

**Electrodynamics, Physics 323
Spring 2008****Final exam**
Instructor: David Cobden**8.20 am, Tuesday 10 June 2008**

Do not turn this page until the buzzer goes at 8.20. You must hand your exam to me before I leave the room at 10.25.

Attempt all the questions.

Please write your name on every page and your SID on the first page.

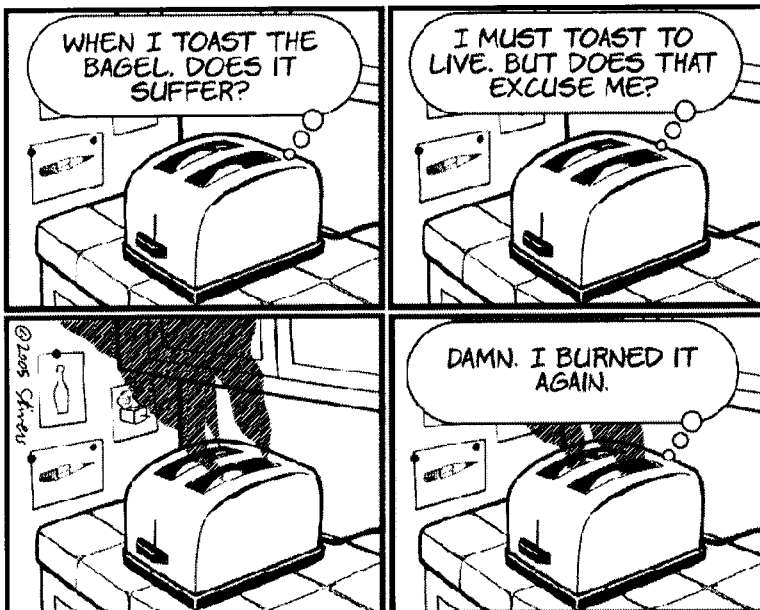
Write all your working on these question sheets. Use this front page for extra working. It is important to show your calculation or derivation. Some of the marks are given for showing clear and accurate working and reasoning.

Watch the blackboard for corrections or clarifications during the exam.

This is a closed book exam. *No books, notes or calculators are allowed.*

$$\nabla \times (\nabla \times \mathbf{X}) = \nabla(\nabla \cdot \mathbf{X}) - \nabla^2 \mathbf{X}$$

Larmor formula: $P = \frac{\mu_0 q^2 a^2}{6\pi c}$



1. [6] Two of Maxwell's equations in matter are (i) $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ and (ii) $\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}_f$. State the others.

$$\vec{\nabla} \cdot \vec{D} = \rho_f \quad \vec{\nabla} \cdot \vec{B} = 0$$

2. [15] A nonzero charge density is induced at time $t = 0$ in a material with conductivity σ and permittivity ϵ . Using continuity of charge and Gauss's law show that the charge density subsequently relaxes exponentially on a timescale $\tau = \epsilon/\sigma$.

$$\frac{\partial \rho_f}{\partial t} = -\vec{\nabla} \cdot \vec{J}_f \quad \vec{J}_f = \sigma \vec{E} \quad \vec{\nabla} \cdot \vec{E} = \frac{\rho_f}{\epsilon}$$

$$\therefore \frac{\partial \rho_f}{\partial t} = -\vec{\nabla} \cdot (\sigma \vec{E}) = -\sigma \vec{\nabla} \cdot \vec{E} = -\frac{\sigma}{\epsilon} \rho_f$$

$$\therefore \frac{\partial \rho_f}{\partial t} = -\frac{\rho_f}{\tau} \quad \tau = \frac{\epsilon}{\sigma}$$

↳ exponential decay

3. [15] Starting from Maxwell's equations (i) and (ii) above and the linear constitutive relations, show that at high enough frequencies the dispersion relation for a monochromatic plane electromagnetic wave of frequency ω in a linear medium can be written $k \approx \frac{\omega}{v} \left(1 + \frac{i}{2\omega\tau}\right)$, where $v = 1/\sqrt{\epsilon\mu}$.

$$(ii) \rightarrow \vec{\nabla} \times \vec{B} = \mu\epsilon \frac{\partial \vec{E}}{\partial t} + \mu\sigma \vec{E}$$

$$\rightarrow \vec{\nabla} \times \frac{\partial \vec{B}}{\partial t} = \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \mu\sigma \frac{\partial \vec{E}}{\partial t}$$

$$\text{by (i)} \quad = -\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} (\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E}$$

$\uparrow$
 0 for transverse waves

$$\therefore \nabla^2 \vec{E} = \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \mu\sigma \frac{\partial \vec{E}}{\partial t}$$

For a monochromatic plane wave $\vec{E} = \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$

$$-k^2 = -\mu\epsilon\omega^2 - i\omega\mu\sigma = -\mu\epsilon\omega^2 \left(1 + \frac{i\sigma}{\epsilon\omega}\right)$$

$$\therefore k = \sqrt{\mu\epsilon}\omega \left(1 + \frac{i}{\omega\tau}\right)^{1/2} \approx \frac{\omega}{v} \left(1 + \frac{1}{2} \frac{i}{\omega\tau}\right) \quad v = \frac{1}{\sqrt{\epsilon\mu}}$$

for $\omega\tau \gg 1$

binomial expansion

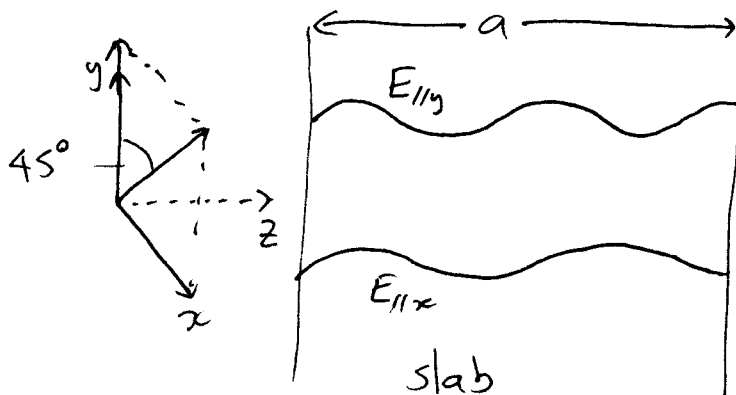
4. [10] Hence find the absorption coefficient (the inverse length scale for decay of power in the wave) in this limit, assuming ϵ and σ are real (this is actually not valid, but please do it anyway!)

$$\rightarrow |E| \propto e^{-\text{Im}(k)r} = e^{-\frac{\sigma}{2\omega} r}$$

$$I(k) = \frac{1}{2\omega r}$$

$$\text{Power} \propto |E|^2 \propto e^{-\frac{\sigma}{\omega} r} \therefore \text{absorption length} \\ \text{absorption coeff } \alpha = \frac{1}{r}$$

5. [10] A linearly polarized laser beam with frequency ω propagating along the z -axis passes normally through a birefringent slab of thickness a , which has different refractive indices n_x and n_y for electric field polarizations in the x and y directions. For what orientation of the input polarization and what thicknesses will the beam that emerges from the other side be entirely circularly polarized?



We need E_x and E_y to have same magnitude, so input at 45° .

... We need $\pm \frac{\pi}{2}$ phase difference

$$\text{so } k_{\parallel y} a - k_{\parallel x} a = n\pi + \frac{\pi}{2}$$

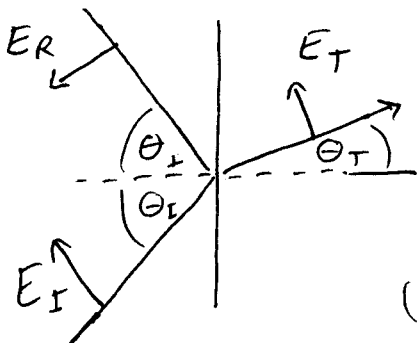
$$k = \frac{n\omega}{c} \therefore (n_y - n_x) \frac{\omega a}{c} = (n + \frac{1}{2}) \pi$$

$$a = \frac{(n + \frac{1}{2}) \pi c}{(n_y - n_x) \omega}$$

6. [15] By applying the boundary conditions for $E_{\parallel}$ and $H_{\parallel}$, derive the Fresnel relation

$$\frac{E_R}{E_I} = \frac{\alpha - \beta}{\alpha + \beta}$$

giving the reflected amplitude E_R for a plane wave of amplitude E_I incident in vacuum on the surface of a dielectric with polarization in the plane of incidence. Here $\alpha = \frac{\cos \theta_T}{\cos \theta_I}$, where θ_I and θ_T are the angles to the normal for the incident and transmitted waves, and $\beta = Z_0/Z_1$, where $Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = \frac{E_I}{H_I}$ is the impedance of free space and Z_1 is the impedance of the dielectric whose refractive index is n .



$$E_{\parallel} : E_I \cos \theta_I - E_R \cos \theta_I = E_T \cos \theta_T \quad (i)$$

$$H_{\parallel} : H_I + H_R = H_T \therefore \frac{E_I}{Z_0} + \frac{E_R}{Z_0} = \frac{E_T}{Z_1} \quad (ii)$$

$$(i) \rightarrow E_I - E_R = E_T \alpha$$

$$(ii) \rightarrow E_I + E_R = E_T \beta$$

$$\beta(i) - \alpha(ii) \rightarrow (\beta - \alpha) E_I - (\beta + \alpha) E_R = 0$$

$$\therefore \frac{E_R}{E_I} = \frac{\beta - \alpha}{\beta + \alpha}$$

(The sign just depends on which direction you draw E_R in).

7. [10] Express the equation for E_R/E_I in terms of θ_i and n only. (Take $\mu = \mu_0$.)

Snell: $\sin \theta_I = n \sin \theta_T \quad \therefore \cos \theta_T = \sqrt{1 - \sin^2 \theta_T} = \sqrt{1 - \frac{\sin^2 \theta_I}{n^2}}$

$Z_1 = \sqrt{\frac{\mu_0}{\epsilon}} \quad v = \frac{c}{n} \rightarrow n = \sqrt{\epsilon_r} \quad \text{so } \beta = \frac{Z_0}{Z_1} = \sqrt{\frac{\epsilon}{\epsilon_0}} = \sqrt{\epsilon_r} = n$

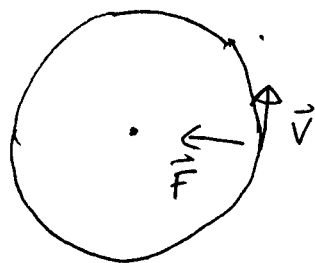
ie. $\frac{E_R}{E_I} = \frac{\sqrt{1 - \frac{\sin^2 \theta_I}{n^2}} - n \cos \theta_I}{\sqrt{1 - \frac{\sin^2 \theta_I}{n^2}} + n \cos \theta_I}$

8. [12] Use the Fresnel relation in Q.6 to deduce the transmission coefficient T (the fraction of incident power transmitted) in terms of α and β .

$T = 1 - R \quad R = \frac{\text{intensity power in reflected wave}}{\text{intensity in incident wave}} = \frac{|E_R|^2}{|E_I|^2}$

$\therefore T = 1 - \left(\frac{\alpha - \beta}{\alpha + \beta} \right)^2 = \frac{4\alpha\beta}{(\alpha + \beta)^2}$

9. [10] A particle of charge q and mass m moving in a uniform magnetic field of magnitude B has initial velocity u perpendicular to the field axis. Show that it moves in a circular orbit and determine the cyclotron frequency ω_c .

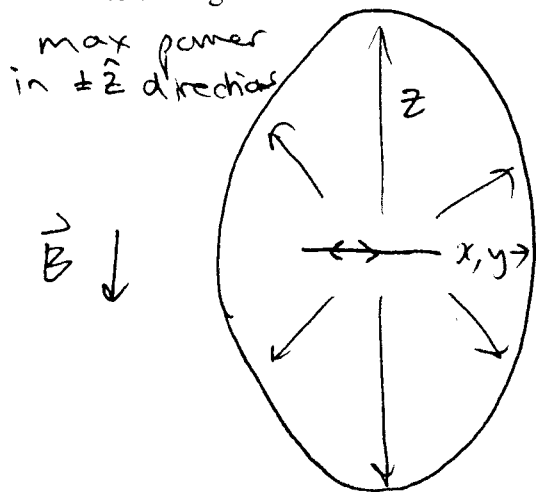


centripetal Lorentz force $\vec{F} = q\vec{v} \times \vec{B}$ is always perp. to $\vec{v}$

Newton: $F = ma$

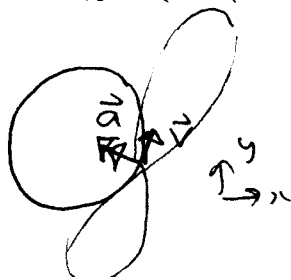
$qvB = \frac{mv^2}{r} = mrv\omega \quad \therefore \omega = \frac{qB}{m}$

10. [8] Sketch the distribution of radiated power in a plane perpendicular to the orbit. (Note: you need to average over the different orientations of the acceleration.)



Power instantaneously radiated as a doughnut perpendicular to $\vec{a}$.

At every point on orbit power is radiated ~~upwards~~, along z-axis, but in the x-y plane



it's a strobing beam with reduced average power.

11. [16] Show that the kinetic energy of the particle decays exponentially due to dipole radiation with a time constant given by

$$\tau = \frac{3\pi m^3 c}{\mu_0 q^4 B^2}.$$

Kinetic energy: $U = \frac{1}{2} m v^2$

Radiation power: $P = \frac{\mu_0 q^2 a^2}{6\pi c}$

Acceleration $a = \frac{v^2}{r} = \frac{q v B}{m}$

$$\begin{aligned} \therefore \frac{dU}{dt} &= -P = -\frac{\mu_0 q^2}{6\pi c} \left(\frac{q v B}{m} \right)^2 = -\frac{\mu_0 q^4 B^2 v^2}{6\pi c m^2} \\ &= -\frac{\mu_0 q^4 B^2 \left(\frac{2U}{m} \right)}{6\pi c m^2} = -\left(\frac{\mu_0 q^4 B^2}{3\pi m^3 c} \right) U = -\frac{U}{\tau} \end{aligned}$$

$\therefore$ exponential decay, $\tau = \frac{3\pi m^3 c}{\mu_0 q^4 B^2}$

12. [10] What are the components of x^μ , $\frac{\partial}{\partial x^\mu}$, A^μ and J^μ in a particular frame S?

$$x^\mu = (ct, x, y, z) \quad J^\mu = (\rho c, J_x, J_y, J_z)$$

$$\frac{\partial}{\partial x^\mu} = \left(\frac{1}{c} \frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \quad A^\mu = \left(\frac{V}{c}, A_x, A_y, A_z \right)$$

13. [6] State the Lorentz gauge condition and the continuity equation for charge in covariant form.

$$\frac{\partial A^\mu}{\partial x^\mu} = 0 \quad \frac{\partial J^\mu}{\partial x^\mu} = 0$$

14. [4] The electromagnetic field tensor is defined as $F^{\mu\nu} = \frac{\partial A^\nu}{\partial x^\mu} - \frac{\partial A^\mu}{\partial x^\nu}$. Show that it is antisymmetric.

$$F^{\mu\nu} = -\frac{\partial A^\mu}{\partial x^\nu} + \frac{\partial A^\nu}{\partial x^\mu} = -\left(\frac{\partial A^\mu}{\partial x^\nu} - \frac{\partial A^\nu}{\partial x^\mu} \right) = -F^{\nu\mu}$$

15. [10] Show that $F^{01} = \frac{E_x}{c}$ and that $F^{23} = B_x$.

$$F^{01} = \frac{\partial A^1}{\partial x^0} - \frac{\partial A^0}{\partial x^1} = \frac{1}{c} \frac{\partial A_x}{\partial t} - \frac{\partial (V/c)}{\partial x} = \frac{1}{c} \left(\frac{\partial A_x}{\partial t} - (\vec{\nabla} V)_x \right) = \frac{E_x}{c}$$

$$F^{23} = \frac{\partial A^3}{\partial x^2} - \frac{\partial A^2}{\partial x^3} = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = (\vec{\nabla} \times \vec{A})_x = B_x$$

16. [10] Fill out the remaining elements: $F^{\mu\nu} = \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & +B_z & -B_y \\ -\frac{E_y}{c} & -B_z & 0 & B_x \\ -\frac{E_z}{c} & B_y & -B_x & 0 \end{pmatrix}$

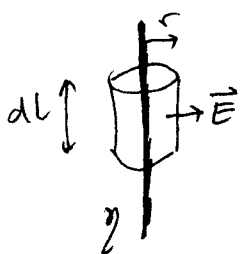
17. [8] Using the fact that $F^{\mu\nu}$ is a 4-tensor, or otherwise, show that $E^2 - c^2 B^2$ is invariant under Lorentz transformations.

$$F_{\mu\nu} F^{\mu\nu} = F_{00} F^{00} + F_{11} F^{11} + \dots = 2 \left(\frac{E_x^2}{c^2} + \frac{E_y^2}{c^2} + \frac{E_z^2}{c^2} \right) + 2 (B_z^2 + B_y^2 + B_x^2)$$

For $\mu=0$ or $\nu=0$, change sign to get $= -\frac{2}{c^2} (E^2 - c^2 B^2)$
 $F_{\mu 0}$ from $F^{\mu 0} \rightarrow E$ components have opposite sign in $F_{\mu 0}$ and $F^{\mu 0}$

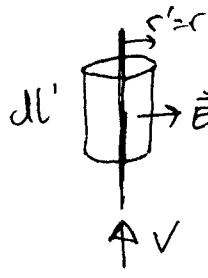
18. [8] In frame S a static infinite line charge of linear density η lies along the z-axis. Find the electric field around it, using old tricks (including cylindrical coordinates).

Symmetry: $\vec{E}$ is radial, $\vec{E} = E_r \hat{r}$
 Gauss: $2\pi r dL \cdot E_r = \frac{\eta dL}{\epsilon_0} \therefore E_r = \frac{\eta}{2\pi \epsilon_0 r}$
 $\vec{E} = \frac{\eta \hat{r}}{2\pi \epsilon_0 r} \quad (\vec{B} = 0)$



19. [6] Another inertial frame S' moves at constant velocity v along the z-axis relative to S. Find the electric field in S' using the same old tricks (remember Lorentz contraction).

Now same charge is in a Lorentz-contracted cylinder, $r' = r$ because it's perpendicular to $\vec{v}$.
 $\therefore \eta' = \gamma \eta \therefore \vec{E}' = \frac{\gamma \eta \hat{r}'}{2\pi \epsilon_0 r'}$



20. [10] Find the magnetic field in S' using Ampere's law. Show that $E^2 - c^2 B^2$ is indeed invariant in this particular case. Symmetry $\rightarrow \vec{B} = B_\phi \hat{\phi}$

Amperean loop: $\int \vec{B} \cdot d\vec{l} = \mu_0 I \rightarrow 2\pi r B_\phi = \mu_0 I$
 $I = \eta' v = \gamma \eta v$
 $\therefore \vec{B} = \frac{\mu_0 I}{2\pi r'} \hat{\phi} = \frac{\mu_0 \eta v \gamma}{2\pi r'} \hat{\phi}$



$$E'^2 - c^2 B'^2 = \left(\frac{\gamma \eta}{2\pi \epsilon_0 r'} \right)^2 - c^2 \left(\frac{\mu_0 \eta \gamma v}{2\pi r'} \right)^2$$

$$= \left(\frac{\gamma \eta}{2\pi \epsilon_0 r'} \right)^2 \left(1 - c^2 \epsilon_0^2 \mu_0^2 v^2 \right) = (E^2 - 0^2) \gamma^2 \left(1 - \frac{v^2}{c^2} \right) = E^2 - B^2 c^2$$