + vt')$$
$$y = y'$$
$$z = z'$$

13. [16] Find the electric and magnetic fields in S′, either by transforming $F^{\mu\nu}$, or using the direct equations given on the cover page, or otherwise. Express them in terms of the S′ coordinates.

$$E'_{\parallel} = E_{\parallel} \Rightarrow E'_x = 0$$
$$B'_{\parallel} = B_{\parallel} \Rightarrow B'_x = 0$$
$$\vec{E}'_{\perp} = \gamma(\vec{E}_{\perp} + \vec{v} \times \vec{B}_{\perp}) \Rightarrow E'_y = \gamma[E_y + (-v)B_z] = \gamma E_0\left(1 - \frac{v}{c}\right)\cos(kx - \omega t)$$
$$E'_z = \gamma(E_z + vB_y) = \gamma(0 + 0) = 0$$
$$\vec{B}'_{\perp} = \gamma\left(\vec{B}_{\perp} - \frac{\vec{v} \times \vec{E}_{\perp}}{c^2}\right) \Rightarrow B'_y = \gamma\left(B_y - \frac{(-v)E_z}{c^2}\right) = 0$$
$$B'_z = \gamma\left(B_z - \frac{vE_y}{c^2}\right) = \gamma \frac{E_0}{c}\left(1 - \frac{v}{c}\right)\cos(kx - \omega t)$$

$$\therefore \vec{E}' = \gamma\left(1 - \frac{v}{c}\right)E_0 \hat{y} \cos\left[k.\gamma(x' + vt') - \omega.\gamma\left(t' + \frac{vx'}{c^2}\right)\right]$$
$$= \gamma\left(1 - \frac{v}{c}\right)E_0 \hat{y} \cos\left[\left(k\gamma - \frac{\gamma\omega v}{c^2}\right)x' - (\omega\gamma - k\gamma v)t'\right]$$
$$= \gamma\left(1 - \frac{v}{c}\right)E_0 \hat{y} \cos\left[k\gamma\left(1 - \frac{v}{c}\right)x' - \omega\gamma\left(1 - \frac{v}{c}\right)t'\right]$$

Similarly $\vec{B}' = \gamma\left(1 - \frac{v}{c}\right)\frac{E_0}{c} \hat{z} \cos\left[k\gamma\left(1 - \frac{v}{c}\right)x' - \omega\gamma\left(1 - \frac{v}{c}\right)t'\right]$

Name solutions

14. [8] What is the frequency ω′ of the wave in S′? Interpret this result.

$$\omega' = \omega\gamma\left(1-\frac{v}{c}\right) = \frac{\omega\left(1-\frac{v}{c}\right)}{\sqrt{1-v^2/c^2}} = \omega\sqrt{\frac{1-v/c}{1+v/c}}$$

relativistic Doppler shift

15. [8] What is the wavelength λ′ of the wave in S? From ω′ and λ′, determine the wave speed in S′.

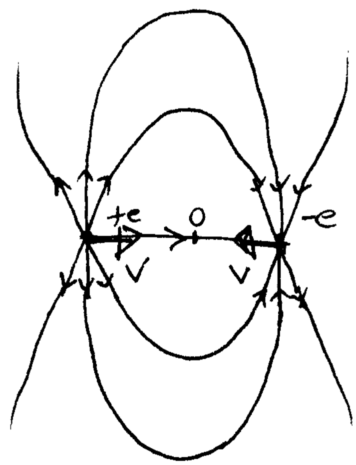
$$k' = k\gamma\left(1-\frac{v}{c}\right) = k\sqrt{\frac{1-v/c}{1+v/c}}$$

$$\lambda' = \frac{2\pi}{k'} = \frac{2\pi}{k}\sqrt{\frac{1+v/c}{1-v/c}} = \frac{2\pi c}{\omega}\sqrt{\frac{1+v/c}{1-v/c}}$$

$$\text{speed} = \frac{\omega'}{k'} = c$$

thanks to relativity!

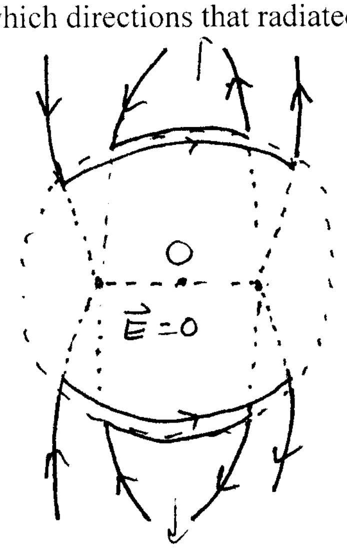
16. [10] An electron and a positron (rest mass *m*) are travelling at equal speed $v = 0.8c$ and opposite directions towards a collision at the origin. Sketch the electric field lines before they collide.



Field of each particle is like Coulomb (radial) but more intense perpendicular to velocity.

17. [10] When they collide the particles annihilate. Sketch the electric fields a short time *t* later, including both the regions inside, outside, and on the surface of a sphere of radius *ct* centered on the origin. Hence deduce in which directions that radiated intensity is maximum.

max intensity in plane through collision perp. to velocity



Inside sphere $r < ct$
$\vec{E} = 0$

Outside, you don't know the collision has happened, so it looks like the particles continued past each other.

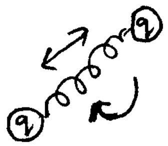
18. [8] What is the total energy emitted as photons? (Neglect the energy of the EM fields before the collision).

All initial relativistic mass-energy is converted to photons

$$E = 2.\gamma mc^2 = \frac{2.mc^2}{\sqrt{1-0.8^2}} = \frac{2mc^2}{0.6} = \frac{10mc^2}{3}$$

Name solutions

19. [10] Two identical particles of charge q and mass m are bound to each other by an attractive central force. The system is set in motion and then left alone in vacuum in zero external field. What kinds of radiation can be emitted (electric or magnetic, dipole or quadrupole), and what modes of their relative motion will generate each type?



In COM frame there's no dipole moment
$\rightarrow$ no electric dipole radiation

Rotation $\rightarrow$ circulating current $\rightarrow$ magnetic moment
but due to cons. of angular mom it is constant
$\rightarrow$ no magnetic dipole radiation

Stretching $\rightarrow$ changing electric quadrupole moment
$\rightarrow$ electric quadrupole radiation is possible.

20. [5] The Lienard-Wiechert scalar potential of a particle of charge q moving on a trajectory $\mathbf{w}(t)$ is $V(\mathbf{r},t) = \dfrac{q}{4\pi\varepsilon_0 R(1-\hat{\mathbf{R}}.\dot{\mathbf{w}}(t_r)/c)}$, where $\mathbf{R} = \mathbf{r} - \mathbf{w}(t_r)$ and $\dot{\mathbf{w}} = d\mathbf{w}/dt$. Give the equation determining the retarded time t_r.

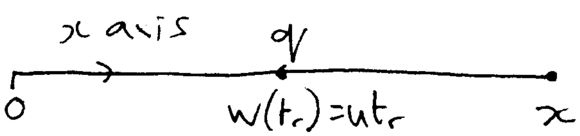
$$c(t-t_r) = |\vec{r} - \vec{w}(t_r)|$$

21. [15] A particle of charge q moves at constant speed u along the x-axis, so $\mathbf{w}(t) = ut\,\hat{\mathbf{x}}$. Determine the scalar potential along the x-axis only.

Along x axis $\vec{R} = R\hat{x}$ $\quad R = x - ut_r$

$$c(t-t_r) = x - ut_r$$

$$\therefore t_r = \frac{x-ct}{u-c} = \frac{ct-x}{c-u}$$



$$\therefore R = x - u\left(\frac{ct-x}{c-u}\right)$$

$$= \frac{x(c-u) - uct + ux}{c-u}$$

$$\therefore R = \frac{c(x-ut)}{c-u} = \frac{x-ut}{1-\frac{u}{c}}$$

$$\therefore V(x,t) = \frac{q}{4\pi\varepsilon_0 R\left(1-\frac{u}{c}\right)}$$

$$= \frac{q}{4\pi\varepsilon_0 \left(\frac{x-ut}{1-u/c}\right)\left(1-\frac{u}{c}\right)}$$

$$= \frac{q}{4\pi\varepsilon_0 (x-ut)}$$

= same as if particle was static at its present position.