Diffraction

Fraunhofer and Fresnel Diffraction

Reports due this week

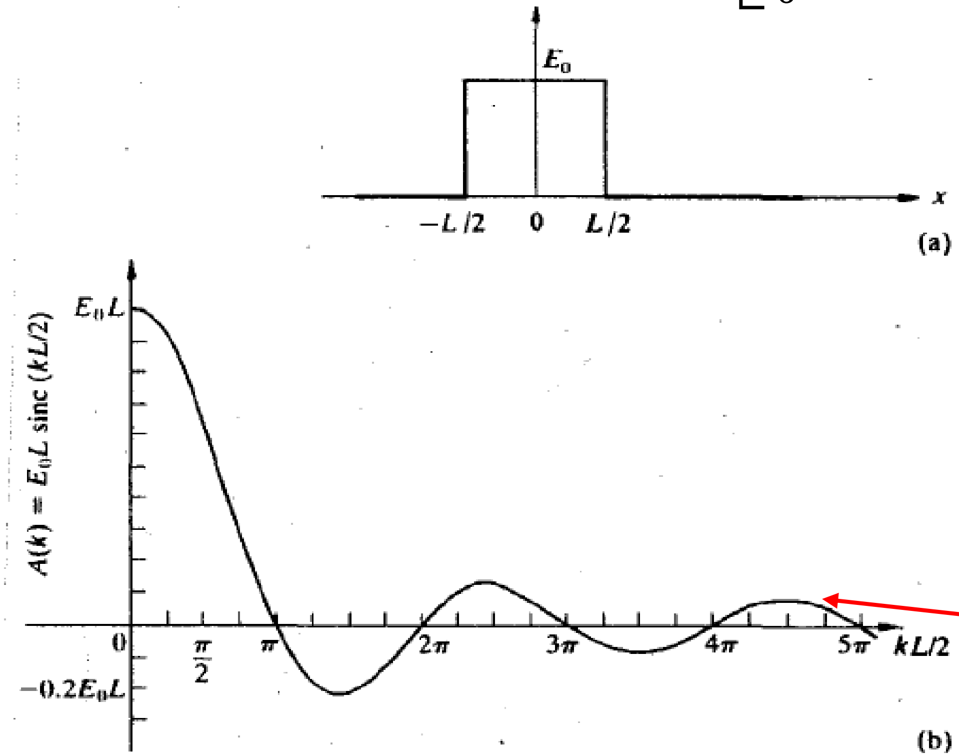
A Fabry-Perot wave meter. Free spectral range: 3000 GHz. Finesse 100,000.

Two light sources have a frequency difference of 4000 GHz. Can we resolve these two light sources?

A sodium atom has a doublet emission with a frequency difference of 0.1 GHz. Can we resolve the fine structure?

Fourier Integrals and Fourier Transforms

$$f(x) = \frac{1}{\pi}\left[\int_0^\infty A(k)\cos(kx)dk + \int_0^\infty B(k)\sin(kx)dk\right]$$



$$A(k) = \int_{-\infty}^{\infty} f(x)\cos(kx)dx$$

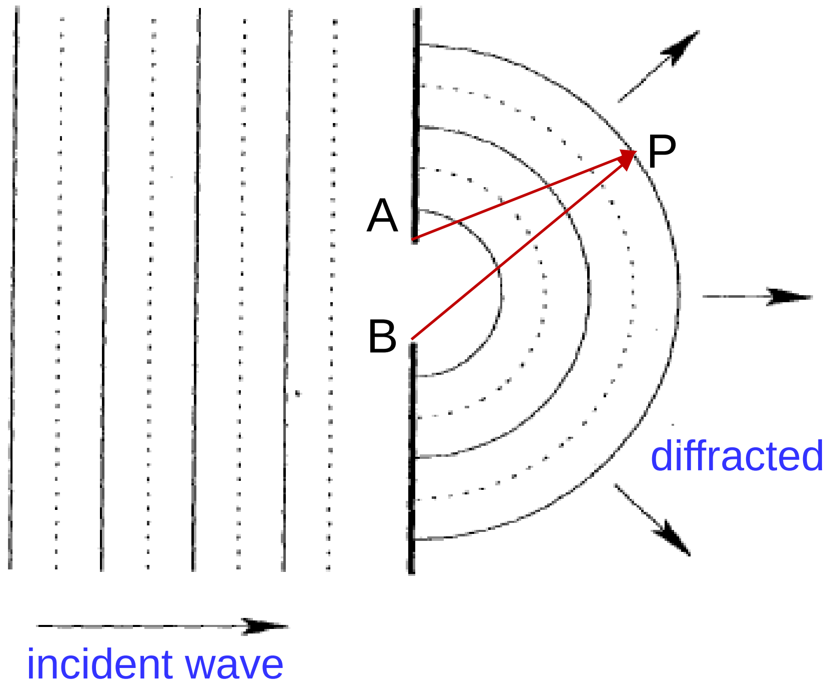
$$B(k) = \int_{-\infty}^{\infty} f(x)\sin(kx)dx$$

$$A(k) = E_0 L \frac{\sin(kL/2)}{kL/2}$$

Hecht, Pg 312, Fig. 7.34
The square pulse and its transform

The sinc function.
Inverse relationship between widths
FT in single-slit Fraunhofer diffraction

Diffraction



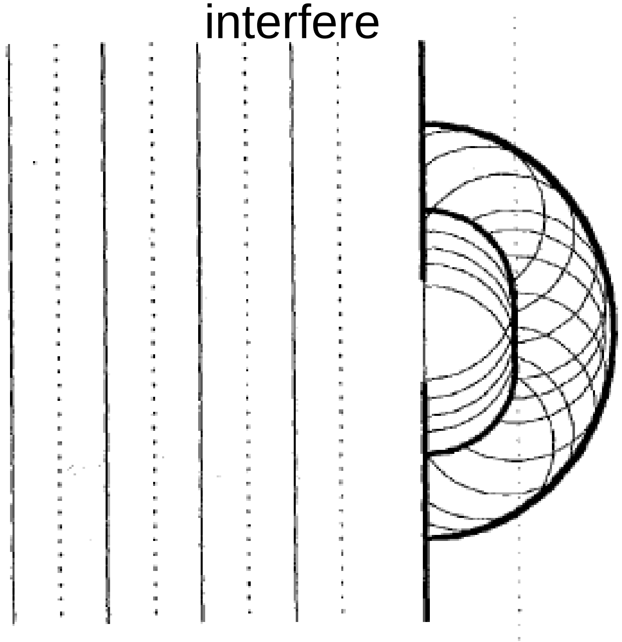
$$\Lambda_{max} = \left|\overline{AP} - \overline{BP}\right| \leq AB$$

Manifestation of the wave nature of light – light “bends” around corners.

When λ ~ smallest dim

$\lambda > AB$ Wave spreads in large angle

$\lambda < AB$ Multitude of wavelets emitted from the aperture interfere



Huygens-Fresnel Principle:

Every unobstructed point of a wavefront acts as a source of spherical secondary wavelets. The field at any point is the superposition of all of the secondary wavelets.

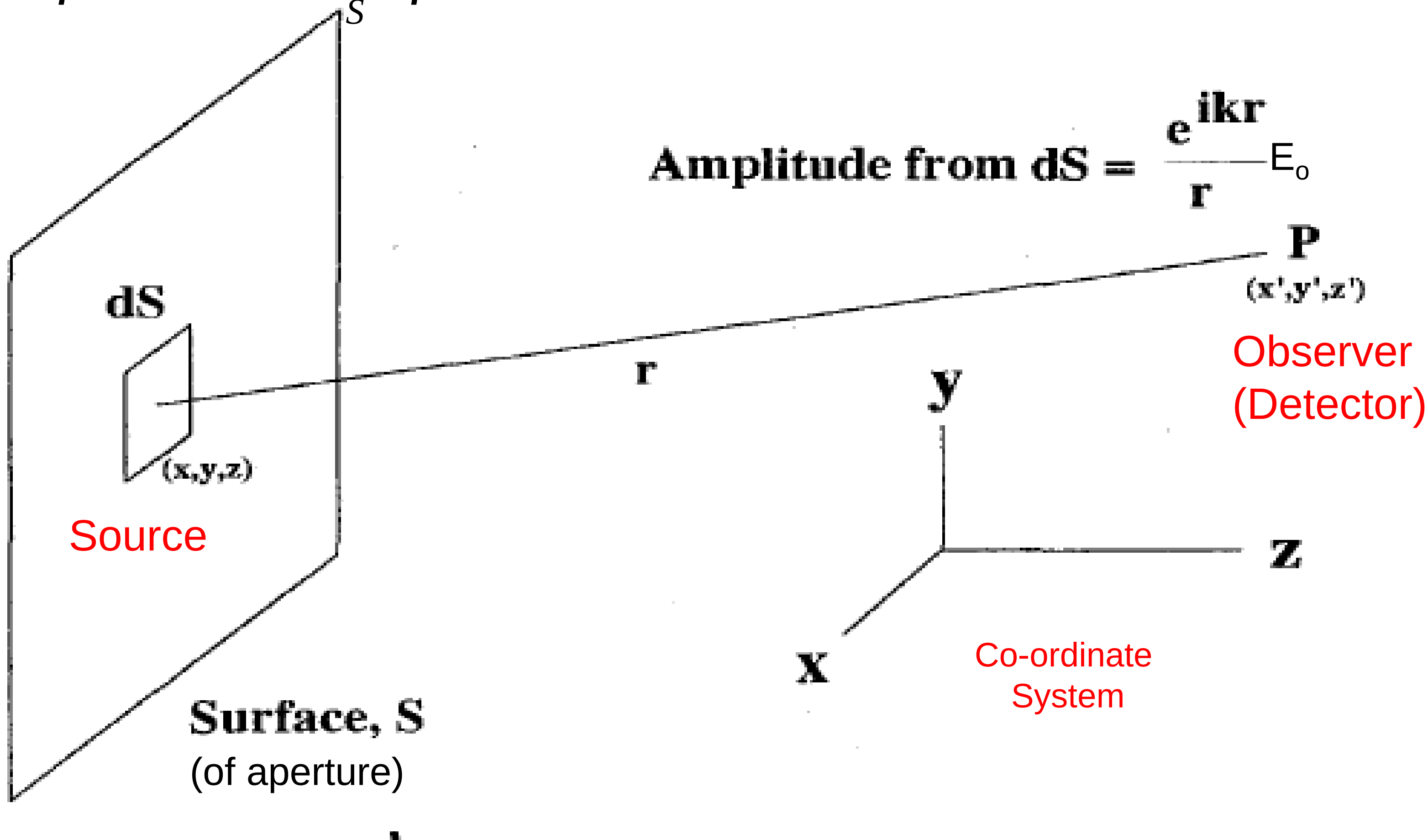
Analytical Approach

$$E_{spher} \propto E_0 \frac{e^{ikr}}{r}$$

Individual Huygens wavelet generated from surface S

$$E = E_o \sum_S \frac{e^{ikr}}{r} \Delta S \rightarrow E_o \int_S \frac{e^{ikr}}{r} dS$$

Superposition of Huygens wavelets at r.
Time dependence is same for all (same frequency)



Analytical Approach - II

Coordinate System:

Source Point (on surface): x, y, z

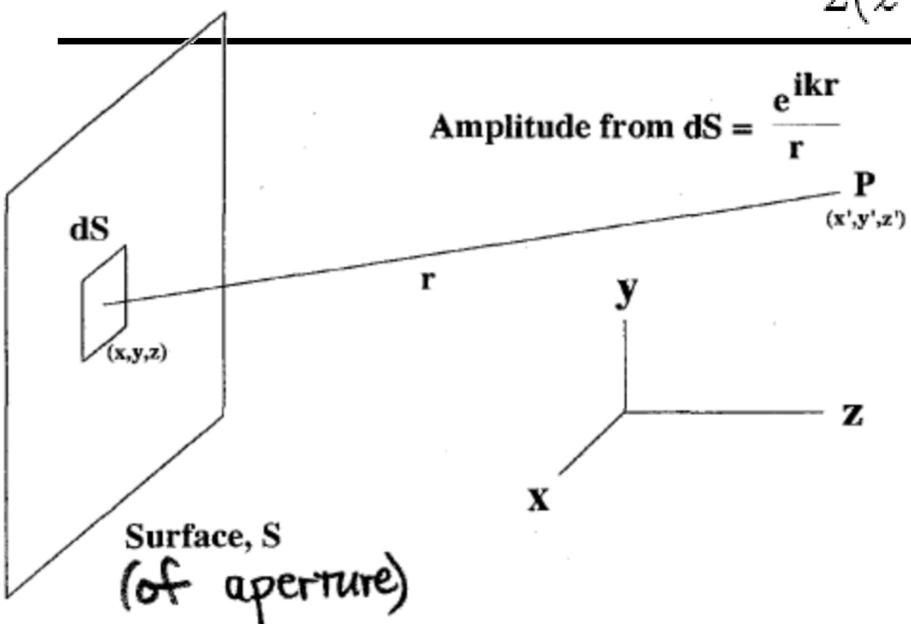
Observation Point (at P): x', y', z'

$$r = \sqrt{(x'-x)^2 + (y'-y)^2 + (z'-z)^2}$$

Assume that $z' - z >> x' - x, y' - y$, and expand the square root:

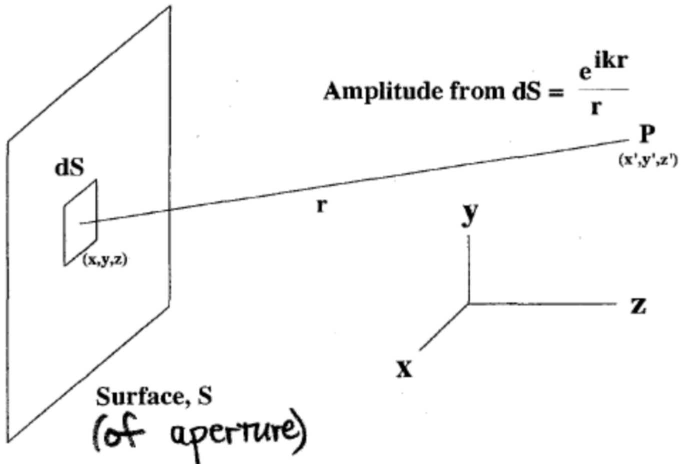
$$r \approx (z'-z)\left\{1 + \frac{1}{2}\frac{(x'-x)^2 + (y'-y)^2}{(z'-z)^2}\right\}$$

$$= (z'-z) + \frac{x'^2 + y'^2}{2(z'-z)} - \frac{(x'x + y'y)}{(z'-z)} + \frac{x^2 + y^2}{2(z'-z)}$$



$$z - z' = d$$

The distance between source plane and detector plane.



Fraunhofer Diffraction: Far field diffraction

$$E \propto \int_S e^{ik\left\{(z'-z)+\frac{x'^2+y'^2}{2(z'-z)}-\frac{x'x+y'y}{(z'-z)}+\frac{x^2+y^2}{2(z'-z)}\right\}}dS$$

$$E \propto \int_S e^{ik\left\{-\frac{x'x+y'y}{d}+\frac{x^2+y^2}{2d}\right\}}dS$$

Can ignore quadratic terms if

$$k\left(\frac{x^2+y^2}{2d}\right) << 1$$

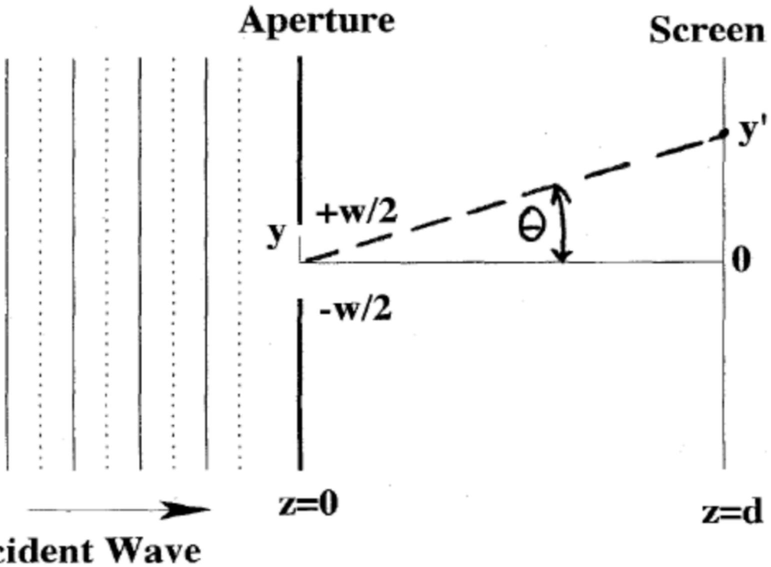
We assume the incoming wave is homogeneous

Criterion for Fraunhofer Diffraction

$$d >> \frac{w^2}{\lambda}$$

w is largest dim of aperture

1-D Fraunhofer Diffraction – Single Slit



$$E \propto \int_{-w/2}^{+w/2} e^{\frac{iky'y}{d}} dy \qquad (k = \frac{2\pi}{\lambda})$$

Huygens' Recipe
Fraunhofer limit

$$I \propto |E|^2 \propto \left| \frac{2d}{iky'} \sin \frac{kwy'}{2d} \right|^2 = w^2 \left[\frac{\sin \frac{kwy'}{2d}}{\frac{kwy'}{2d}} \right]^2$$

SINC sq. function

In general, related to Fourier Transform of aperture function

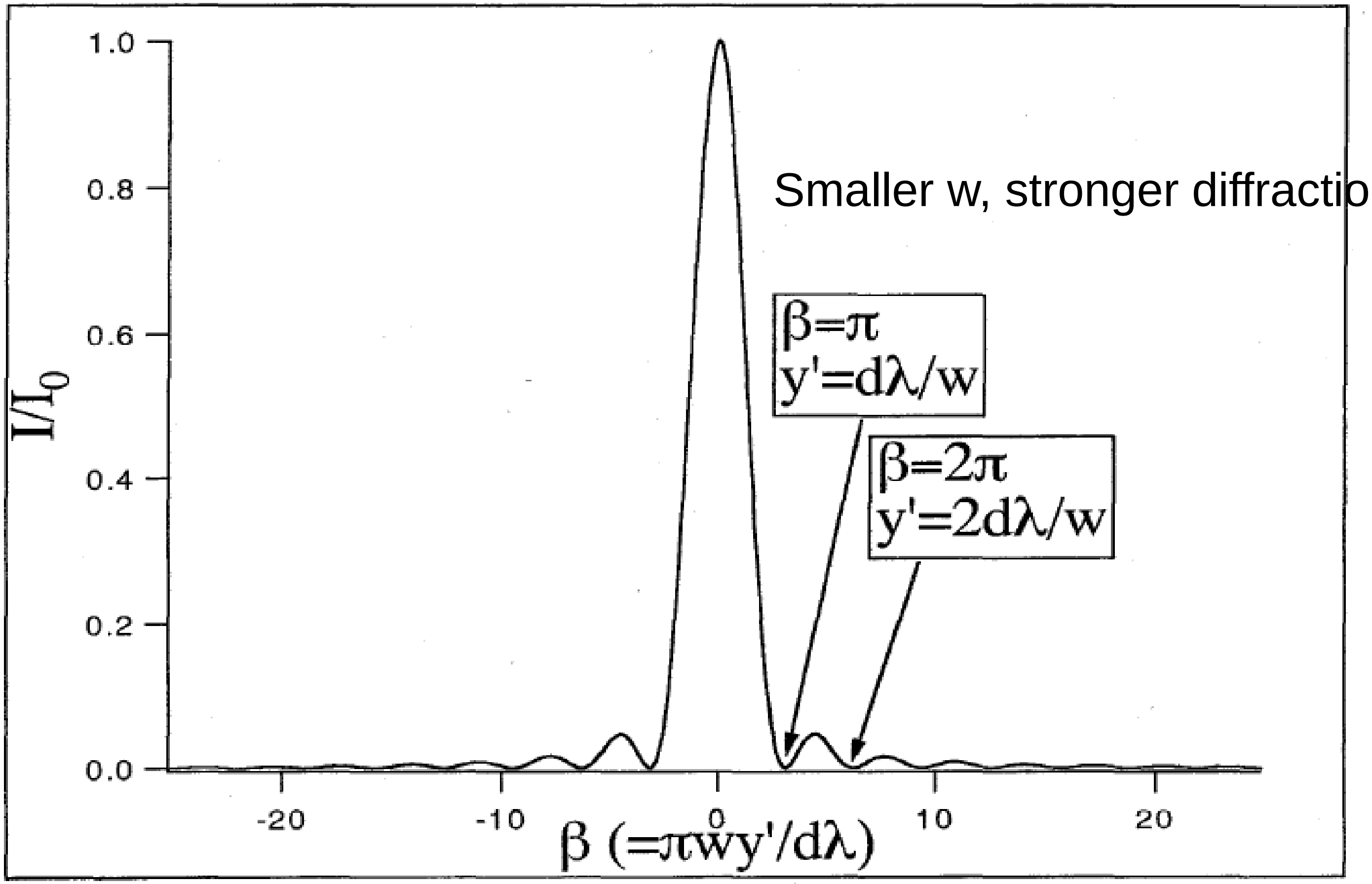
$$I(y') = I_0 \left(\frac{\sin \beta}{\beta} \right)^2, \qquad \beta \equiv \frac{kwy'}{2d}$$

$$E \propto \left[\frac{d}{iky'} e^{\frac{iky'y}{d}} \right]_{y=-w/2}^{y=+w/2} = \frac{d}{iky'} [e^{\frac{ikwy'}{2d}} - e^{\frac{-ikwy'}{2d}}] = \frac{2d}{iky'} \sin \frac{kwy'}{2d}$$

Hecht page 452-469

Single Slit Diffraction Pattern

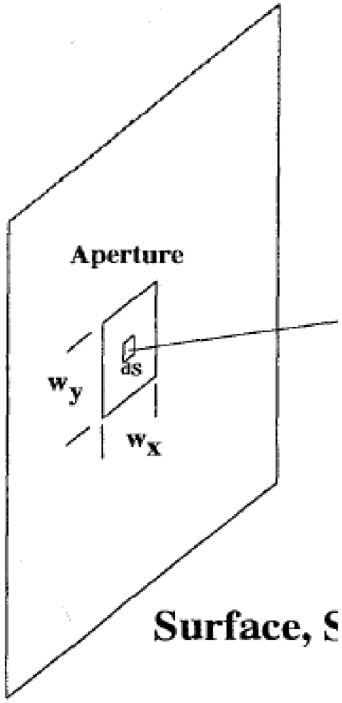
$$I(y') = I_0 \left(\frac{\sin \beta}{\beta}\right)^2, \qquad \beta = \frac{kwy'}{2d}$$



Smaller w, stronger diffraction

Can determine width w from minima location

Fraunhofer Diffraction – Rectangular Aperture



$$E = \frac{E_o}{d} \iint e^{\frac{-ik(xx'+yy')}{d}} ds$$

Slit widths = w_x **(along x),** w_y **(along y)**

From Huygens-Fresnel Principle:

$$E \propto \int_S \frac{e^{ikr}}{r} dS \approx \int_S e^{ik\frac{x'x+y'y}{d}} dS = \int_{-w_y/2}^{+w_y/2} \int_{-w_x/2}^{+w_x/2} e^{ik\frac{x'x+y'y}{d}} dx dy$$

For *Rectangular Aperture* **x and y directions are** *Independent* **:**

$$E \propto \left(\int_{-w_y/2}^{+w_y/2} e^{\frac{iky'y}{d}} dy \right) \left(\int_{-w_x/2}^{+w_x/2} e^{\frac{ikx'x}{d}} dx \right)$$

These are familiar integrals from **1-dimensional case**:

$$E \propto \left(\frac{\sin \beta_x}{\beta_x} \right) \left(\frac{\sin \beta_y}{\beta_y} \right), \quad \beta_x \equiv \frac{kw_x x'}{2d}, \beta_y \equiv \frac{kw_y y'}{2d}$$

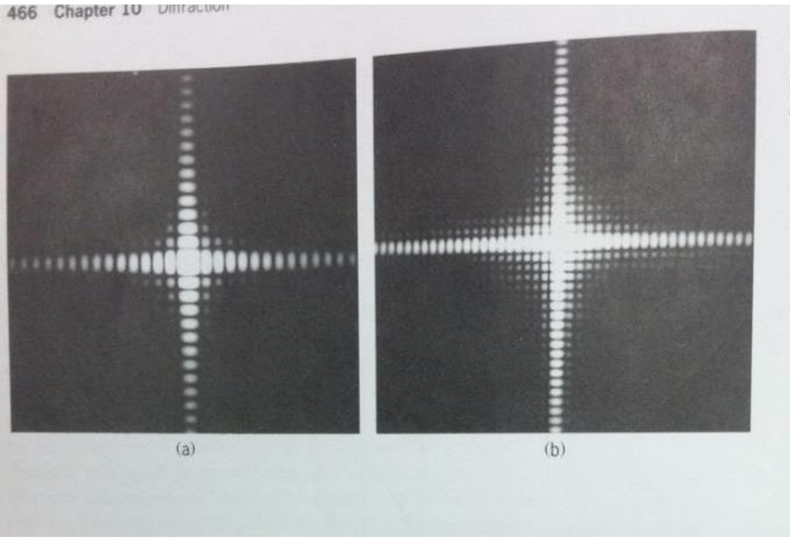
Intensity $I \propto |E|^2$

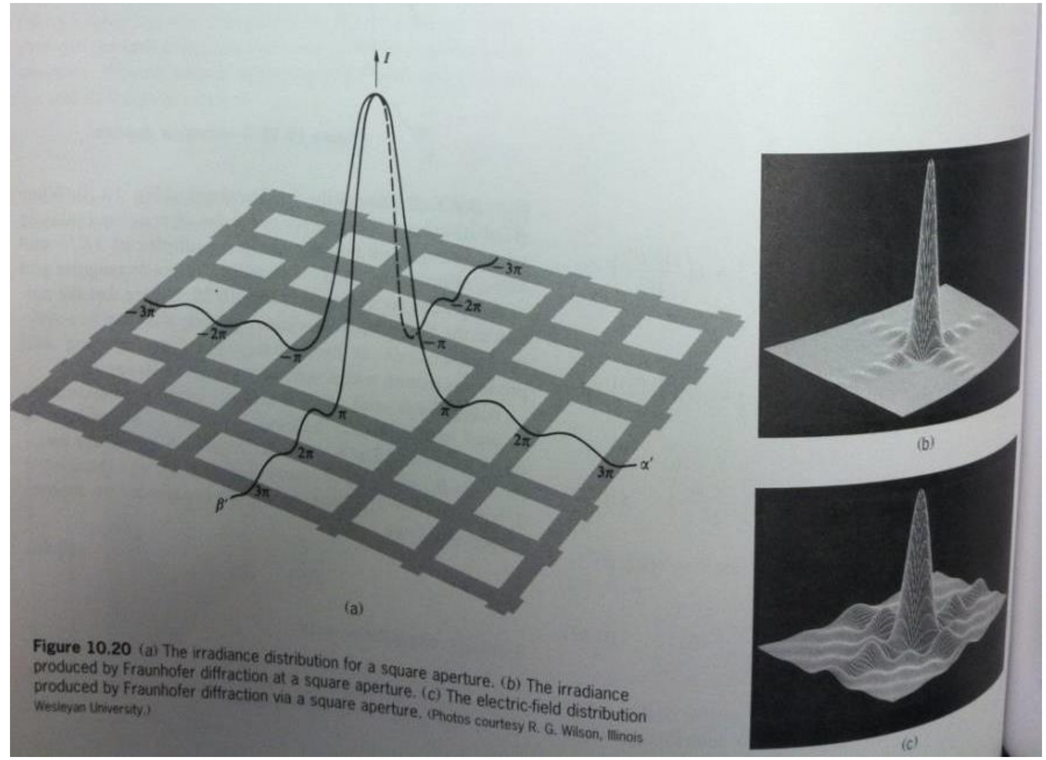
$$I = I_0 \left(\frac{\sin \beta_x}{\beta_x} \right)^2 \left(\frac{\sin \beta_y}{\beta_y} \right)^2$$

Hecht page 452-469

Fraunhofer Diffraction – Rectangular Aperture

$$I = I_0 \left(\frac{\sin \beta_x}{\beta_x}\right)^2 \left(\frac{\sin \beta_y}{\beta_y}\right)^2$$



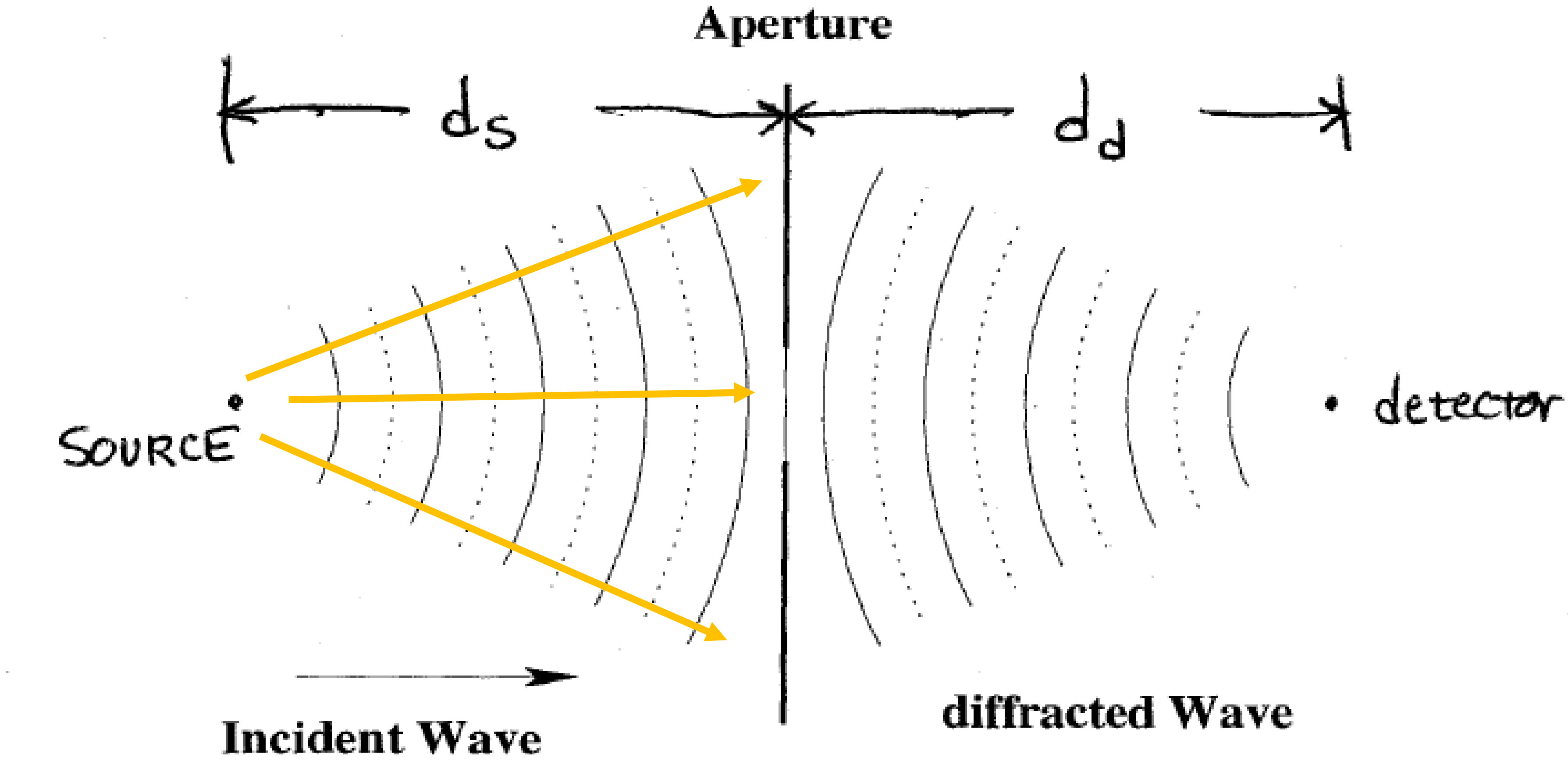


$$\frac{I_m}{I(0)} = \frac{1}{\beta_i^2}, \beta_i = (n + 1/2)\pi$$

$$\beta_x \equiv \frac{k w_x x'}{2d}, \beta_y \equiv \frac{k w_y y'}{2d}$$

Fresnel Diffraction

Occurs when $w^2/d\lambda >> 1$
Near Field Effect ⬄
Wavefront curvature
is important.



Fresnel Diffraction

All Waves are Spherical and **Aperture is Rectangular:** (x_1, x_2) by (y_1, y_2)

Source Wave: $E \propto \frac{e^{ikr_s}}{r_s}$, where r_s = distance from **source** to **wavefront**.

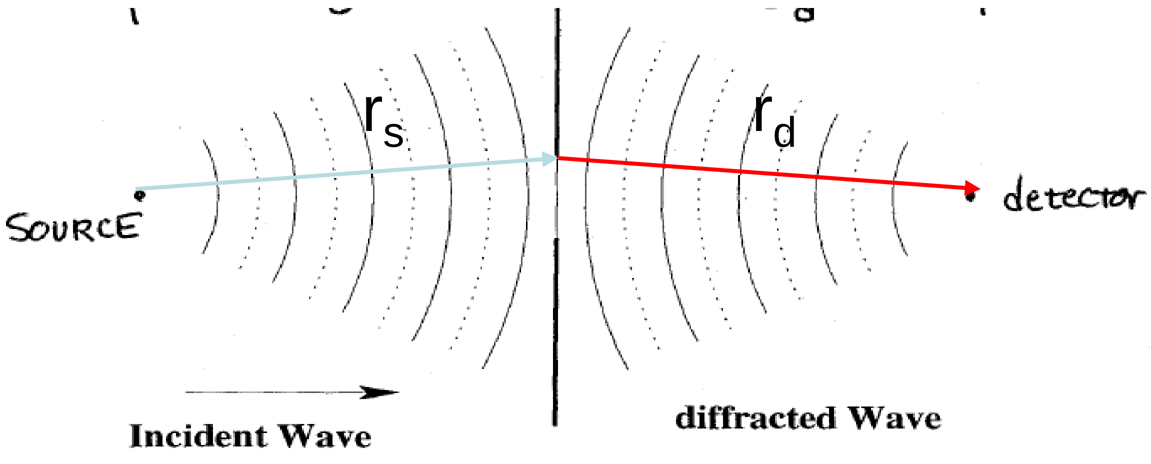
Diffracted Wave: $E \propto \frac{e^{ikr_d}}{r_d}$, where r_d = distance from **observer** to **wave-front**.

Using Huygen's Principle surface element dS produces contribution dE seen by observer:

$$dE \propto \frac{1}{r_s r_d} e^{ik(r_s+r_d)} dS$$

Total field at observing point:

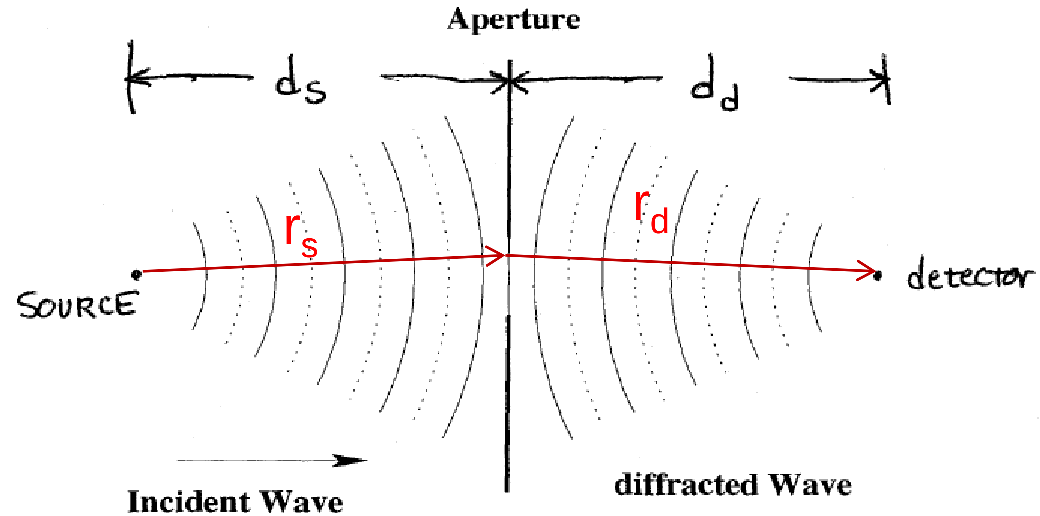
$$E \propto \frac{1}{r_s r_d} \int_S e^{ik(r_s+r_d)} dS$$



Fresnel Diffraction

Occurs when $w^2/d\lambda >> 1$
Near Field Effect ⇔
Wavefront curvature
is important.

(keep quadratic terms)



If d_s = distance of source to **aperture plane** and d_d is distance from observer to **aperture plane**:

$$ik(r_s + r_d) \approx ik\left(d_s + d_d + (x^2+y^2)\frac{d_s+d_d}{2d_sd_d}\right) = ik(d_s+d_d) + i\pi u^2/2 + i\pi v^2/2$$

where x and y are in aperture plane (as in Fraunhofer case) and:

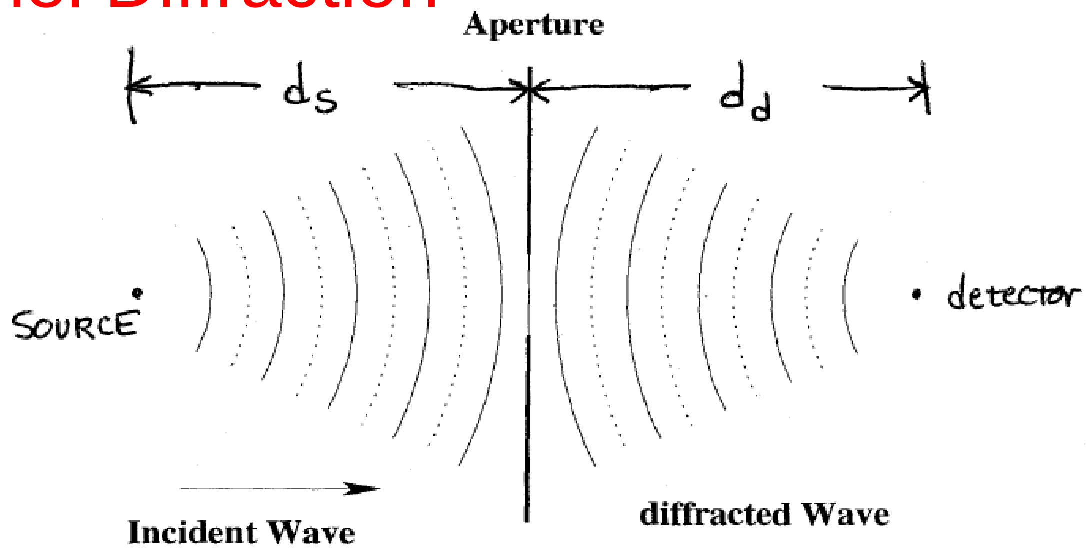
$$u \equiv x\left[\frac{2(d_s+d_d)}{\lambda d_s d_d}\right]^{1/2} \text{ and } v \equiv y\left[\frac{2(d_s+d_d)}{\lambda d_s d_d}\right]^{1/2}$$

$$\frac{1}{d} = \frac{1}{d_s} + \frac{1}{d_d} \qquad u = x\left[\frac{2}{\lambda d}\right]^{1/2}$$

$$E \propto \frac{1}{r_s r_d}\int_S e^{ik(r_s+r_d)}dS \approx \frac{1}{d_s d_d}\int_{x_1}^{x_2}\int_{y_1}^{y_2} e^{ik(r_s+r_d)}dx\,dy \propto \int_{u_1}^{u_2} e^{i\pi u^2/2}du \int_{v_1}^{v_2} e^{i\pi v^2/2}dv$$

Fresnel Diffraction

Occurs when $w^2/d\lambda >> 1$
Near Field Effect ⇔
Wavefront curvature
is important.



Intensity (2 dimensions)

Either source or
Detector in near field

$$I \propto \left| \int_{u_1}^{u_2} e^{i\pi u^2/2} du \right|^2 \left| \int_{v_1}^{v_2} e^{i\pi v^2/2} dv \right|^2$$

$$u \equiv x \left[\frac{2(d_s + d_d)}{\lambda d_s d_d} \right]^{1/2} \text{and} \quad v \equiv y \left[\frac{2(d_s + d_d)}{\lambda d_s d_d} \right]^{1/2}$$

One Dimension:

$$\frac{1}{d} = \frac{1}{d_s} + \frac{1}{d_d}$$

$$I \propto \left| \int_{u_1}^{u_2} e^{i\pi u^2/2} du \right|^2, \quad u \equiv x \left[\frac{2(d_s + d_d)}{\lambda d_s d_d} \right]^{1/2}$$

$$u = x \left[\frac{2}{\lambda d} \right]^{1/2}$$

NOTE: This expression is for an *observation point* at $x' = 0$: the origin of the *observer's* coordinate system.

Fresnel Integrals

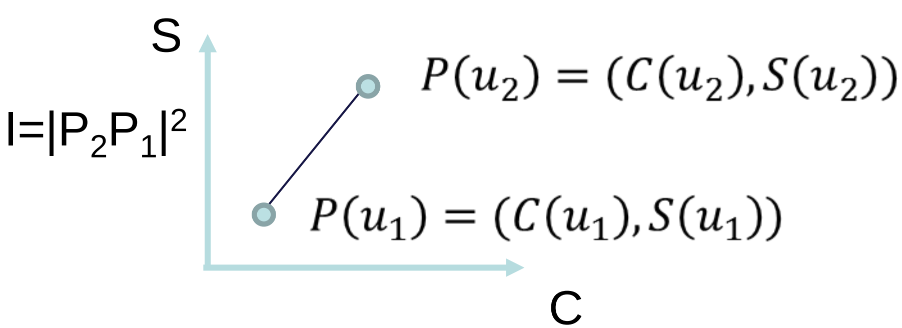
$$\textbf{FRESNEL COSINE INTEGRAL} \equiv C(u) = \int_0^u \cos(\pi u'^2/2)du'$$

$$\textbf{FRESNEL SINE INTEGRAL} \equiv S(u) = \int_0^u \sin(\pi u'^2/2)du'$$

$$\textbf{Thus,} \quad \int_0^u e^{i\pi u'^2/2}du' = \int_0^u \cos(\pi u'^2/2)du' + i\int_0^u \sin(\pi u'^2/2)du$$
$$= C(u) + iS(u)$$

$$I \propto \left|\int_{u_1}^{u_2} e^{i\pi u^2/2}du\right|^2 = \left|\int_{u_1}^{0} e^{i\pi u^2/2}du + \int_0^{u_2} e^{i\pi u^2/2}du\right|^2$$
$$= \left|-\int_0^{u_1} e^{i\pi u^2/2}du + \int_0^{u_2} e^{i\pi u^2/2}du\right|^2$$
$$= |C(u_2) - C(u_1) + i(S(u_2) - S(u_1))|^2$$
$$= [C(u_2) - C(u_1)]^2 + [S(u_2) - S(u_1)]^2$$

$$\textbf{Thus,} \quad I \propto [C(u_2) - C(u_1)]^2 + [S(u_2) - S(u_1)]^2$$



The Cornu Spiral

CORNU SPIRAL: graphic representation of $C(u)+iS(u)$ Properties of Cornu Spiral:

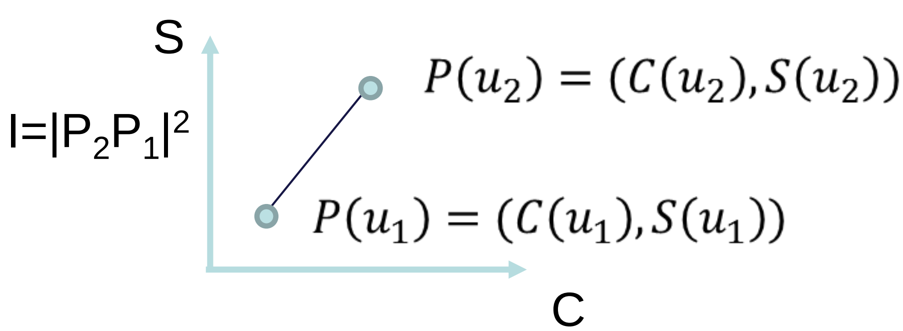
Once width is fixed, the length of the curve is fixed

1. **Length along curve** is proportional to **slit width**:

$$(dl)^2 = (dC(u))^2+(dS(u))^2 = \cos^2(\pi u'^2/2)(dw)^2+\sin^2(\pi u'^2/2)(dw)^2 = (dw)^2$$

2. **Intensity** is proportional to *square* of length of *chord* between end-points of slit.

3. **Move Observation Point** by moving **end points** of chord in opposite direction.

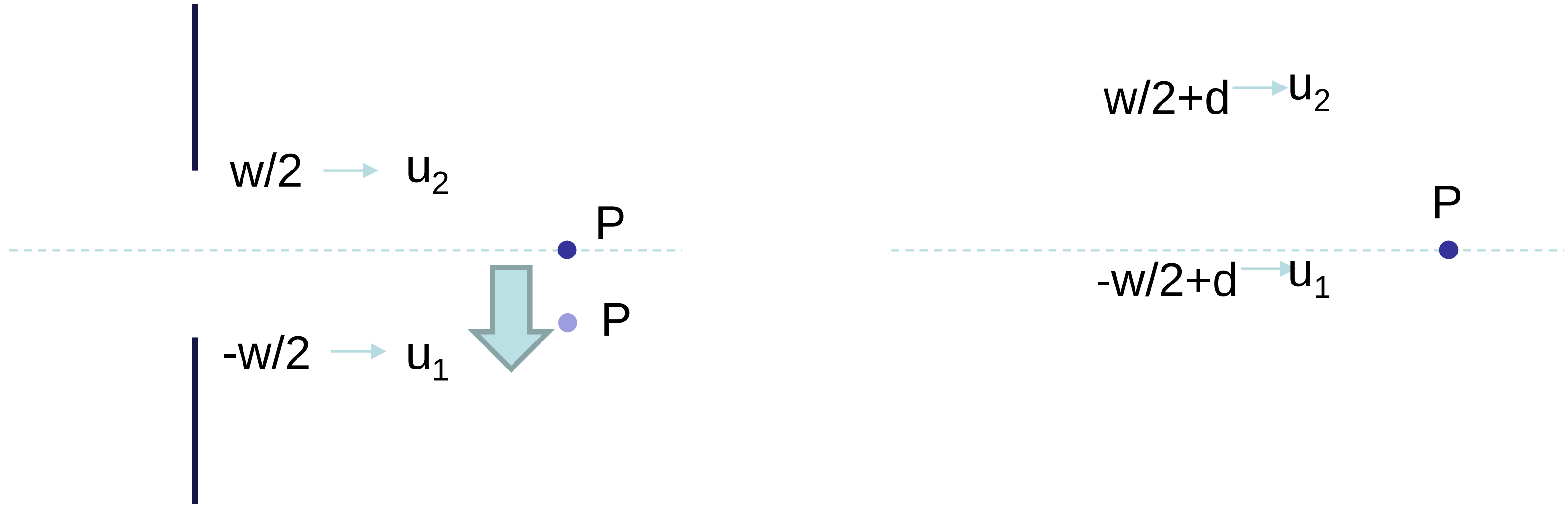


Fresnel Diffraction

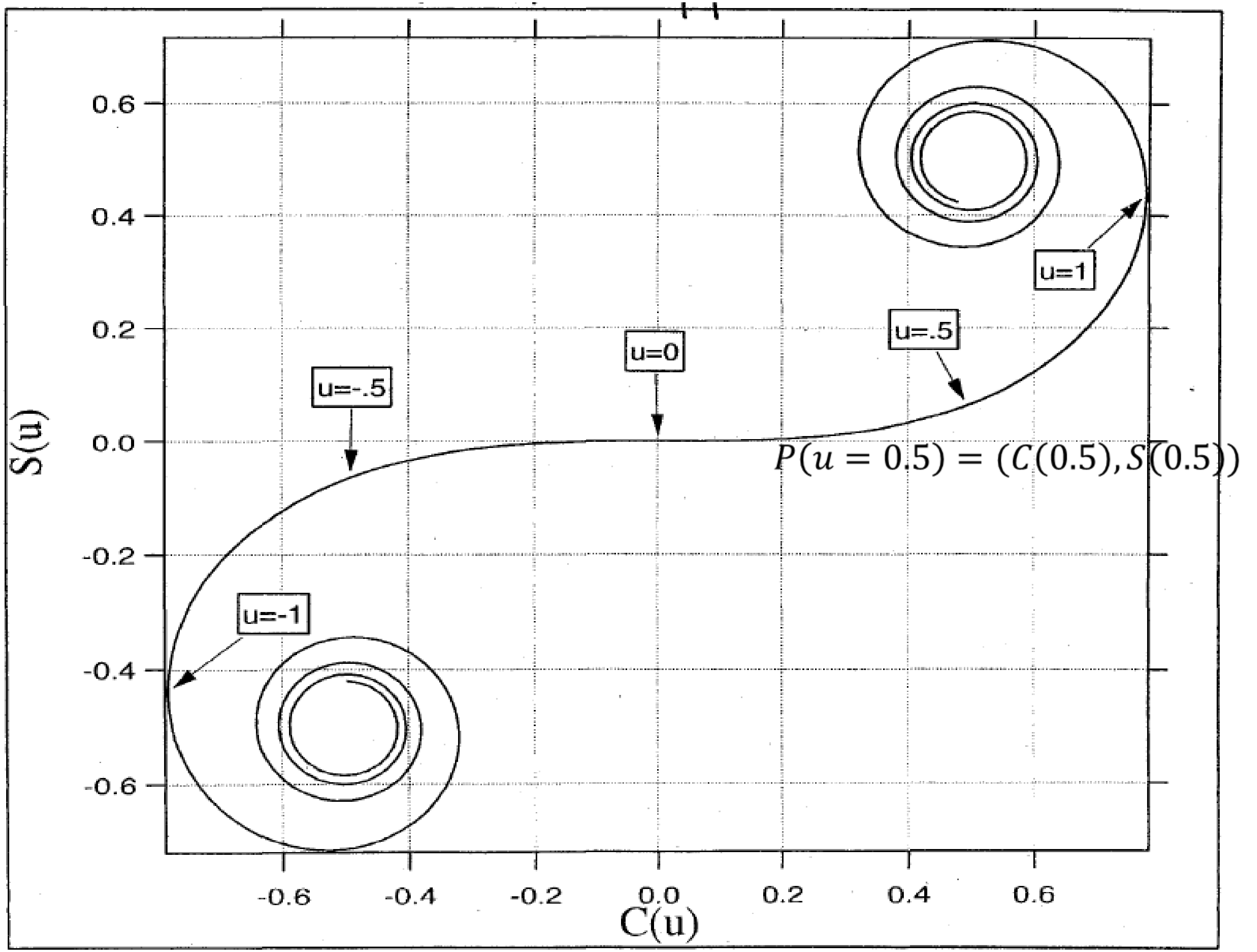
3. **Move Observation Point** by moving **end points** of chord in opposite direction.

$$\textbf{Thus,}\quad I \propto [C(u_2) - C(u_1)]^2 + [S(u_2) - S(u_1)]^2$$

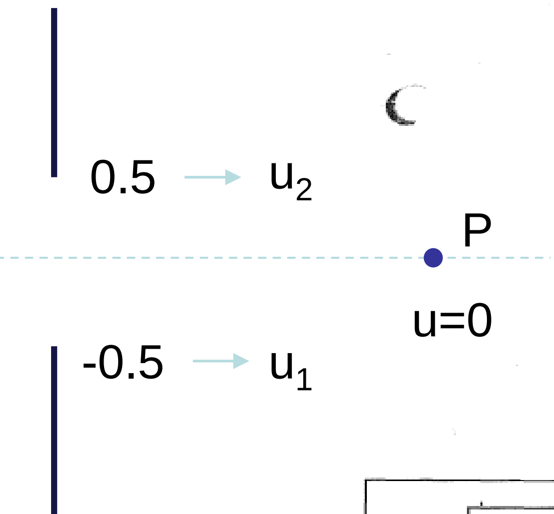
All the equations are derived for X' = 0.

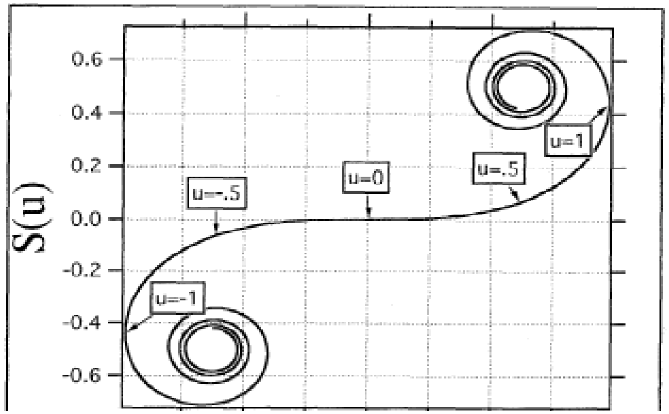


The Cornu Spiral



1. **Length along curve** is proportional to **slit width**:





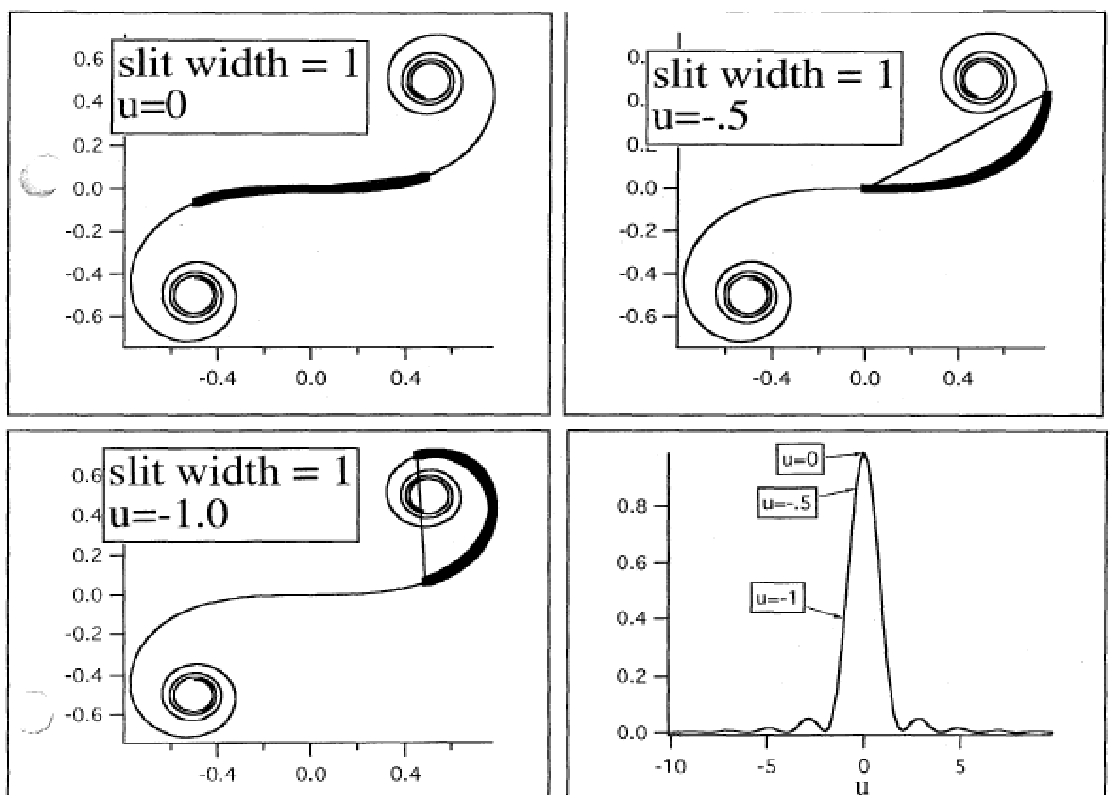
Normalized slit width $= w\left(\frac{2}{d\lambda}\right)^{1/2}$

where $\frac{1}{d} = \frac{1}{d_s} + \frac{1}{d_d}$

source ... tor

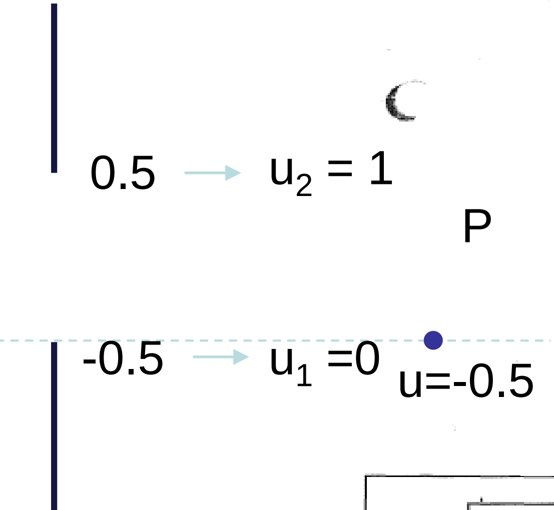
$$\frac{1}{d} = \frac{1}{d_s} + \frac{1}{d_d}$$

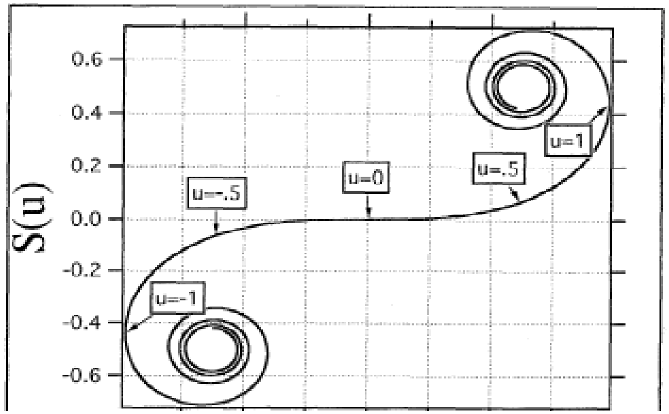
$$u = x\left[\frac{2}{\lambda d}\right]^{1/2}$$



Fraunhofer Diffraction on Cornu Spiral

1. **Length along curve** is proportional to **slit width**:





Normalized slit width $= w\left(\frac{2}{d\lambda}\right)^{1/2}$

where $\frac{1}{d} = \frac{1}{d_s} + \frac{1}{d_d}$

source, detector

$$\frac{1}{d} = \frac{1}{d_s} + \frac{1}{d_d}$$

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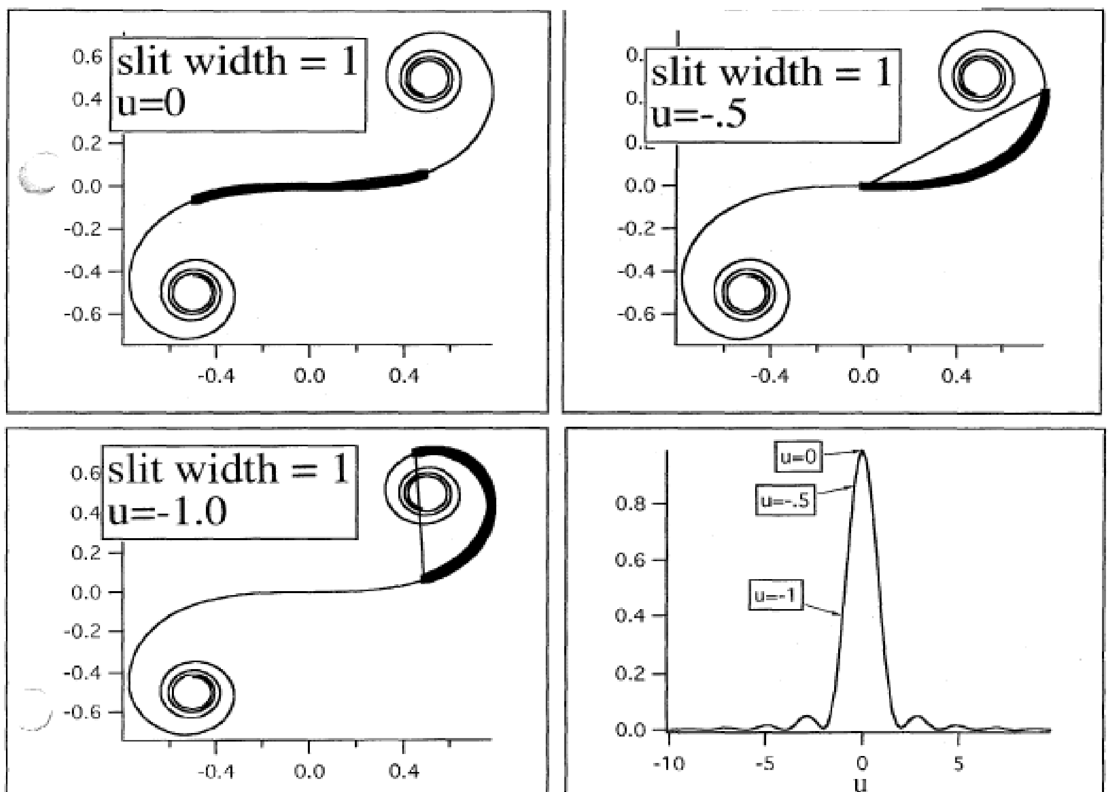

Fraunhofer Diffraction on Cornu Spiral

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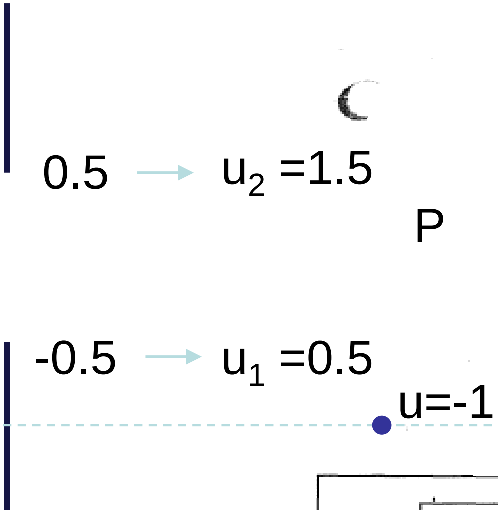
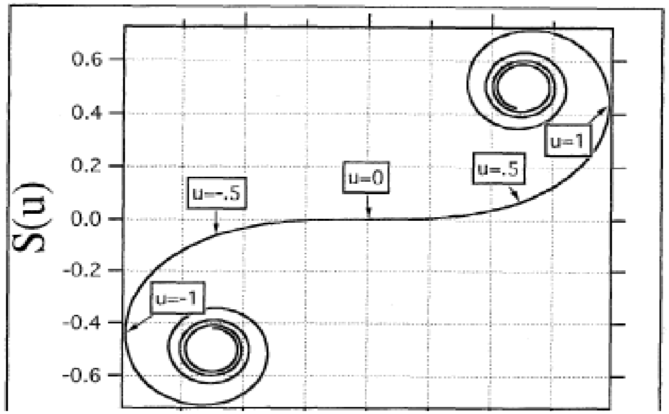




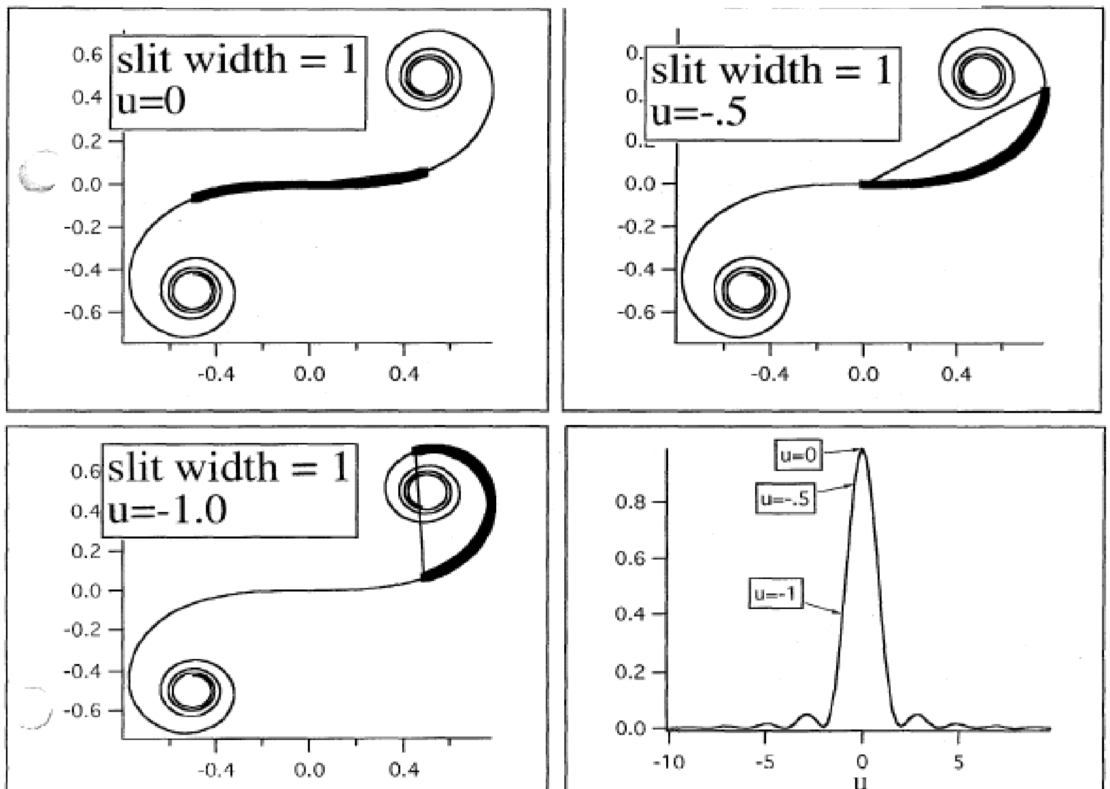
Normalized slit width $= w\left(\frac{2}{d\lambda}\right)^{1/2}$

where $\frac{1}{d} = \frac{1}{d_s} + \frac{1}{d_d}$

source, detector

$$\frac{1}{d} = \frac{1}{d_s} + \frac{1}{d_d}$$

$$u = x\left[\frac{2}{\lambda d}\right]^{1/2}$$



Fraunhofer Diffraction on Cornu Spiral

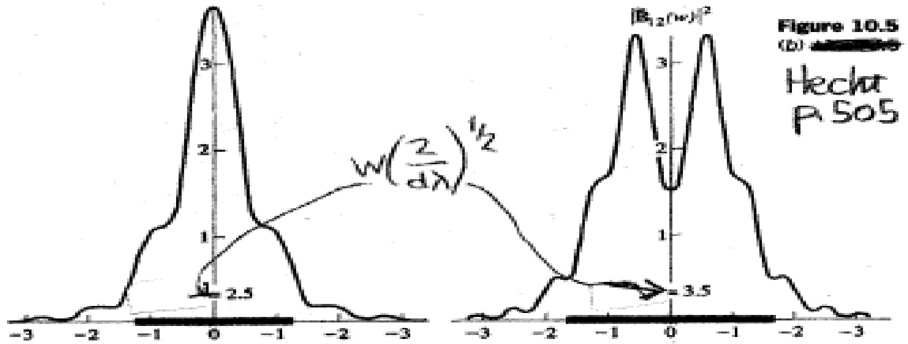


Figure 10.5
(b) [redacted]

Hecht p.505

horizontal axis $u = x\left(\frac{2}{d\lambda}\right)^{1/2}$

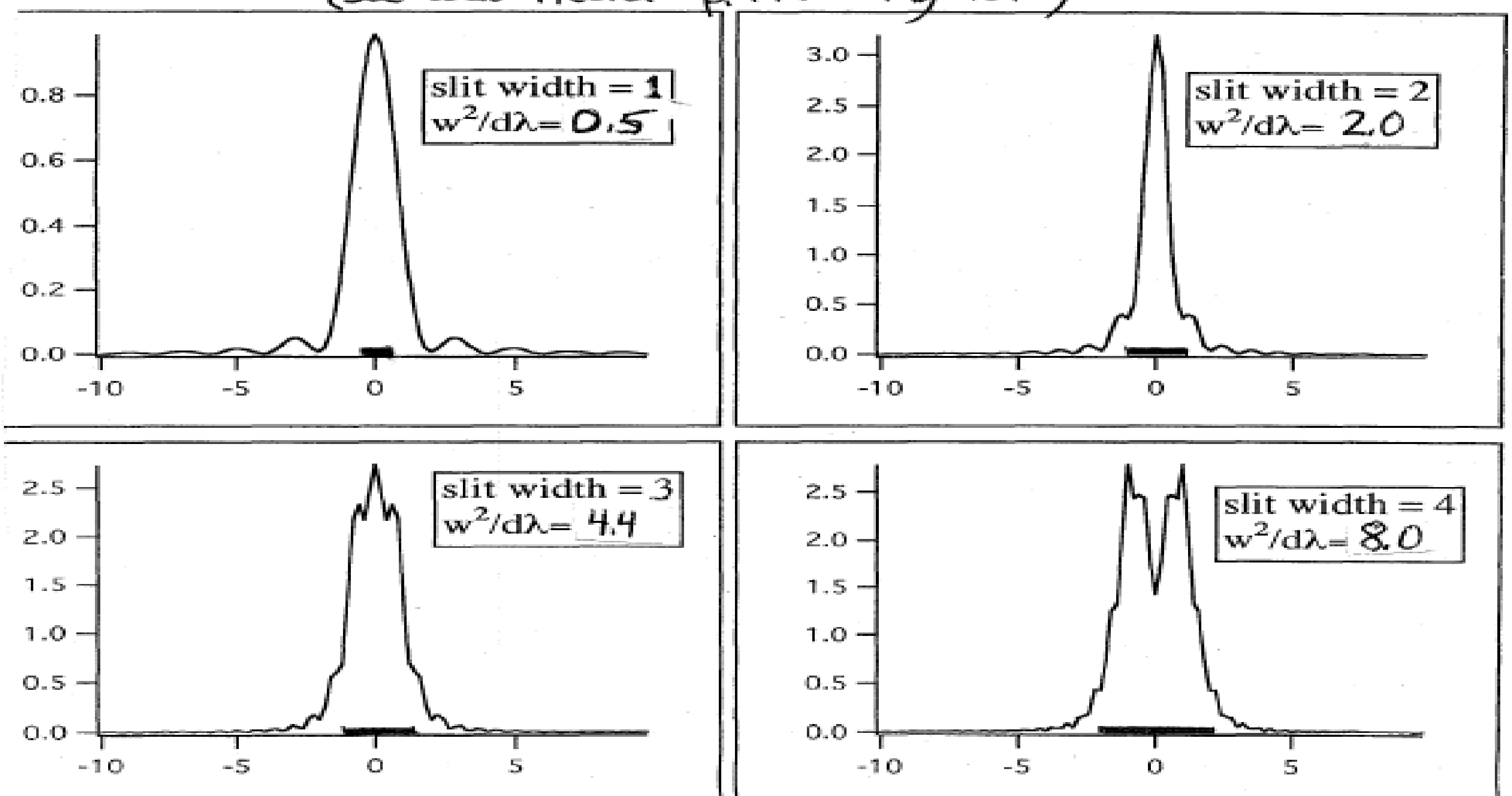
normalized slit width $= w\left(\frac{2}{d\lambda}\right)^{1/2}$

where $\frac{1}{d} = \frac{1}{d_s} + \frac{1}{d_d}$

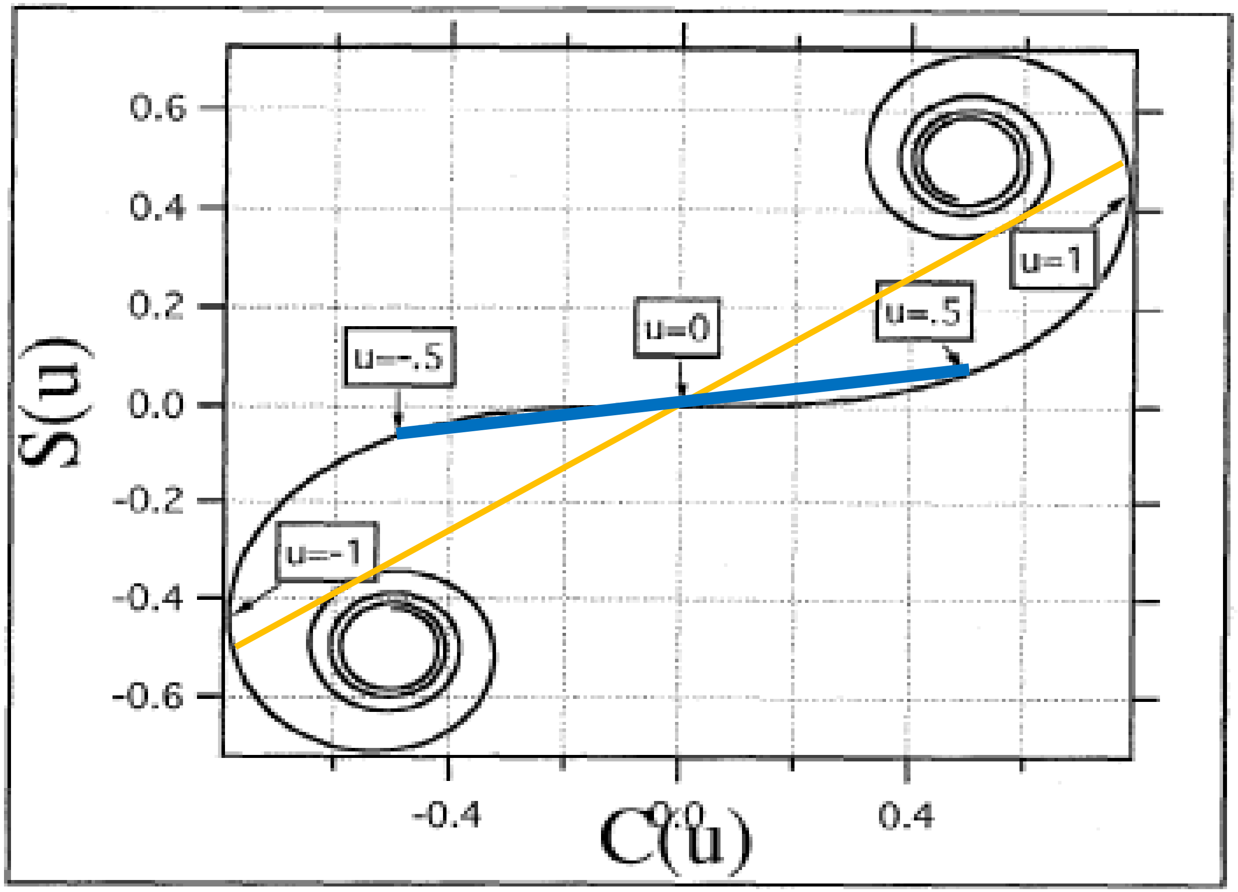
source → d_s, detector → d_d

Below, we show the progression from w= 1(Fraunhofer) to w=4 (Fresnel). (horizontal bar is geometric shadow of slit)

(see also Hecht p.447 Fig 10.2)



p.10



Normalized slit width $= w\left(\frac{2}{d\lambda}\right)^{1/2}$

where $\frac{1}{d} = \frac{1}{d_s} + \frac{1}{d_d}$

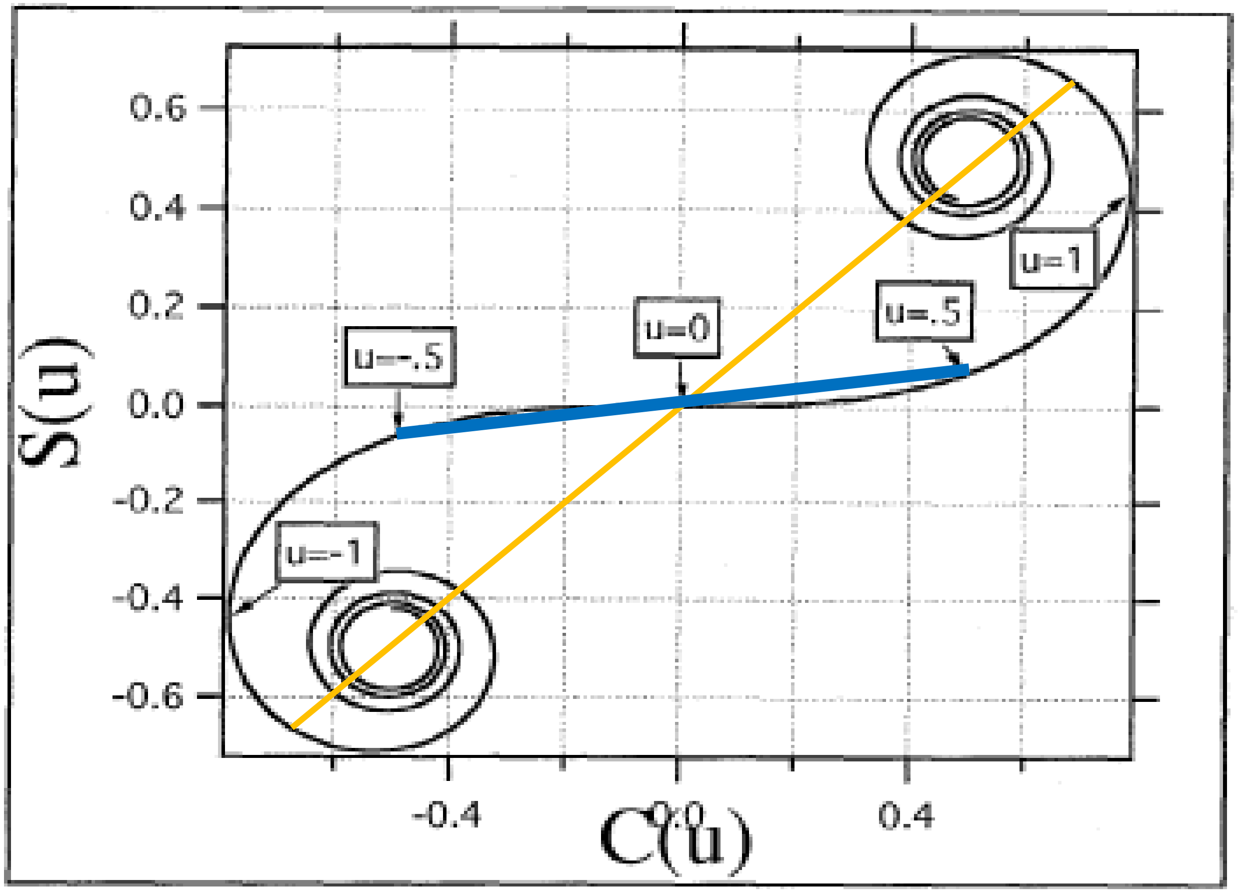
source

detector

Detector is at the u=0.

Start with slit width =1. We continuously increase the slit width.

p.10



Normalized slit width $= w\left(\frac{2}{d\lambda}\right)^{1/2}$

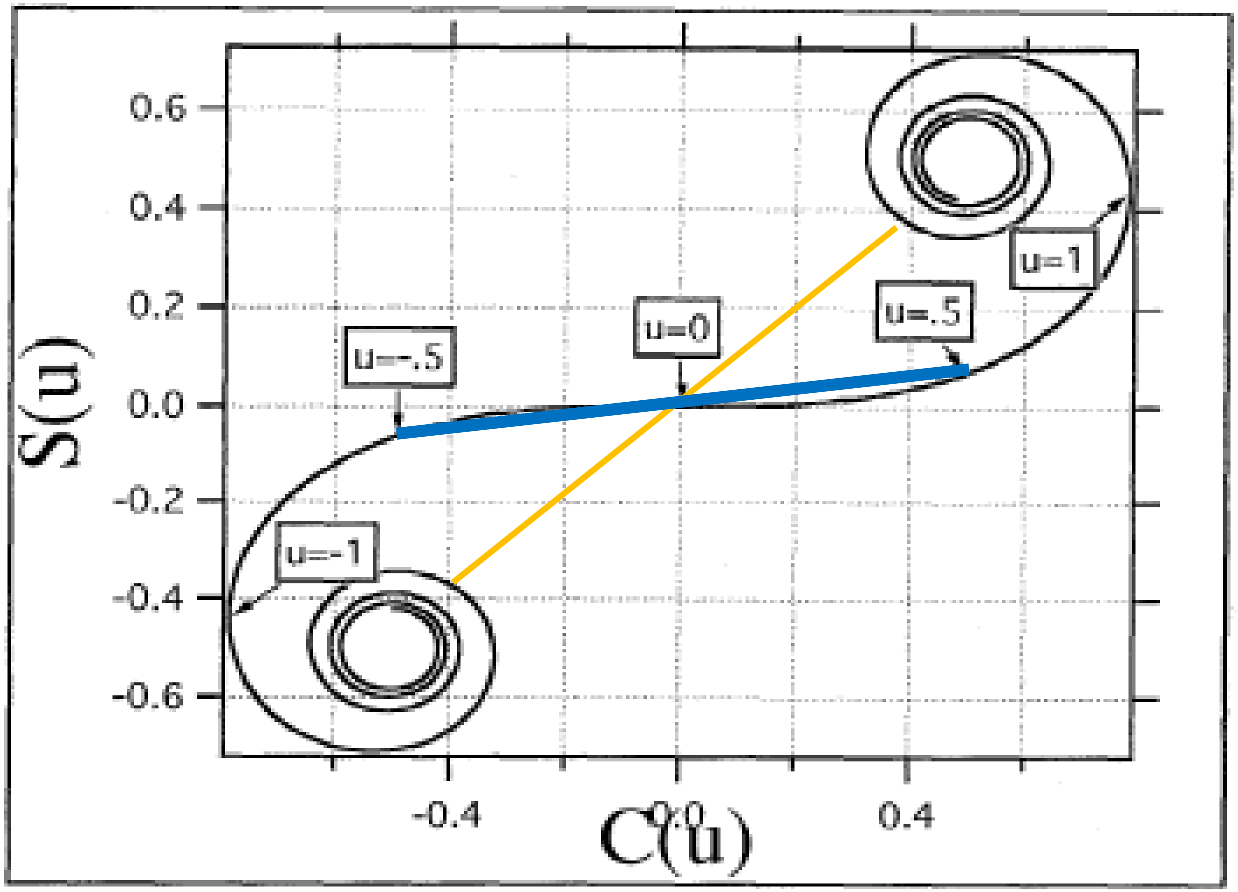
where $\frac{1}{d} = \frac{1}{d_s} + \frac{1}{d_d}$

source (d_s), detector (d_d)

Detector is at the u=0.

Start with slit width =1. We continuously increase the slit width.

p.10



Normalized slit width $= w\left(\frac{2}{d \cdot \lambda}\right)^{1/2}$

where $\frac{1}{d} = \frac{1}{d_s} + \frac{1}{d_d}$

source

detector

Detector is at the u=0.

Start with normalized slit width =1. We continuously increase the slit width.