Fourier Analysis and Interferometers

Phys 331, Lecture 2, October 06, 2014

Important: Speed of light repot is due this week.

Today Lecture: Michelson Interferometer

Optics Lab — Physics 331 Sign-up Sheet

	Monday1:30 ***AD***	***Tuesday 1:30*** ***AA***	***Wednesday1:30*** ***AB***	***Friday 1:30*** ***AC***
Fraunhofer & Fresnel Diffraction				
Fabry-Perot Interferometer				
Michelson Interferometer				
Concave Diffraction Grating				
Reflection from a Dielectric Surface				
Faraday Rotation				
Holography *Must complete Fabry-Perot or Michelson first*. 1-d diffraction experiment				

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Please remember what you have signed up

Fourier Analysis

Fourier series for periodic functions:

$$f(x) = \frac{A_0}{2} + \sum_{m=0}^{\infty} A_m \cos(mkx) + \sum_{m=0}^{\infty} B_m \sin(mkx)$$

$$A_m = \frac{2}{\lambda}\int_0^{\lambda} f(x)\cos(mkx)dx$$

Wave number
(spatial frequency)

k=2π/ λ=2 πσ

$$B_m = \frac{2}{\lambda}\int_0^{\lambda} f(x)\sin(mkx)dx$$

$$mk = \frac{2\pi}{\lambda / m} = k_m$$

Determine A and B through Fourier analysis.

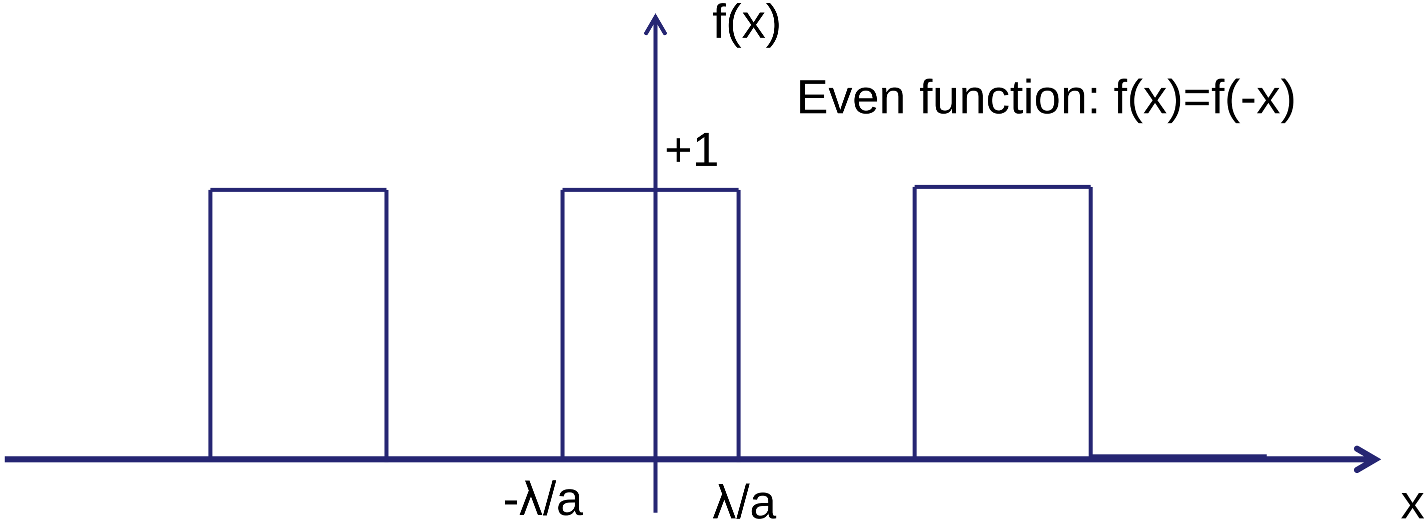
m=1 fundamental frequency
m>1 high order harmonics

Fourier's Theorem: a function f(x), having a spatial period λ, can be synthesized by a sum of harmonic functions whose wavelengths are integral of submultiples of λ.

Hecht, Pg 304

Decomposing Periodic Functions

Decomposing a Square Wave



$$f(x) = \frac{A_0}{2} + \sum_{m=0}^{\infty} A_m \cos(mkx) + \sum_{m=0}^{\infty} B_m \sin(mkx)$$

$$f(x) = \sum_{0}^{\infty} A_m \cos(mkx)$$

"EVEN" Square Wave

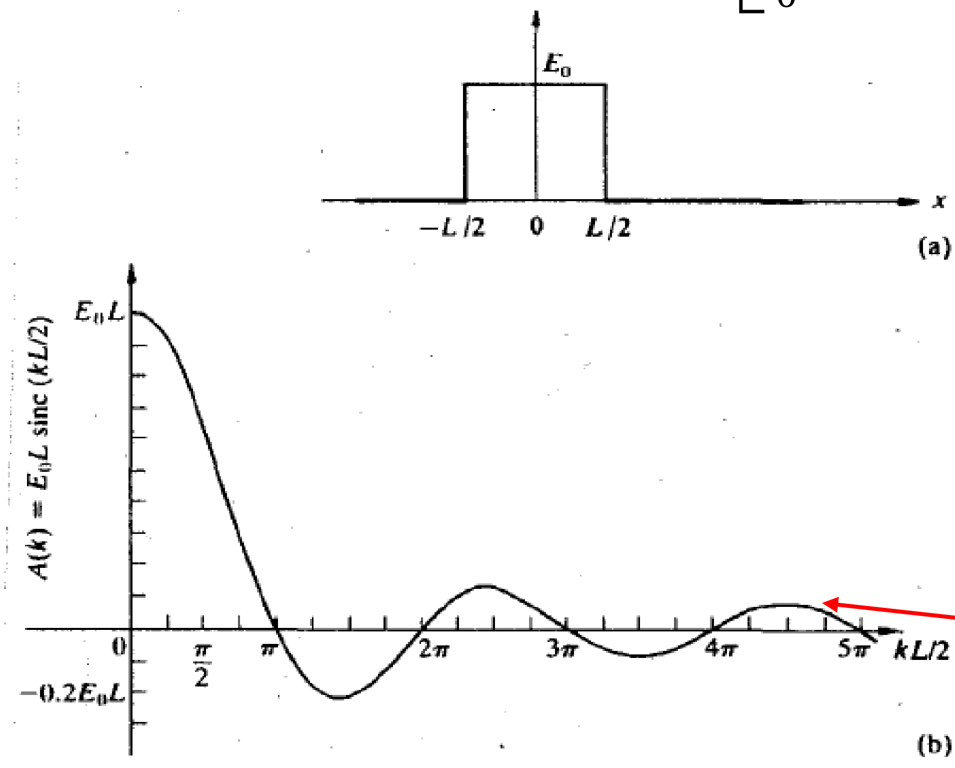
$$A_0 = 0; A_m = \frac{4}{a}\left(\frac{\sin m2\pi/a}{m2\pi/a}\right)$$

$$B_m = 0$$

sinc(u)=sin(u)/u

Fourier Integrals and Fourier Transforms

$$f(x) = \frac{1}{\pi}\left[\int_0^\infty A(k)\cos(kx)\,dk + \int_0^\infty B(k)\sin(kx)\,dk\right]$$



$$A(k) = \int_{-\infty}^{\infty} f(x)\cos(kx)\,dx$$

$$B(k) = \int_{-\infty}^{\infty} f(x)\sin(kx)\,dx$$

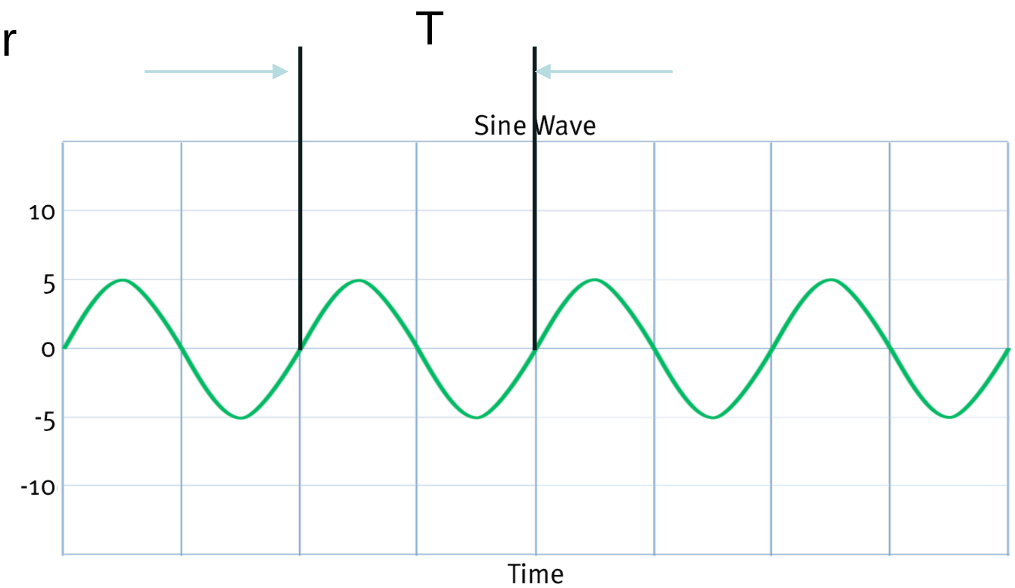
$$A(k) = E_0 L \frac{\sin(kL/2)}{kL/2}$$

The sinc function.
Inverse relationship between widths
FT in single-slit Fraunhofer diffraction

Hecht, Pg 312, Fig. 7.34
The square pulse and its transform

Wave Propagation of Light

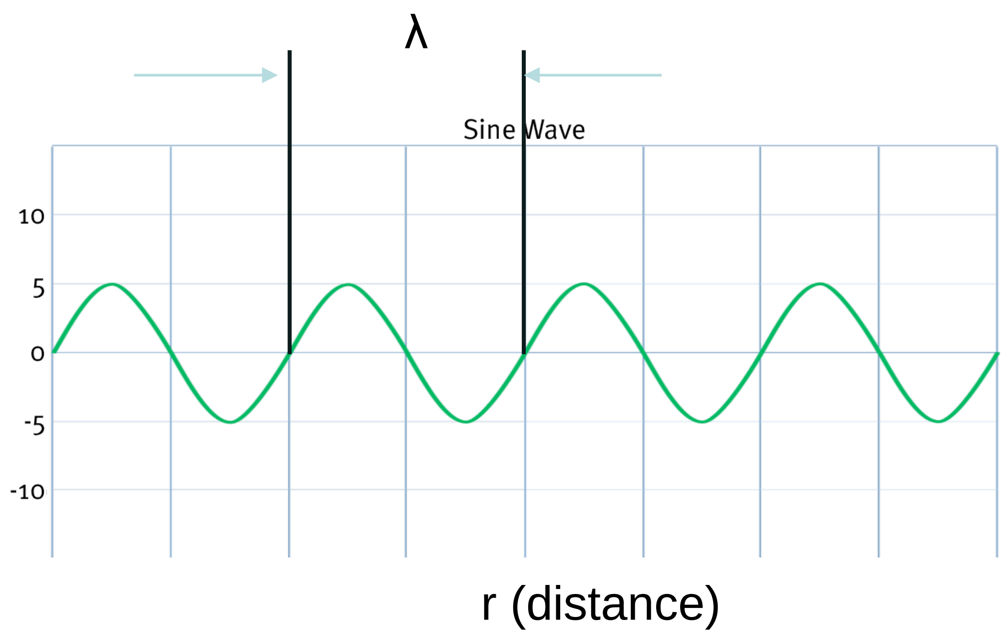
Fix r



$$Ae^{i(k.r-\omega t)}$$

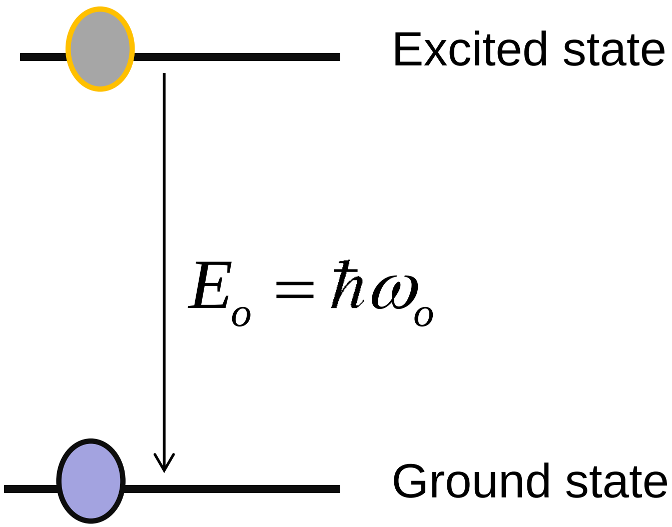
$$\omega=\frac{2\pi}{T}=2\pi\nu$$

C*T=λ



$$k=\frac{2\pi}{\lambda}=2\pi\sigma$$

Coherence Time and Coherence Length



$$Ae^{i(k.r-\omega t)}$$

$$k = 2\pi / \lambda$$

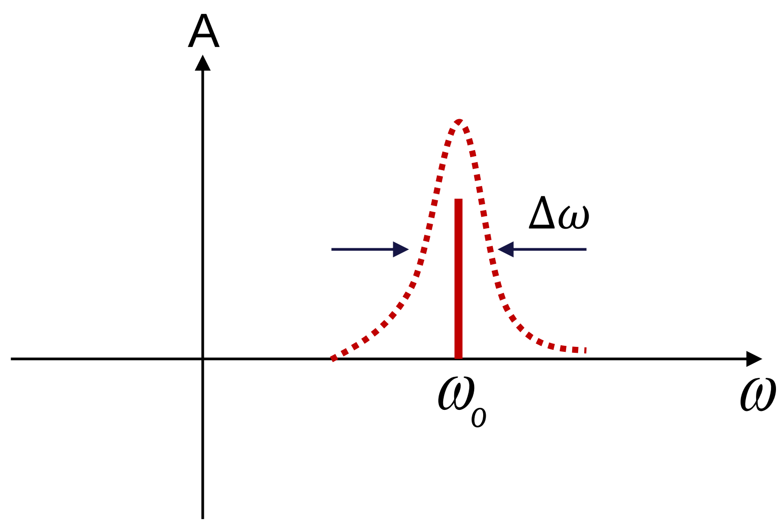
$$\frac{\Delta\omega}{2\pi} = \Delta\nu$$ is the bandwidth

$$\delta t_c = 1/\Delta\nu$$ is the coherence time

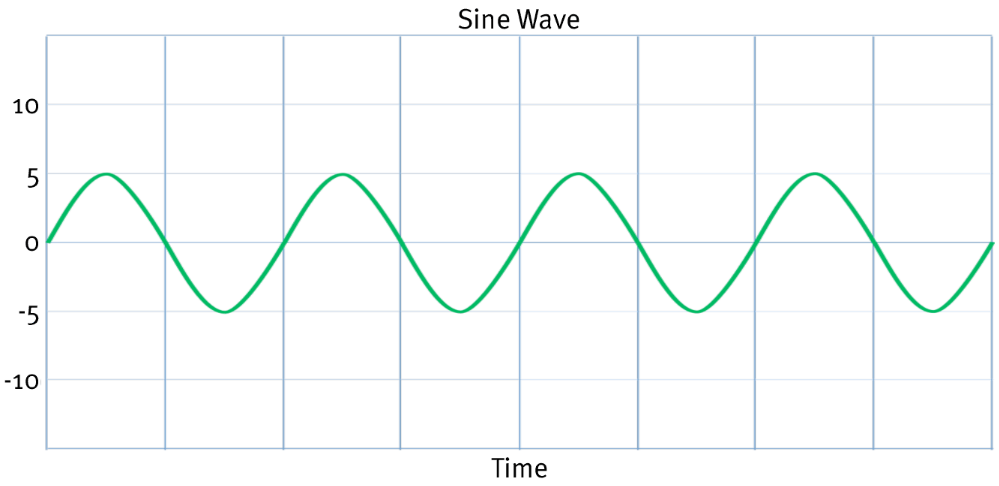
$$c \times \delta t_c = L_c$$ is the coherence length

$$E = E_o \pm \Delta E/2$$

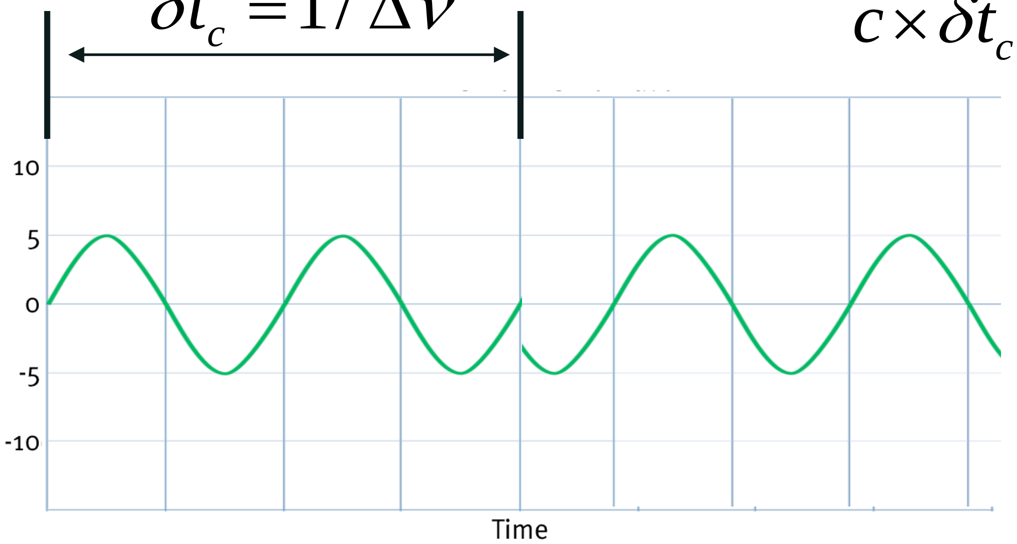
$$\omega = \omega_o \pm \Delta\omega/2$$



Wave Propagation of Light

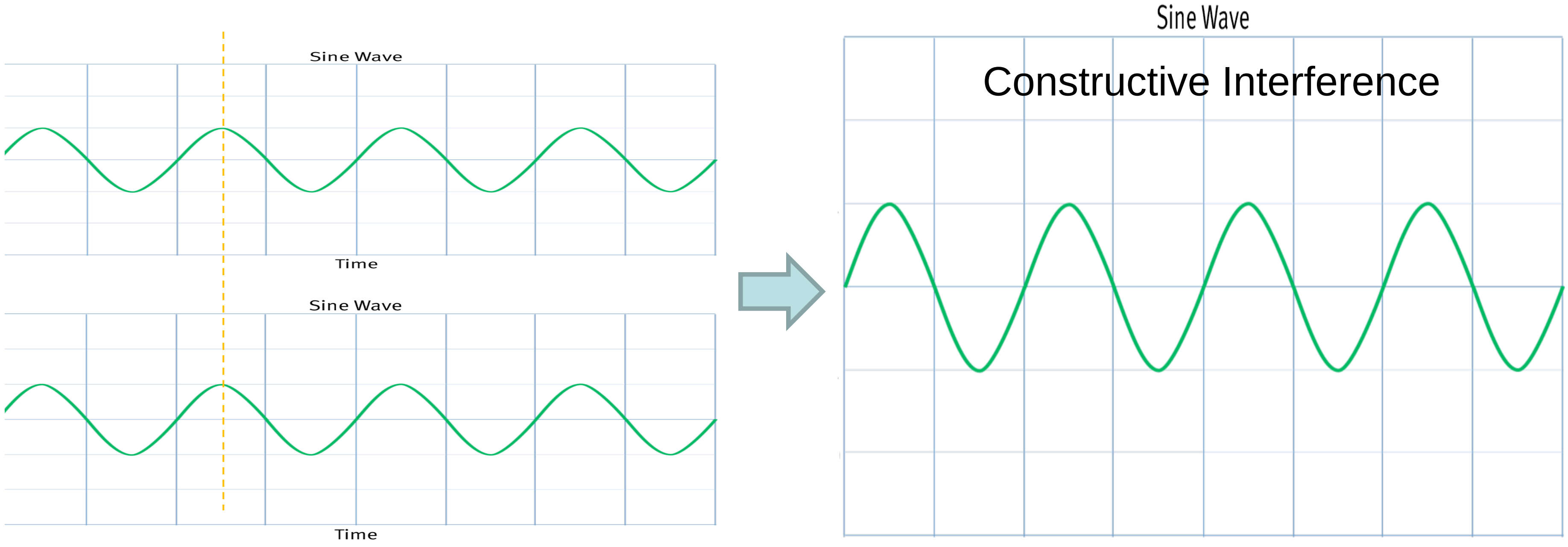


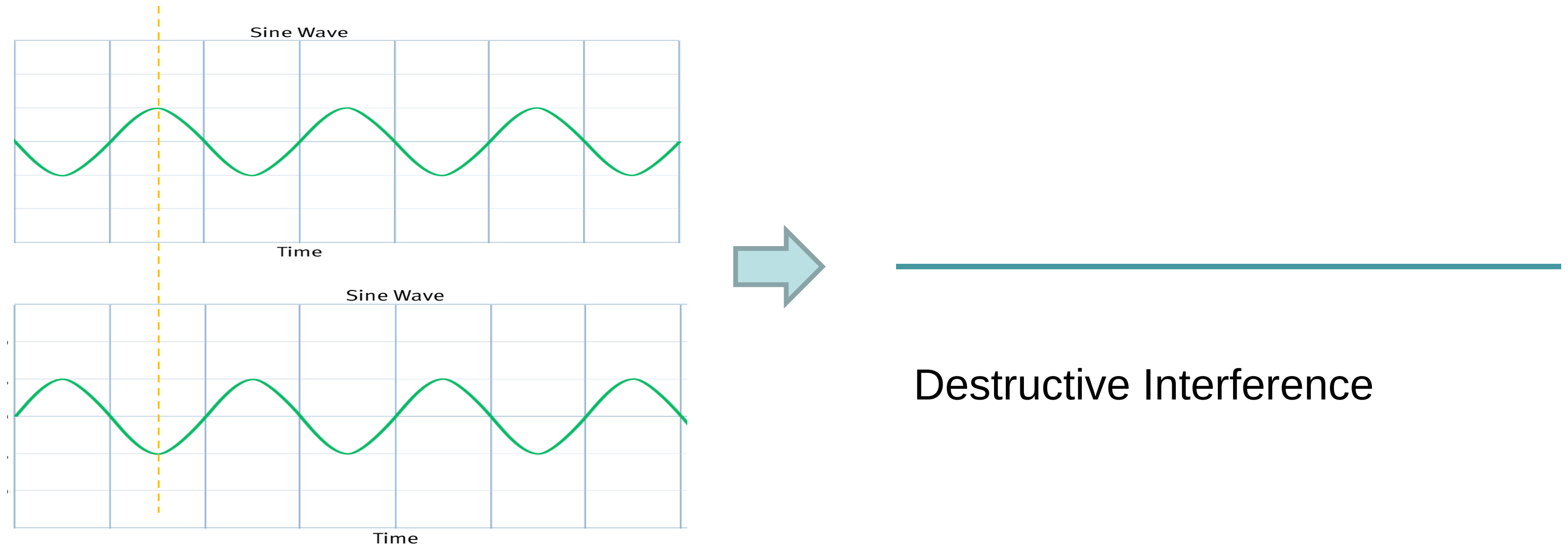
$$\delta t_c = 1/\Delta\nu$$



$c \times \delta t_c = L_c$ is the coherence length

Simple explanation of Optical Interference





Laser, Hg Lamp, and white light

Which is the most coherent light source of the three?

Laser: $c*t=3*10^{8}*10^{-6}=300m>2d$

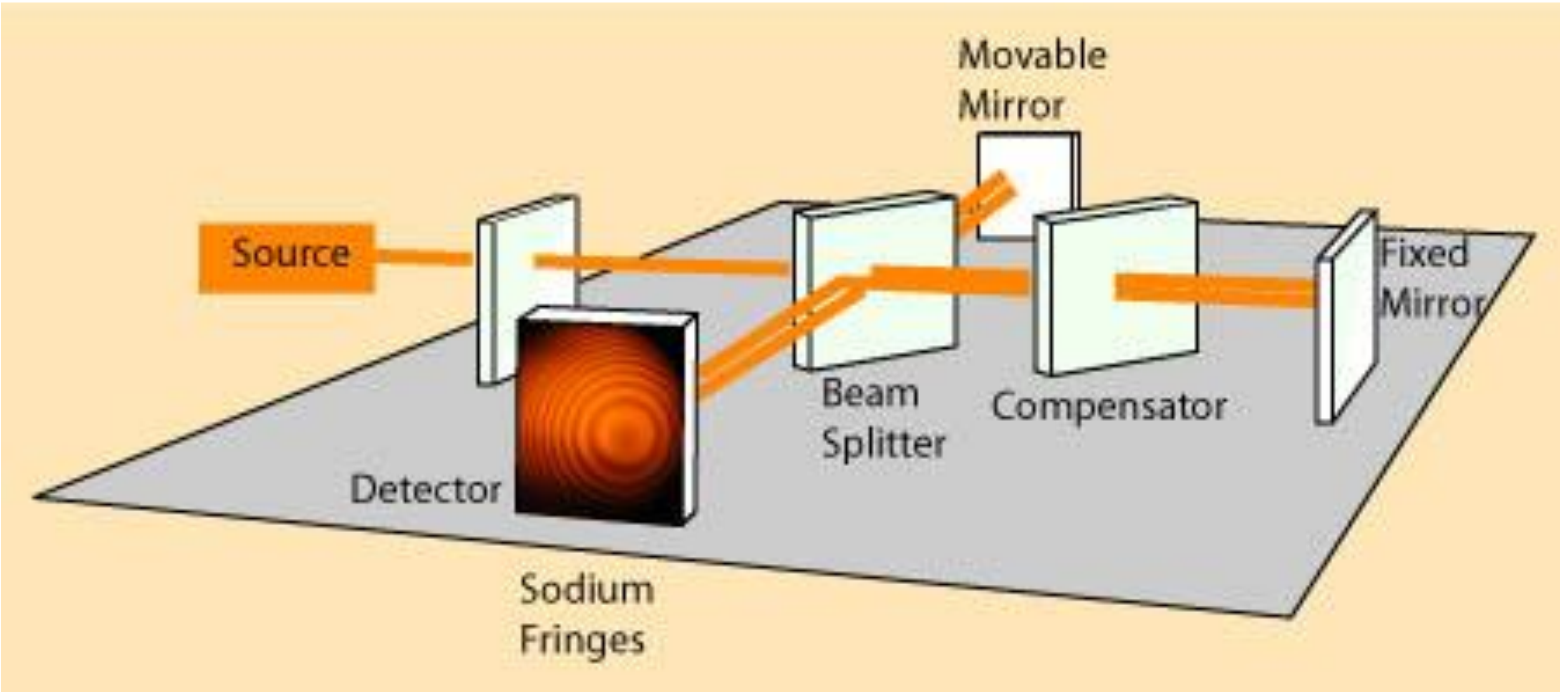
Hg lamp: $c*t=3*10^{8}*10^{-12}=300\mu m>2d$

White light: $c*t=3*10^{8}*10^{-14}=3\mu m>2d$

Michelson Interferometer

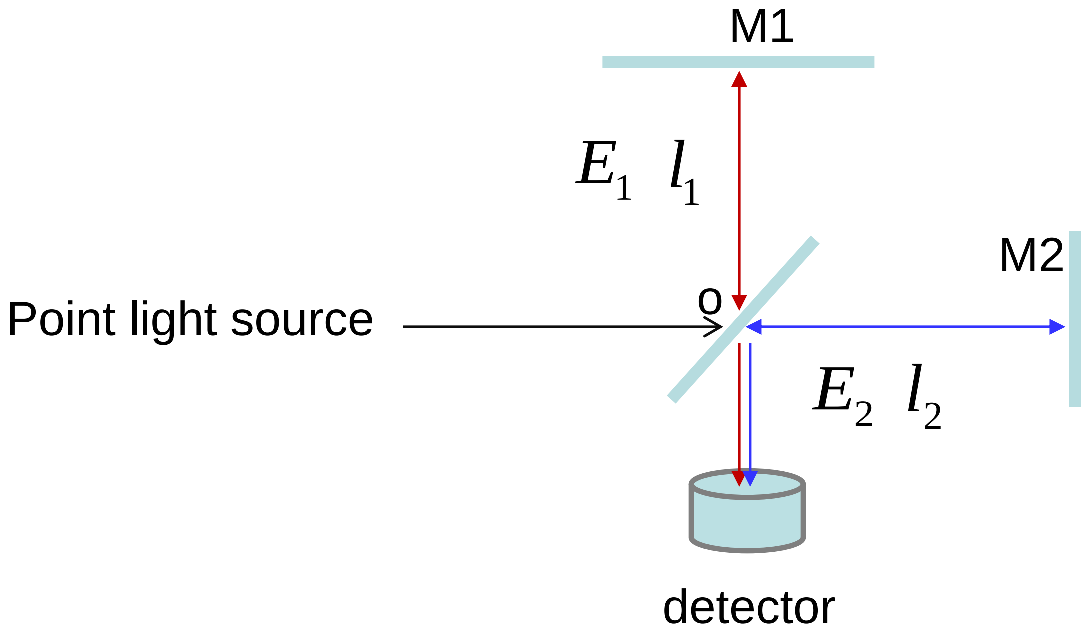
(1) Measure wavelength; (2) Coherence length

Amplitude splitting interfrometer E^{ikr} k.r represents the optical phase



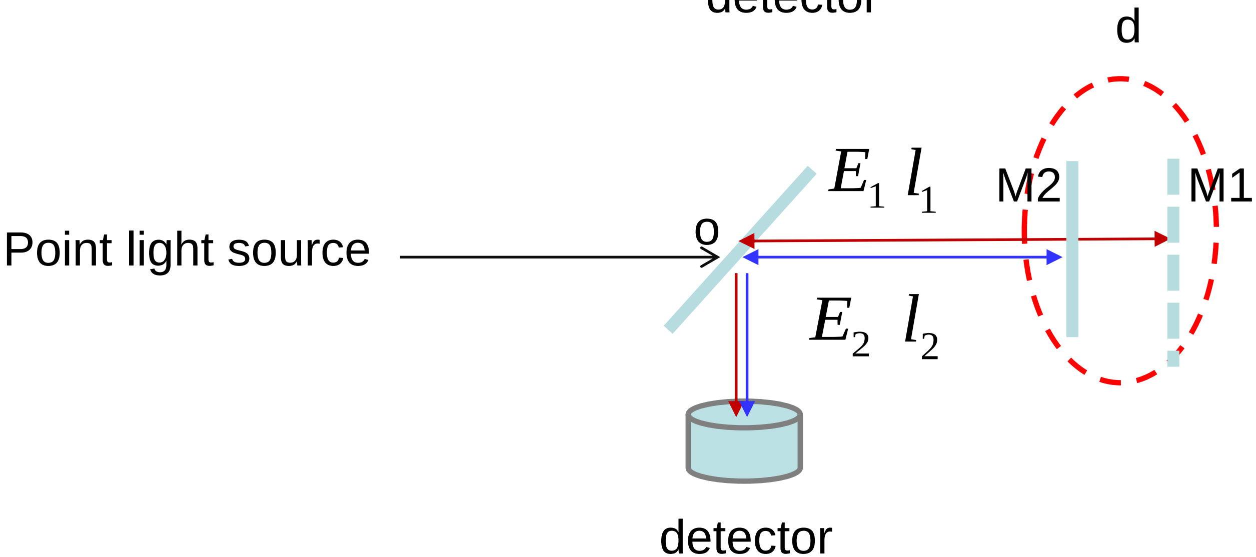
https://www.youtube.com/watch?v=j-u3IEgcTiQ

Michelson Interferometer: Conceptual Rearrangement



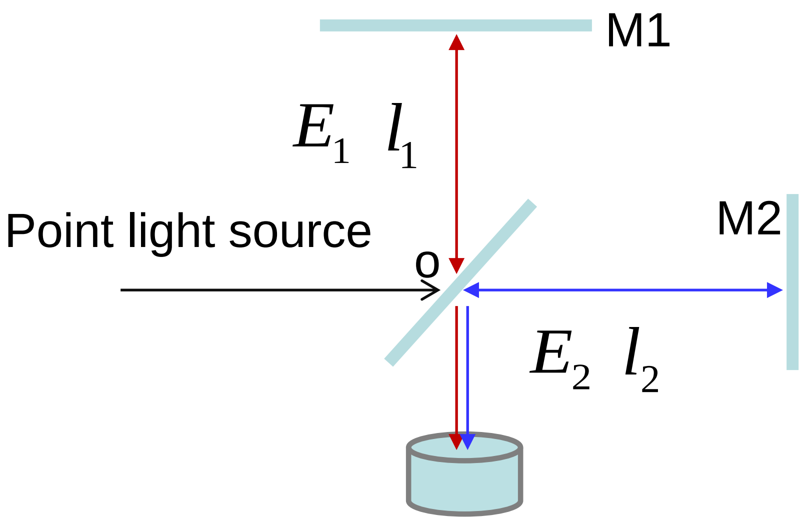
Optical phase =k.r

Optical path difference:
$2(l_2 - l_1)=2d$



Optical phase =2k.d

Michelson Interferometer – simplified picture



Phase difference $e^{i\delta}$

$$E_1 = A_1 e^{ik.(2l_1)}$$

$$E_2 = A_2 e^{ik.(2l_2)}$$

reflection

Optical phase =2k.d + π

$$\delta = \frac{2\pi}{\lambda}.2(l_2 - l_1) + \pi$$

$$I = |E|^2 = A^2$$

$$|E_1 + E_2|^2 = A_1^2 + A_2^2 + 2A_1A_2\cos\delta$$

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\delta$$

Built-in π phase shift between two optical paths (OM1 and OM2) due to the reflection. See Hecht 116-117;

Michelson Interferometer – simplified picture

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\delta$$

$$\delta = \frac{2\pi}{\lambda}.2d + \pi$$

Constructive interference

$$I_{\max} = I_1 + I_2 + 2\sqrt{I_1 I_2}$$

If $\delta = 2m\pi$

$$k.2d = (2m-1)\pi$$

or $2d = \left(m - \frac{1}{2}\right)\lambda$

Destructive interference

$$I_{\min} = I_1 + I_2 - 2\sqrt{I_1 I_2}$$

$$\delta = (2m+1)\pi$$

$$k.2d = 2m\pi$$

$$2d = m\lambda$$

Michelson Interferometer – simplified picture

Constructive interference

$$I_{\max} = I_1 + I_2 + 2\sqrt{I_1 I_2}$$

Destructive interference

$$I_{\min} = I_1 + I_2 - 2\sqrt{I_1 I_2}$$

Question, in order to get the best contrast between Imax and Imin, what are the values of I_1 and I_2 ? i.e. what type of beam splitter shall we use?

Interference Contrast

$$I_{\max} = I_1 + I_2 + 2\sqrt{I_1 I_2}$$

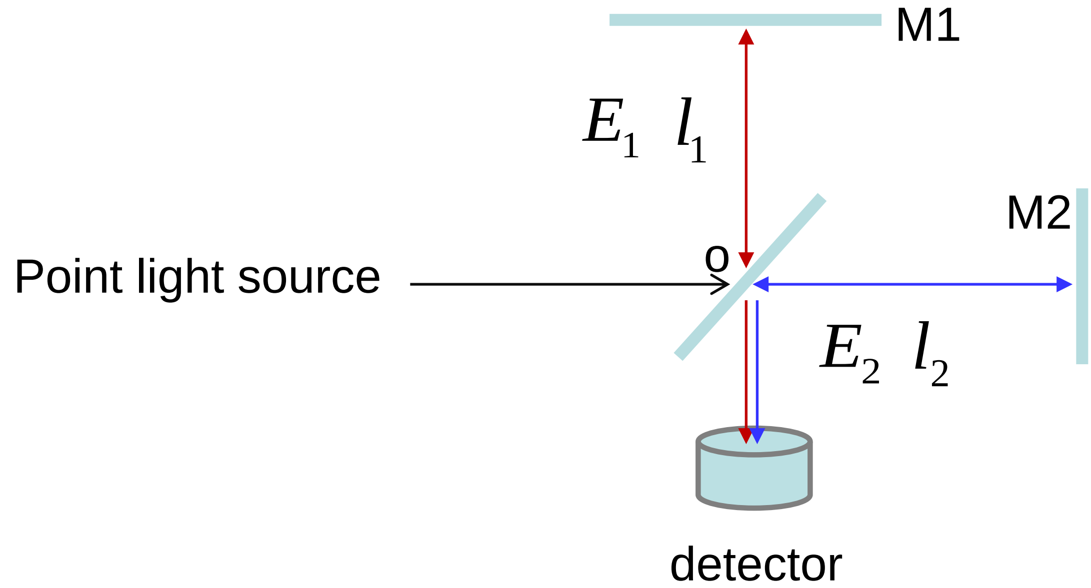
$$I_{\min} = I_1 + I_2 - 2\sqrt{I_1 I_2}$$

$$contrast = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} = 2\frac{\sqrt{I_1 I_2}}{I_1 + I_2}$$

$$I_1 = I_2$$

Contrast=1

Michelson Interferometer – simplified picture



$$2d = \left(m - \frac{1}{2}\right)\lambda$$

$$I_{max} = I_1 + I_2 + 2\sqrt{I_1 I_2}$$

Constructive interference

From the m to the (m+1) order of constructive interference, the optical path dereference is 2d=λ

$$I_1 = I_2 = \frac{1}{2} I_o \qquad \delta = \frac{2\pi}{\lambda}.2d = 2\pi\sigma\Delta \qquad \sigma = \frac{1}{\lambda}$$

$$\Delta = 2d$$

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\delta$$

Intensity for monochromiatic source: $\mathrm{I} = \mathrm{I_o}(1 + \cos 2\pi\sigma\Delta)$

For a spectrally broad source: $\mathrm{I}(\Delta) = \int_0^{\infty} \mathrm{I}(\mathrm{k})(1 + \cos 2\pi\sigma\Delta)\, d\sigma$

This is a Fourier Transform between position (Δ) and momentum (σ).

Inverse Fourier Transform -> optical frequency

Michelson Interferometer (Fourier Transform Spectrometer)

$$I(\Delta) = \int_0^{\infty} I(\sigma)(1+\cos 2\pi\sigma\Delta)\,d\sigma$$

Thorne p.186

$$I(\Delta) = \int_{-\infty}^{+\infty} I(\sigma)\,d\sigma + \int_{-\infty}^{+\infty} I(\sigma)\cos(2\pi\sigma\Delta)\,d\sigma$$

Constant background

Fourier Transform between the frequency domain and interference pattern

An example: a doublet

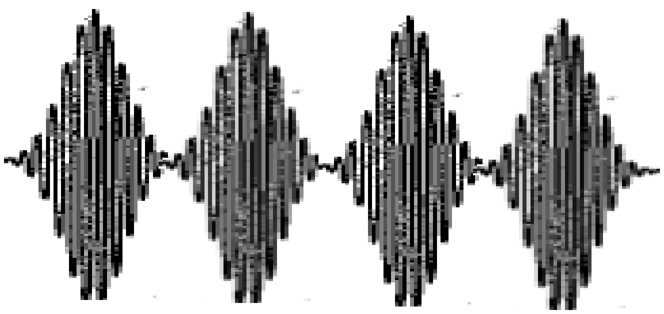
$$I(\sigma) = I_o\left[\delta(\sigma-(\sigma_o+s)) + \delta(\sigma-(\sigma_o-s))\right]$$

$$\delta_{ab} \begin{cases} 1 & a=b \\ 0 & a \neq b \end{cases}$$

$$I(\Delta) = 2I_o + \int_{-\infty}^{+\infty} I(\sigma)\cos(2\pi\sigma\Delta)\,d\sigma$$

$$= 2I_o + I_o\left(\cos(2\pi\Delta(\sigma_o+s)) + \cos(2\pi\Delta(\sigma_o-s))\right)$$

$$= 2I_o(1+\cos(2\pi\Delta\sigma_o)\cos(2\pi\Delta s))$$



Michelson Interferometer

Various I(σ)

$$I(\Delta)=\int_{-\infty}^{+\infty} I(\sigma)d\sigma+\int_{-\infty}^{+\infty} I(\sigma)\cos(2\pi\sigma\Delta)d\sigma$$

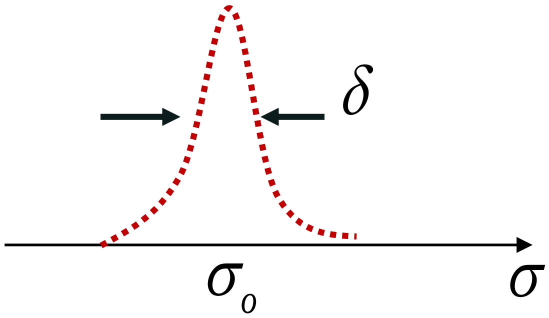
Single frequency with infinite coherence length

$$I(\Delta)=I_o(1+\cos 2\pi\sigma_o\Delta)$$

Doppler broadened lines

$$I(\sigma)=I_o e^{\frac{-4\ln 2(\sigma-\sigma_o)^2}{(\delta\sigma)^2}}$$

$$I(\Delta)=I_o(1+e^{\frac{-(\pi\delta\Delta)^2}{4\ln 2}}\cos 2\pi\sigma_o\Delta)$$



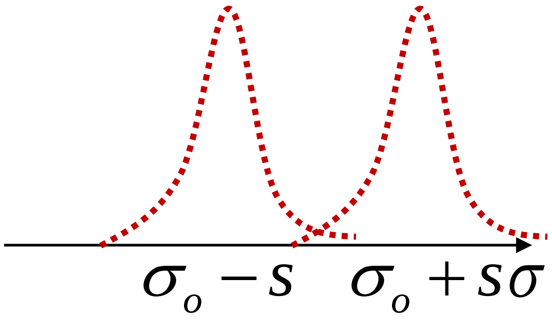
Two sharp lines

$$I(\sigma)=I_o\left[\delta(\sigma-(\sigma_o+s))+\delta(\sigma-(\sigma_o-s))\right]$$

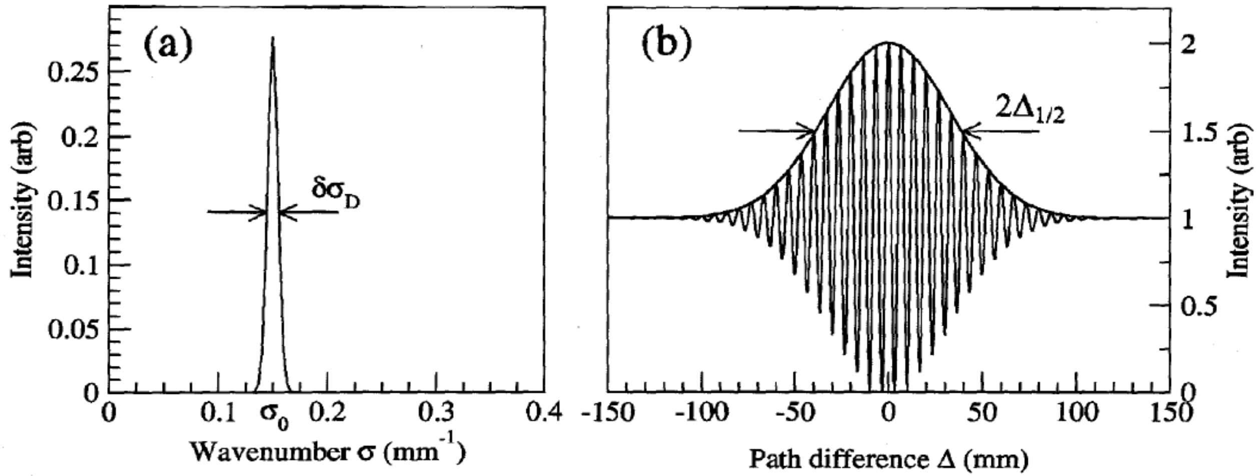
$$I(\Delta)=2I_o(1+\cos(2\pi\Delta\sigma_o)\cos(2\pi\Delta s))$$

Two equal intensity Doppler broadened lines

$$I(\Delta)=I_o(1+e^{\frac{-(\pi\delta\Delta)^2}{4\ln 2}}\cos(2\pi\Delta\sigma_o)\cos(2\pi\Delta s))$$



FOURIER TRANSFORMS IN MICHAELSON EXPERIMENT



Gaussian distribution in wavenumber gives

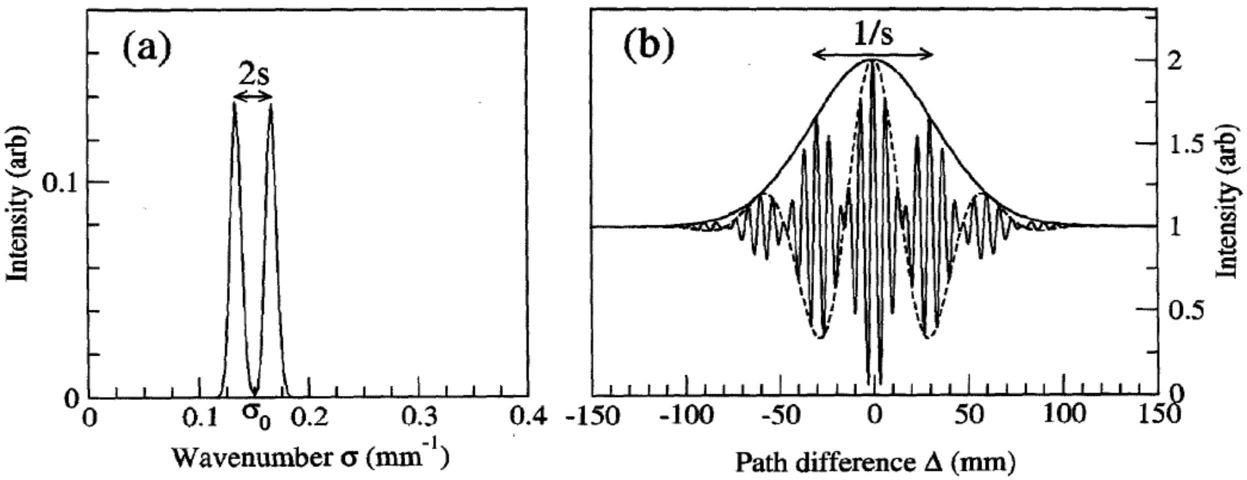
this intensity distribution in path differences

Doppler broadened lines $I(\sigma) = I_o e^{\frac{-4\ln 2(\sigma-\sigma_o)^2}{(\delta\sigma)^2}}$

$$I(\Delta) = I_o\left(1 + e^{\frac{-(\pi\delta\Delta)^2}{4\ln 2}} \cos 2\pi\sigma_o\Delta\right)$$

$$\Delta_{1/2} = \frac{2\ln 2}{\pi\delta}$$

Michelson Interferometer



Doublet Gaussian in wavenumber gives beats in intensity distribution in path differences

(The wavenumbers are well outside of visible range)

Two equal intensity Doppler broadened lines

$$I(\Delta) = I_o(1 + e^{\frac{-(\pi\delta\Delta)^2}{4\ln 2}} \cos(2\pi\Delta\sigma_o)\cos(2\pi\Delta s))$$

Video

https://www.youtube.com/watch?v=j-u3IEgcTiQ

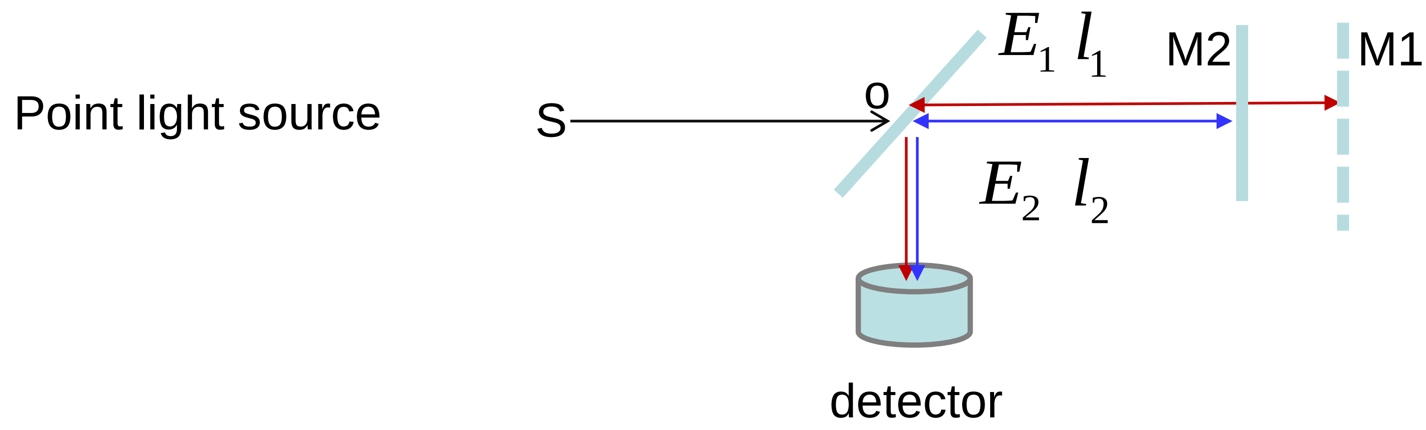
Is this correct?

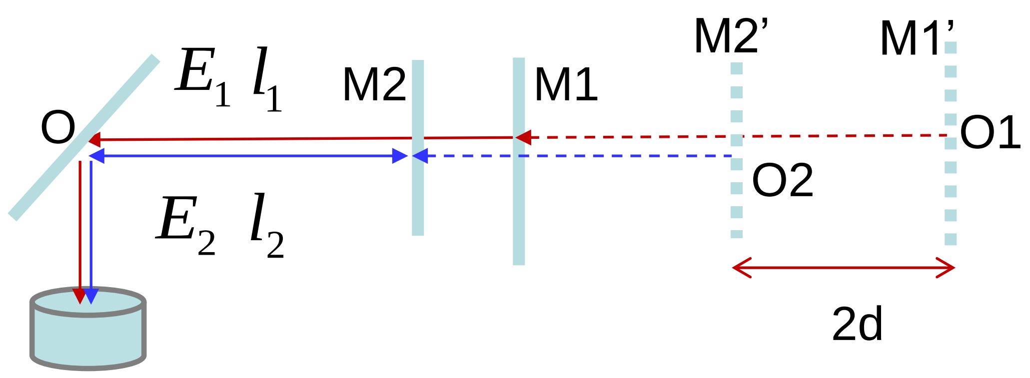
$$2d\cos(\theta)=m\lambda$$

$$2(d+\Delta)\cos(\theta)=(m+N)\lambda$$

$$2\Delta\cos(\theta)=N\lambda$$

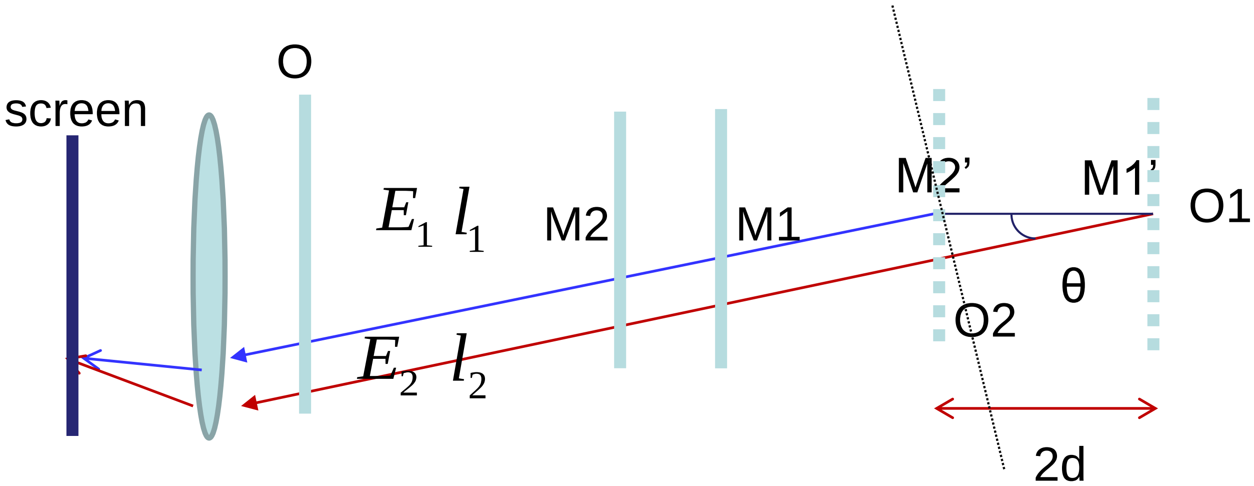
Michelson Interferometer: Conceptual Rearrangement





(O1O) – (O2O)=2(M1O – M2O) = 2d

Michelson IFM – Finite Light Spot



$$\delta = 2d\cos\theta$$

Destructive interference when round trip optical path difference is a multiple of the wavelength. When 2dcosθ = mλ

Laser: $c * t = 3*10^8 * 10^{-6} = 300m > 2d$

Hg lamp: $c * t = 3*10^8 * 10^{-12} = 300\mu m > 2d$

White light: $c * t = 3*10^8 * 10^{-14} = 3\mu m > 2d$

No interference when path difference > coherence length (Think Fourier)

Circular fringes: For *fixed* mirror positions, *path difference* depends only on θ:

$$2d = m\lambda \quad \text{central dark fringe}, \theta = 0$$

$$2d\cos\theta_1 = (m-1)\lambda$$

$$2d\cos\theta_2 = (m-2)\lambda$$

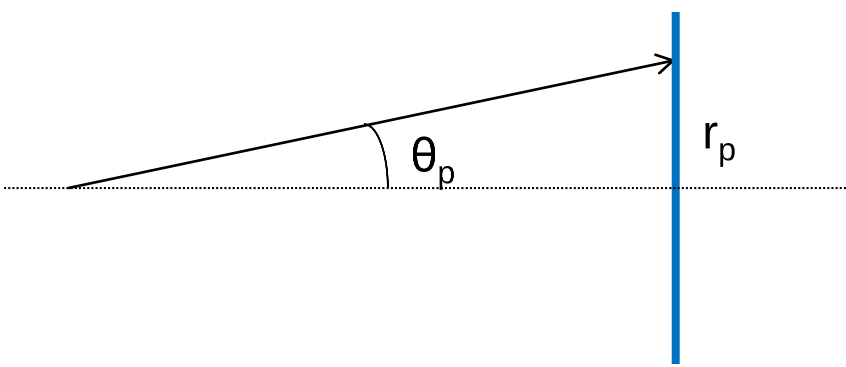
$$\vdots$$

$$2d\cos\theta_p = (m-p)\lambda$$

$$\implies 2d(1-\cos\theta_p) = p\lambda$$

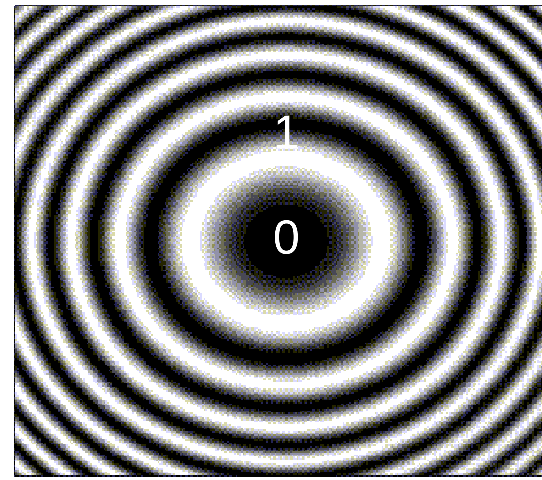
If $\theta_p \ll 1$, $\cos\theta_p = 1 - \theta_p^2/2$, therefore:

$$\theta_p = \left(\frac{p\lambda}{d}\right)^{1/2}, \quad \text{radius of } pth \text{ fringe}$$



Destructive Interference

$2d\cos\theta = m\lambda$



First part of Michelson Lab. What happens when d→0?