Phys. 334 **Day 2 Lecture Notes**

Capacitors and AC (Alternating Current) Circuits:

Capacitors, (symbol ) are as common and useful as resistors, and are found in almost all circuits that have time varying voltages and currents. We will see that capacitors can be thought of as frequency dependent resistors.

Capacitors are made of 2 conducting plates separated by an insulator, with a wire attached to each plate for electrical connection. The plates store electrical charge.

By definition, the charge, Q, stored on a capacitor is given by:

$Q = CV$ where C is the capacitance in Farads (pF, nF, μF are common).

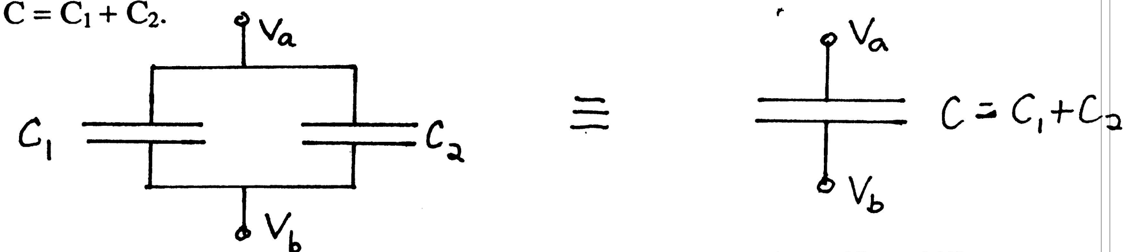
Note **V is the voltage difference** between the two plates of the capacitor and there is +Q charge stored one one plate and -Q stored on the other.

A 1 Farad capacitor with 1 Volt across it stores 1 Coulomb of charge. Because in electronics we are interested in the current, $I = dQ/dt$, it follows that:

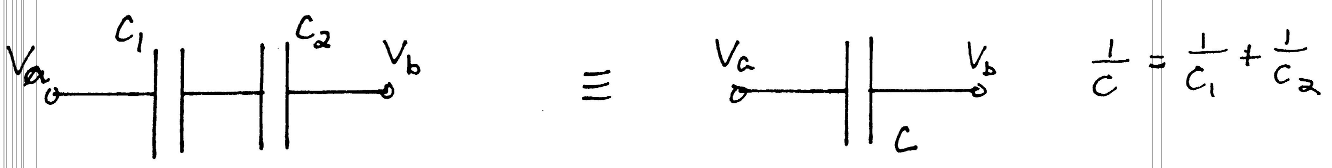
$I = dQ/dt = C\, dV/dt$: C = 1 Farad and dV/dt = 1 Volt/s gives I = 1 Amp

Note: $I \neq 0$ only when V is changing with time, so for static (constant voltage or DC) circuits, a capacitor acts like an open circuit.

For Capacitors in parallel: we have $Q_1 = C_1(V_a - V_b)$ and $Q_2 = C_2(V_a - V_b)$ so that the total charge stored, $Q = Q_1 + Q_2 = (C_1 + C_2)(V_a - V_b)$ which says the total capacitance is $C = C_1 + C_2$.



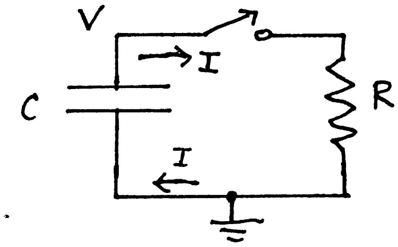
For Capacitors in series: you can similarly show that $1/C = 1/C_1 + 1/C_2$.



Note: this is opposite for what we had for resistors in series and parallel. This is because for V = IR and V = Q/C, R is like 1/C.

Time Varying Signals: We will think of most time varying signals as sinusoidal, ie $\propto \sin(\omega t)$. This is because many times the signals are actually sine waves and even when they are not, Fourier analysis tells us that we can represent any function of time as a sum of sine waves: if we understand the behavior of sine waves, we can construct any time varying signal. Note that for $\sin(\omega t)$, only the product ωt appears: we can think of either ω or t as the variable, that is, work in frequency space (variable ω, $V(\omega)$, $I(\omega)$) or time space (variable t, $V(t)$, $I(t)$). Both approaches are useful. We will begin by considering time as the variable.

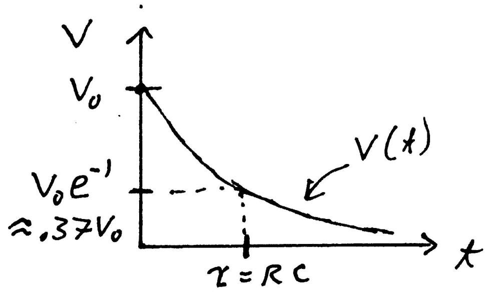
A. Consider:



We start with a charge on the capacitor, $Q_0 = CV_0$ and then close the switch at time t = 0. Once we choose the direction of I as shown, we have:
$V = IR$ and $I = -C\, dV/dt$, with $V(t=0) = V_0$

Combining the 2 equations: $dV/dt = -V(t)/RC$
Which is the differential equation of the exponential function:

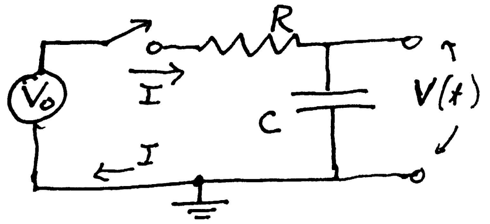
<u>$V(t) = V_0 \exp(-t/RC)$</u>



So a charged capacitor will discharge through a resistor exponentially with a time constant $\tau = RC$

$R(\Omega) \times C(\text{Farads}) = \tau(\text{seconds})$

B. Now consider:

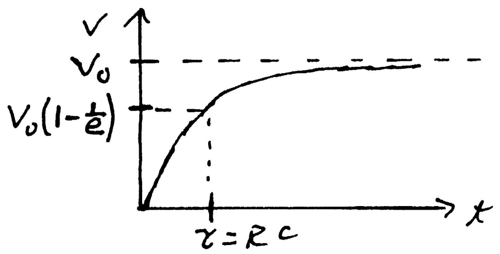


We start with 0 charge on the capacitor (V = 0) and at time t = 0, close the switch. Now,

$I = (V_0 - V)/R = C\, dV/dt$, or $dV/dt = -(V - V_0)/RC$

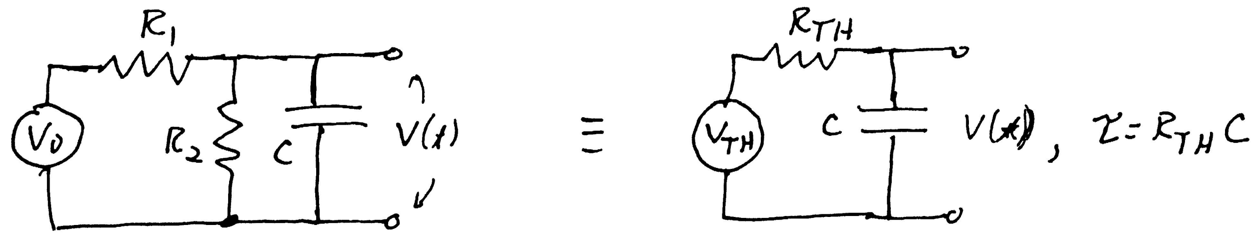
with $V(t = 0) = 0$.

The solution is: <u>$V(t) = V_0(1 - \exp(-t/RC))$</u>

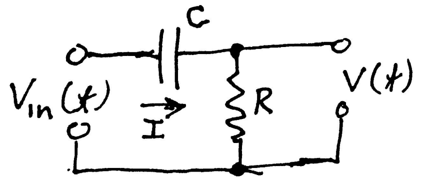


So here, the capacitor charges up exponentially with the same time constant $\tau = RC$.

Any time resistors and capacitors are together in a circuit, their combined effect is to limit the rate at which V's and I's can change to exponential functions with time constant $\tau = RC$. For more complicated circuits, such as the one shown below, you can use use the Thevenin equivalent circuit to find the appropriate R to use with one of our solutions to A. or B. above.



Differentiator:

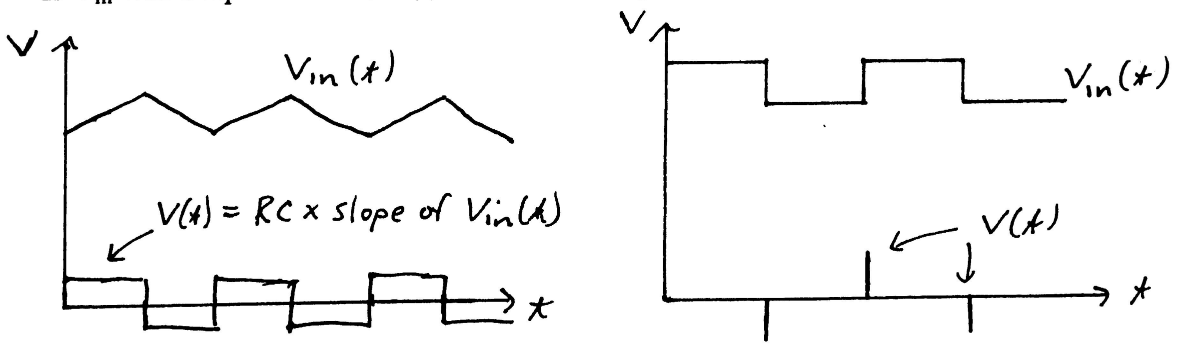


$I = C\, d(V_{in} - V)/dt = V/R$ or $dV_{in}/dt = V/RC + dV/dt$

If we choose RC sufficiently small, or equivalently ensure that $V << V_{in}$, then $dV/dt << V/RC$ and we can neglect dV/dt relative to V/RC and find:

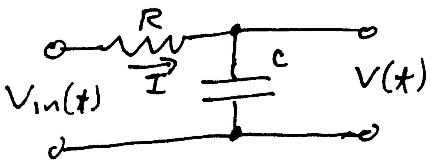
$dV_{in}/dt \approx V/RC$ or $\underline{V(t) \approx RC\, dV_{in}/dt}$

That is, the output signal, V(t), is proportional to the time derivative of the input signal, V_{in}.
For example, if $V_{in}(t)$ is a triangle wave, V(t) would be a square wave.
If V_{in} was a square wave, V(t) would be a sequence of spikes.



Note, if $V_{in}(t) = \sin(\omega t)$, then $V(t) = \omega RC \cos(\omega t)$ and the condition that $V << V_{in}$ becomes $\omega RC << 1$.

Integrator:



If we interchange the position of the R and C, we have:

$I = C\, dV/dt = (V_{in} - V)/R$ or $dV/dt = (V_{in} - V)/RC$

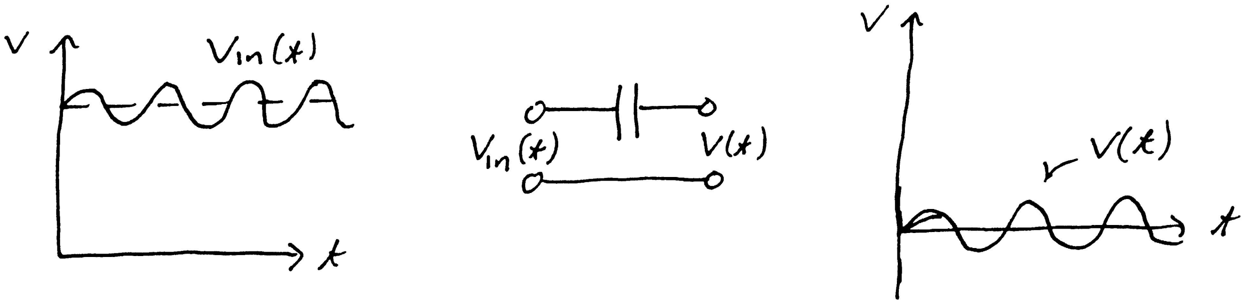
If we choose RC large enough, dV/dt will be small and V will remain $<< V_{in}$, ie $(V_{in} - V) \approx V_{in}$.

Then, $dV/dt \approx V_{in}/RC$ or $\underline{V(t) = \frac{1}{RC}\int_0^t V_{in}(t)dt + k}$ where k is a constant.

That is, V(t) is the time integral of $V_{in}(t)$.
For example, if $V_{in}(t)$ is a square wave, V(t) will be a triangle wave.

If $V_{in}(t) = \sin(\omega t)$, then $V(t) = -\cos(\omega t)/\omega RC$ and the condition that $V << V_{in}$ becomes $\omega RC >> 1$.

Decoupling AC from DC: Two parts of a circuit often need to operate at different DC voltages, but need to be connected for AC. Because a capacitor is open circuit for DC currents, it can be used to 'decouple' DC from AC:



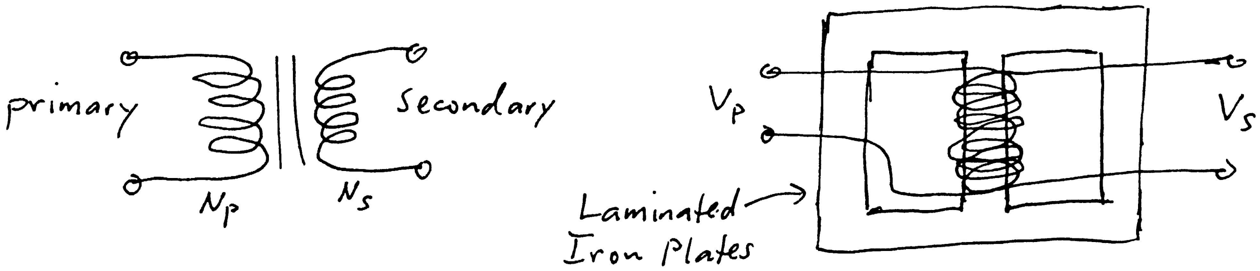
INDUCTORS: (Symbol ) Inductors are typically tightly would coils of wire (often with an iron core). They store magnetic energy (capacitors store electric energy). Inductors are not nearly as common as capacitors and, in fact, are rarely found outside of RF (radio-frequency) circuits.

Inductance, L, is defined by: $\phi = L \times I$, where ϕ is magnetic flux through a loop of wire and I is the current flowing in the wire. Faraday's Law tells us that $d\phi/dt = EMF = V$, which leads to:

$$\underline{V = L\, dI/dt} \qquad \text{L in Henry's}$$

Again, V refers to the potential difference (EMF) generated between the two ends of the inductor. $V \neq 0$ only if the current is changing (an inductor acts like a short circuit or piece of wire for constant currents). Note, inductors are in some sense the opposite of capacitors, where $I = C\, dV/dt$.

Tranformers: The magnetic field produced by one coil can be arranged to change the magnetic flux through a second coil, producing a voltage (EMF) across the second coil. The first coil, called the primary has N_p loops of wire, while the second, called the secondary has N_s loops. If the primary and secondary coils physically overlap (as the usually do), you can show that $V_s/V_p = N_s/N_p$ where V_s and V_p are the voltages (AC) across the secondary and primary coils, respectively. Tranformers are used to both step up ($N_s > N_p$) AC voltages and step them down ($N_s < N_p$). We won't be doing much with transformers but you should know that they are an essential component of almost everything that plugs into a wall plug, where the 120 V (60 Hz) source is usually stepped down to a safer voltage.



IMPEDENCE and AC CIRCUIT ANALYSIS: Electronics is most useful with time varying signals. We can solve the differential equations for the circuits as we have done above, but this is cumbersome and often not very illuminating. A better approach is to analyze the circuit for a sinusoidal voltage or current at a single frequency. Then, Fourier's Theorem allows us to reconstruct any periodic time varying signal:

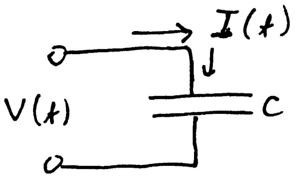
$$F(t) = a_0/2 + \Sigma_k\, [a_k \cos(\omega t) + b_k \sin(\omega t)]$$

Where the coefficients a_k and b_k can be determined, given F(t).

Note: ω = angular frequency (in radians/sec) = $2\pi f$
And f = frequency (in Hz or cycles/sec)

The concept of 'impedence' is very important. It gives us a type of Ohm's Law for capacitors and inductors for signals at a single frequency. Impedence, as we'll see, is a generalized resistance (a resistance that is a complex number).

Consider:



Let $V(t) = V \sin(\omega t)$

Then $I(t) = C\, dV/dt = \omega CV \cos(\omega t) = \omega CV \sin(\omega t + 90^0)$

That is, the current is also sinusiodal at the same frequency but is shifted in phase by 90^0.

If we consider the amplitudes of the voltage and current alone, $|\sin(\omega t)| = |\cos(\omega t)| = 1$

And $|I| = \omega C\, |V|$ or $|V| = |I|/\omega C$, that is, the capacitor acts like a resistor whose resistance is $1/\omega C$ (a frequency dependent resistor). But there was also a phase change by 90^0 between the current and voltage that we lost by considering the amplitudes alone. In general, we need to keep track of the amplitude and phase of the signals. To do this, we will use complex numbers, which have both an amplitude and phase.

1. We start with the fact that $\exp(j\omega t) = \cos(\omega t) + j\sin(\omega t)$ where $j^2 = -1$ (we use j instead of i so we don't confuse i with a current).
   And we recall that any complex number $z = a + jb$, where a and b are real numbers.

2. We now allow all the V's and I's in a circuit to be represented by complex numbers. (For now, we will represent the complex signal with a subscript, c.)

   $$V_c(t) = V_c \exp(j\omega t) = (V_{real} + jV_{imag}) \exp(j\omega t)$$

   $$I_c(t) = I_c \exp(j\omega t) = (I_{real} + jI_{imag}) \exp(j\omega t)$$

3. Physical voltages and currents are understood to be the real part of the associated complex number: $Re[a + jb] = a$. For example,

   $$V(t) = Re[V_c(t)] = Re[(V_R + jV_I)(\cos(\omega t) + j\sin(\omega t))] = V_R\cos(\omega t) - V_I\sin(\omega t)$$

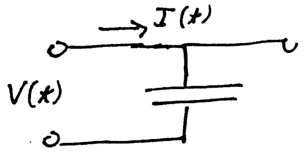
(recall that $z_1 z_2 = (a_1 + jb_1)(a_2 + jb_2) = (a_1 a_2 - b_1 b_2) + j(a_1 b_2 + a_2 b_1)$)

4. We must remember that this description applies to sinusoidal signals at a single frequency. It will allow us to analyze circuits with resistors, capacitors, and inductors. Note all of the time dependence of $V_c(t)$ and $I_c(t)$ is in the $\exp(j\omega t)$ term.

5. **Define Impedance = Z as $Z = V_c(t)/I_c(t)$** (like $R = V/I$)

Note Z is a complex number (we leave off the subscript c) and we see that Z is a complex number resistance.

Let's return to our capacitor example: Now we let $V(t) = \mathrm{Re}[V_c(t)] = \mathrm{Re}[V_c \exp(j\omega t)]$



Then, $I(t) = \mathrm{Re}[I_c(t)] = C\, dV/dt = C\, \mathrm{Re}[j\omega V_c \exp(j\omega t)]$

$Z = V_c(t)/I_c(t) = (V_c \exp(j\omega t))/(Cj\omega V_c \exp(j\omega t)) = 1/j\omega C$

Note, $Z = 1/j\omega C$ is the same result we obtained when we considered the amplitudes of V and I alone, except for the factor of j which represents the 90^0 phase change between V and I.

RESULTS WITHOUT PROOF:

Kirchoff's laws were based upon the conservation and energy and charge alone. They and all of the results obtained from them are valid for complex numbers as well as real numbers. In particular:

1. $\Sigma\, I_c(t)$ into a point = 0

2. $\Sigma\, V_c(t)$ around a closed loop = 0

3. For impedances (Z's) in series: $Z_T = Z_1 + Z_2 + \ldots Z_N$

4. For impedances in parallel: $1/Z_T = 1/Z_1 + 1/Z_2 + \ldots 1/Z_N$

5. For Resistors: $Z = R$

   For Capacitors: $Z = 1/j\omega C$

   For Inductors: $Z = j\omega L$

6. All of the results we have obtained for resistors (voltage dividers, dynamic resistance) apply for capacitors and inductors if we replace the resistance by the impedance.

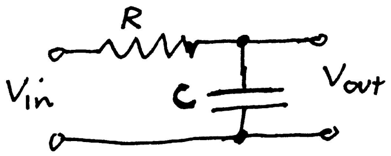
7. The power consumed in a circuit (now AC) is $P = \mathrm{Re}[V_c^* I_c] = [V_c I_c^*]$
   Where * means complex conjugation: if $z = a + jb$, $z^* = a - jb$.

8. At this point, we drop the subscript c from our complex voltages and currents and understand that when we use impedance, we are referring to situations where the signals are sinusoidal and voltages and currents can be complex numbers.

RC FILTERS:

By combining R's and C's (or L's), it is possible to make frequency dependent voltage dividers (which are called filters, because they filter out certain frequencies).

Consider:



$I = V_{in}/Z_{Total} = V_{in}/(R + 1/j\omega C) = j\omega C V_{in}/(1 + j\omega RC)$

And $V_{out} = IZ_{cap} = I/j\omega C = V_{in}/(1 + j\omega RC)$

Or: $V_{out}/V_{in} = 1/(1 + j\omega RC) = (1 - j\omega RC)/(1 + \omega^2R^2C^2)$

So we see that the output voltage is reduced in amplitude (in a frequency dependent way) from the input voltage and shifted in phase.

Often, we are interested in the relative amplitudes of the input and output alone. The amplitude of a complex number is defined as: $|z| = [z\, z^*]^{1/2}$

For the circuit above, $|V_{out}/V_{in}| = [1/(1 + j\omega RC) \times 1/(1 - j\omega RC)]^{1/2} = 1/[1 + \omega^2R^2C^2]^{1/2}$

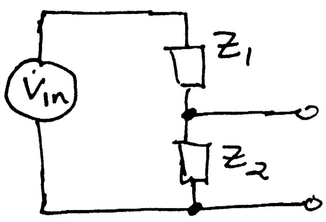
Note: For $\omega RC << 1$, $|V_{out}/V_{in}| \approx 1$, while for $\omega RC >> 1$, $|V_{out}/V_{in}| \approx 1/\omega RC$,

That is, for small frequencies, the output is equal to the input (the signal is passed on) while for high frequencies, the output signal is attenuated in proportion to $1/\omega$.

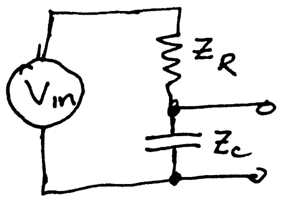
The above circuit is called a 'Low Pass Filter' because it passes low frequencies and attenuates high frequencies. The frequency at which the filter changes from transmission to attenuation is when $\omega RC = 1$, or $\omega = 1/RC$. (Note, this is the angular frequency.)

This frequency is called the 'corner frequency' or 'knee' of the filter, but is most often called the '3db point' of the filter: 3db = 3 decibels $\approx \sqrt{2}$ change in amplitude, and when $\omega = 1/RC$, $|V_{out}/V_{in}| = 1/\sqrt{2}$, ie the output is attenuated by $\sqrt{2}$.

We could have calculated the above result much more simply by thinking of the capacitor as a resistor with a complex resistance. Consider a voltage divider where we replace the resistors by impedances (complex resistors). In the circuit below, the Z's can be resistors, Capacitors, or inductors. We get our original voltage divider result with Z's replacing the R's (because the result came from current conservation and Ohm's Law).

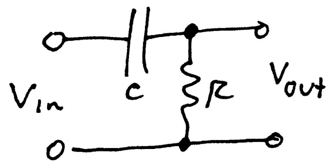


$$V_{out} = \frac{Z_2}{Z_1 + Z_2} V_{in}$$



$$V_{out} = \frac{Z_c}{Z_c + Z_R} V_{in} = \frac{1/j\omega C}{1/j\omega C + R} V_{in} = \frac{1}{1 + j\omega RC} V_{in}$$

We now use this simpler approach for the circuit where we exchange the position of the resistor and capacitor:



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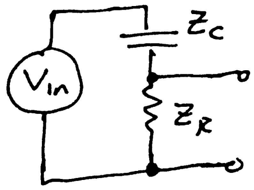


$$V_{out} = \frac{Z_R}{Z_R + Z_c} V_{in} = \frac{R}{R + 1/j\omega C} V_{in}$$

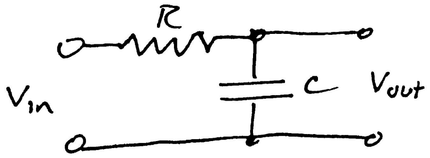
$$\Rightarrow V_{out} = \frac{j\omega RC}{1 + j\omega RC} V_{in} \quad \text{or} \quad \underline{\frac{V_{out}}{V_{in}} = \frac{j\omega RC}{1 + j\omega RC}}$$

For $\omega RC >> 1$, $|V_{out}/V_{in}| \approx 1$ while for $\omega RC << 1$, $|V_{out}/V_{in}| \approx \omega RC << 1$

This circuit is called a 'High Pass Filter" because it passes through high frequencies and attenuates low frequencies. Again the 3db point is when $\omega RC = 1$, or $\omega = 1/RC$.

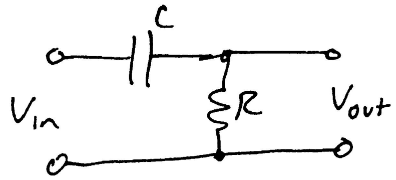
The behavior of these two filters is easily understood if you think of them as voltage dividers and realize that the 'resistance' of the capacitor ($1/j\omega C$) is very large for small frequencies and very small for high frequencies. The voltage drop across the capacitor is determined by how large its impedance is relative to the resistance of the resistor.

Note:



= Low Pass Filter (attenuates for $\omega >> \frac{1}{RC}$)

= Integrator for $\omega >> \frac{1}{RC}$



= High Pass Filter (attenuates for $\omega << \frac{1}{RC}$)

= Differentiator for $\omega << \frac{1}{RC}$

Filter roll off: For a low pass filter when $\omega >> 1/RC$, $|V_{out}/V_{in}| \approx 1/\omega RC$ and when ω doubles, $|V_{out}/V_{in}|$ decreases by a factor of 2 which a decrease of 6db. The filter is said to 'roll off' by '6db per octave' (an octave is a factor of 2 in frequency).

Similarly, for a high pass filter when $\omega << 1/RC$, the filter again rolls off by 6db per octave.

You can make filters that roll off faster than 6 db per octave. For example, you could place a second low pass filter after the first low pass filter (this is called 'cascading' filters) and together they can attenuate a signal by a factor of 4 for a factor of two increase in frequency (the roll off becomes 12 db per octave). You must take care, however, that the second filter does not affect the operation of the first. That is, you must ensure that Z_{in} of the second filter is >> Z_{out} of the first.

BODE PLOTS: The behavior of a circuit as a function of frequency is usually represented in a 'Bode Plot'. This is a graph of $|V_{out}/V_{in}|$ on a log scale vs. the frequency on a log scale. The voltage ratio is expressed in decibels (db):

$$\text{db} = 20 \log_{10}(|V_{out}/V_{in}|)$$

Some common usages: 20 db is a factor of 10
40 db is a factor of 100
60 db is a factor of 1000
6 db is a factor of 2
3 db is a factor of $\sqrt{2} = 1.41$

On a Bode plot, because of the log-log scales, the transition between the different regimes of a filter ($\omega RC < 1$ and $\omega RC > 1$) appears as a 'corner' or 'knee' in the plot. Bode plots for our Low and High Pass filters are shown below.