Phys. 428, Lecture 6

LECTURE	DATE	INSTRUCTOR	TOPIC
1	March 29	PK	Overview: Imaging equation, inverse problem
2	April 5	PK	2D-LSI imaging systems, X-ray physics: formation and interaction
3	April 12	WH	X-ray detection and imaging systems
4	April 19	WH	X-ray computed tomography (CT) systems
5	April 26	WH	X-ray CT part 2. Contrast Agents
6	May 3	PK	Image reconstruction and image quality
7	May 10	PK	Nuclear decay schemes and isotopes
8	May 17	PK	Gamma cameras: components and systems
9	May 24	PK	Tomography in molecular imaging: SPECT scanners
10	May 31	PK	Positron emission tomography (PET) and hybrid PET/CT scanners

Mid-Term & Weekly questions

- Since we are behind in the material, there will be <u>no midterm</u>
- Instead the final project will be subdivided into graded stages

Class Project

- Pick:
  - An imaging modality covered in class (x-ray, mammography, gamma camera imaging, CT, PET, SPECT)
  - A pathology/disease (+ optional linked treatment)
- Answer 4 questions:
  - What is the biology of the imaging?
  - What is the physics of the imaging?
  - What are competing imaging (and non-imaging) methods, and what are the trade-offs?
  - What is the relative cost effectiveness?

- **1+ page outline Tuesday May 10 (20%)**
  (title, imaging modality, disease/treatment, some references relevant to project and each of the 4 questions)
- Background summary Tuesday May 17 (15%)
  (2+ page outline, all references with capsule summaries)
- Rough draft Tuesday May 24 (15%)
  (introduction, starts for all 4 questions, summary discussion)
- Final version Tuesday June 7 (50%)
  - Minimum 10 pages not counting references (11 pt Arial font, 1.5 inch margins, 2x line spacing)
  - Be sure to cite all references, figures, and claims of fact)

Discussion of Questions from Last Lecture

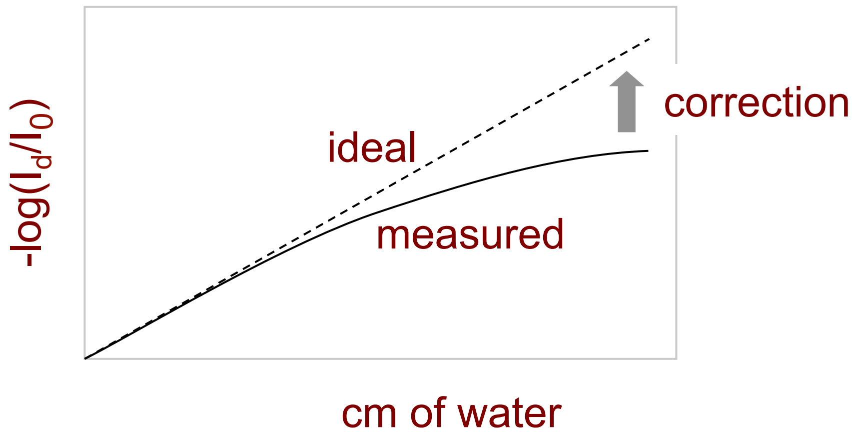
- Are all contrast agents metabolized and excreted in from the body, or do any last in the body for a significant period of time?
  - Currently used iodinated agents are cleared almost completely by glomerular filtration. With reduced renal function, there is vicarious excretion primarily in bile and through the bowel. Circulatory half life is 1–2 hours, assuming normal renal function.
- What are the way(s) to minimize the Beam Hardening effect?

Water based correction

$$I_d = \int_0^{E_{\max}} S_0(E) E e^{-\int_0^d \mu(s,E)ds} dE$$

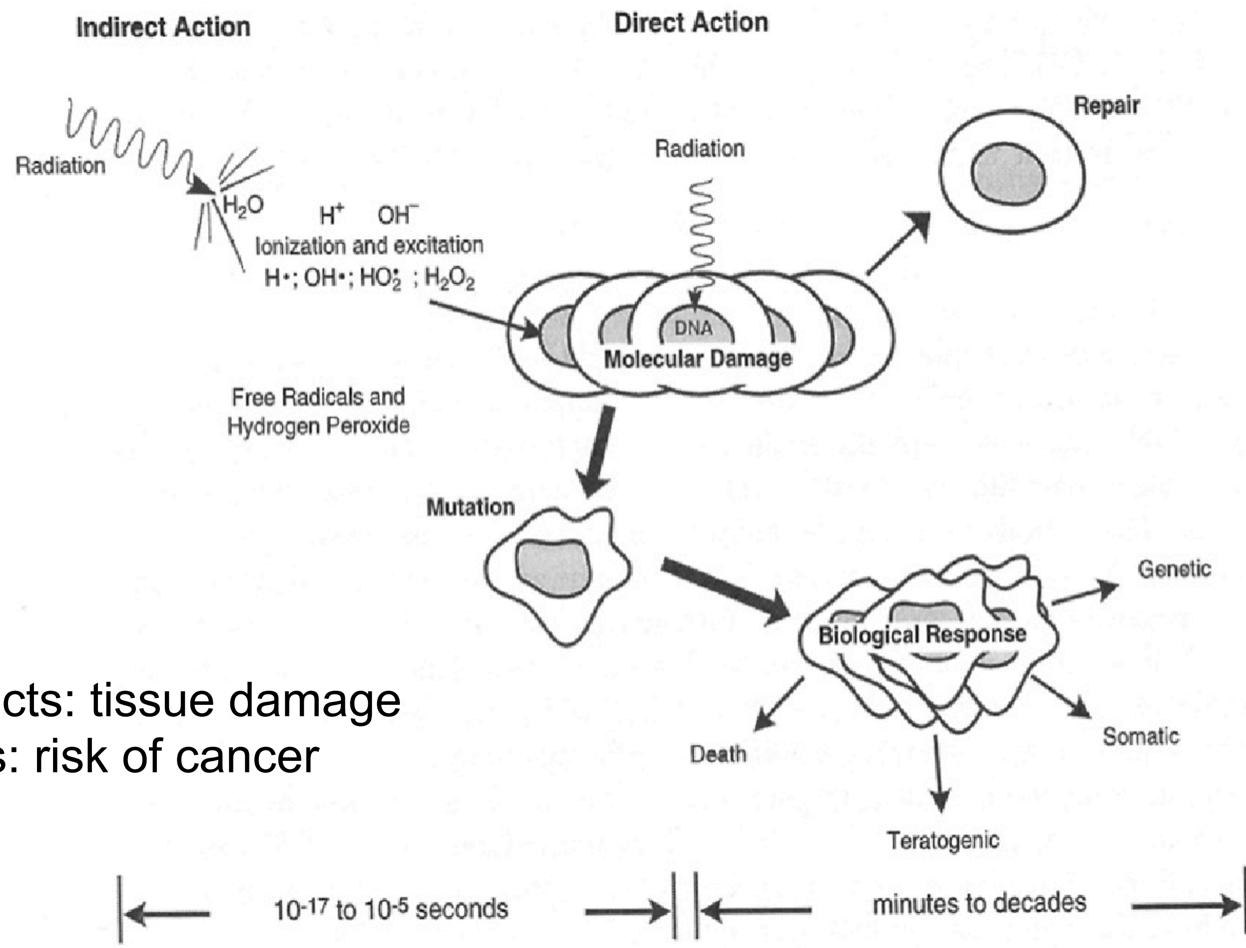
$$\approx I_0 e^{-\mu_W L} \quad \text{(assume all water)}$$

$$\mu_W L = -\log(I_d / I_0)$$



- [what is] the difference between the indirect action and direct action? Also, why is the direct action repairable and indirect not?

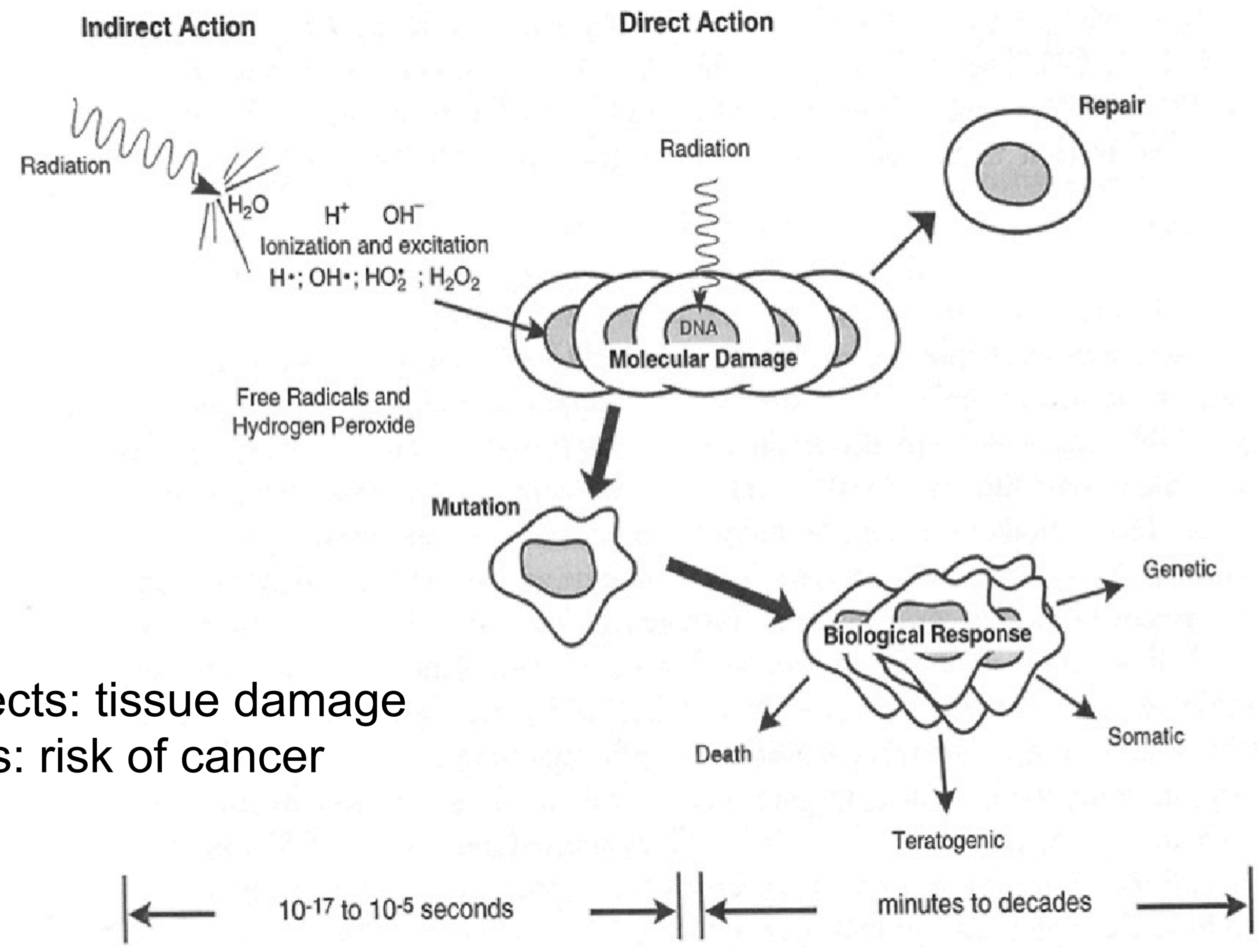
Effects of ionizing radiation



Deterministic effects: tissue damage
Stochastic effects: risk of cancer

Bushberg et al. The Essential Physics of Medical Imaging. 2002

Effects of ionizing radiation

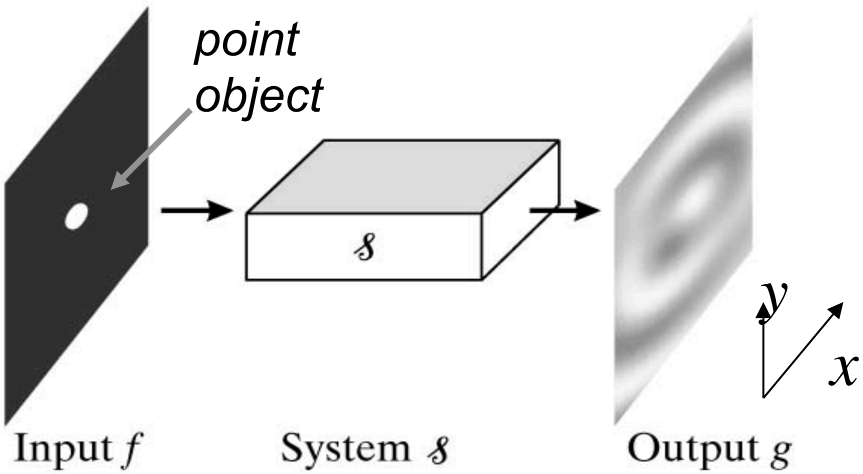


Deterministic effects: tissue damage
Stochastic effects: risk of cancer

Bushberg et al. The Essential Physics of Medical Imaging. 2002

Impulse Response

- Linear, shift-invariant (LSI) systems are the most useful
- First we start by looking at the response of a system using a point source at location (ξ,η) as an input



input $f_{\xi\eta}(x, y) \triangleq \delta(x-\xi, y-\eta)$

output $g_{\xi\eta}(x, y) \triangleq h(x, y; \xi, \eta)$

- The output *h()* depends on location of the point source (ξ,η) and location in the image (*x,y*), so it is a 4-D function
- Since the input is an impulse, the output is called the *impulse response function,* or the *point spread function* (PSF) - *why?*

Impulse Response of Linear Shift Invariant Systems

- For LSI systems $\mathscr{S}\left[f(x-x_0,y-y_0)\right]=g(x-x_0,y-y_0)$
- So the PSF is $\mathscr{S}\left[\delta(x-x_0,y-y_0)\right]=h(x-x_0,y-y_0)$
- Through something called the superposition integral, we can show that

$$g(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\,h(x,y;\xi,\eta)\,d\xi\,d\eta$$

- And for LSI systems, this simplifies to:

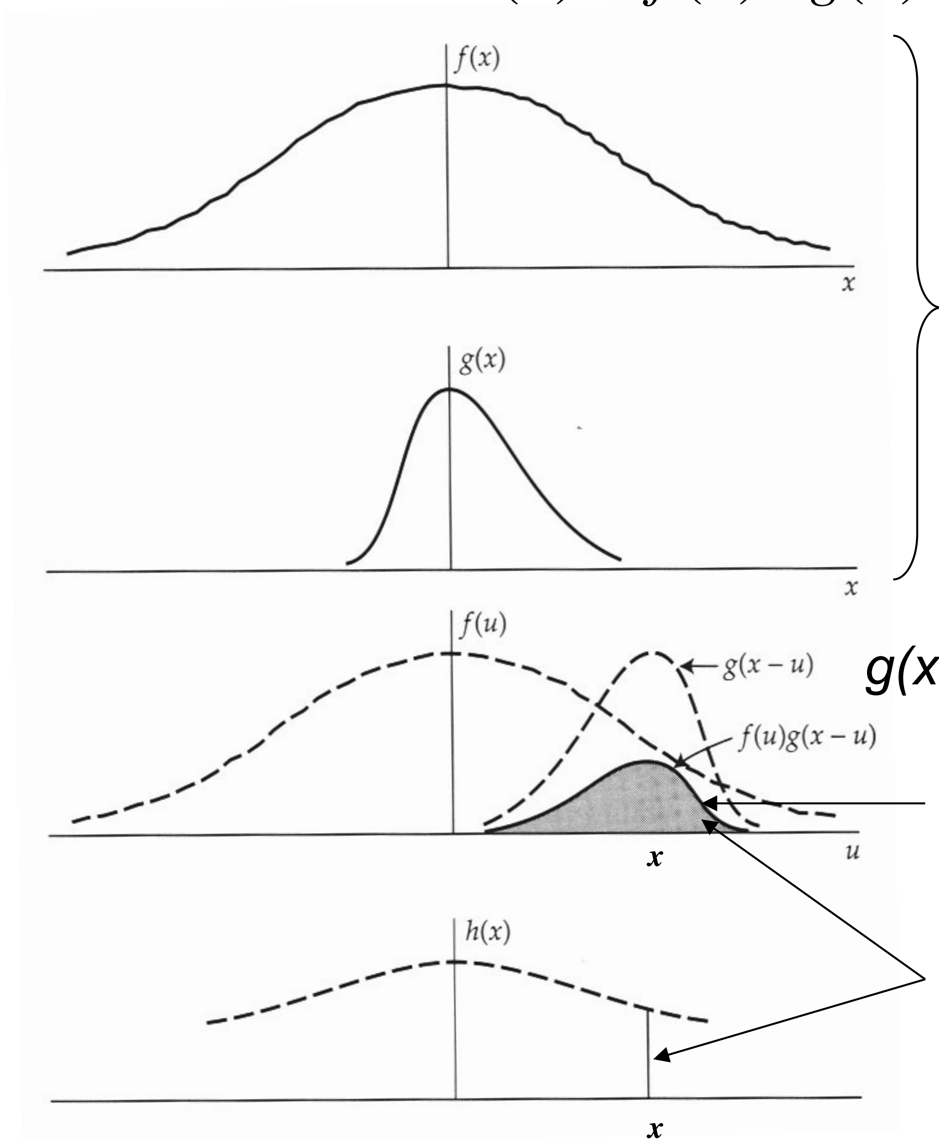
$$g(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\,h(\xi-x,\eta-y)\,d\xi\,d\eta$$

- The last integral is a convolution integral, and can be written as

$$g(x,y)=f(x,y)*h(x,y)\quad(\text{or}\,f(x,y)**h(x,y))$$

Review of convolution

- Illustration of $$h(x) = f(x) * g(x) = \int_{-\infty}^{\infty} f(u)g(x-u)du$$



original functions

g(x-u), reversed and shifted to x

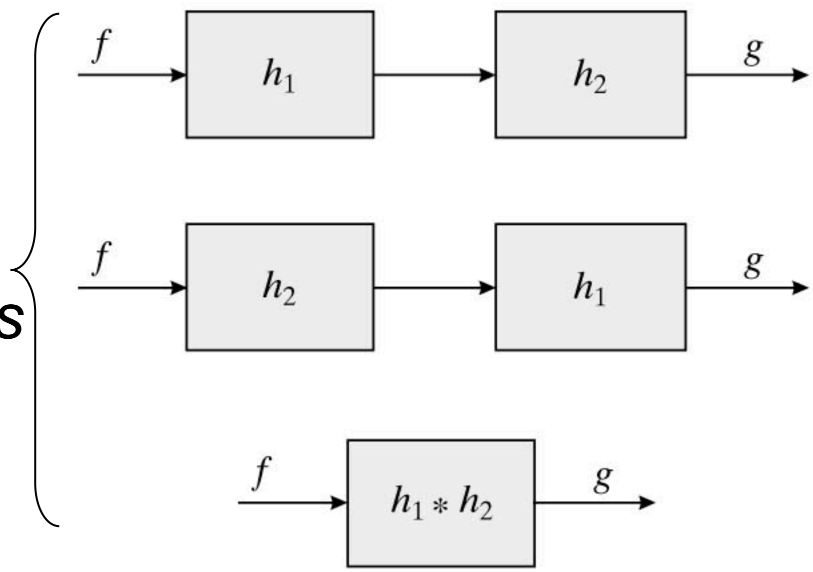
curve = product of f(u)g(x-u)

area = integral of f(u)g(x-u)
= value of h() at x

Properties of LSI Systems

- The convolution integral has the basic properties of
  1. Linearity (definition of a LSI system)
  2. Shift invariance (ditto)
  3. Associativity $$g(x,y) = h_2(x,y) * \left[h_1(x,y) * f(x,y)\right] = \left[h_2(x,y) * h_1(x,y)\right] * f(x,y)$$
  4. Commutativity $$h_1(x,y) * h_2(x,y) = h_2(x,y) * h_1(x,y)$$

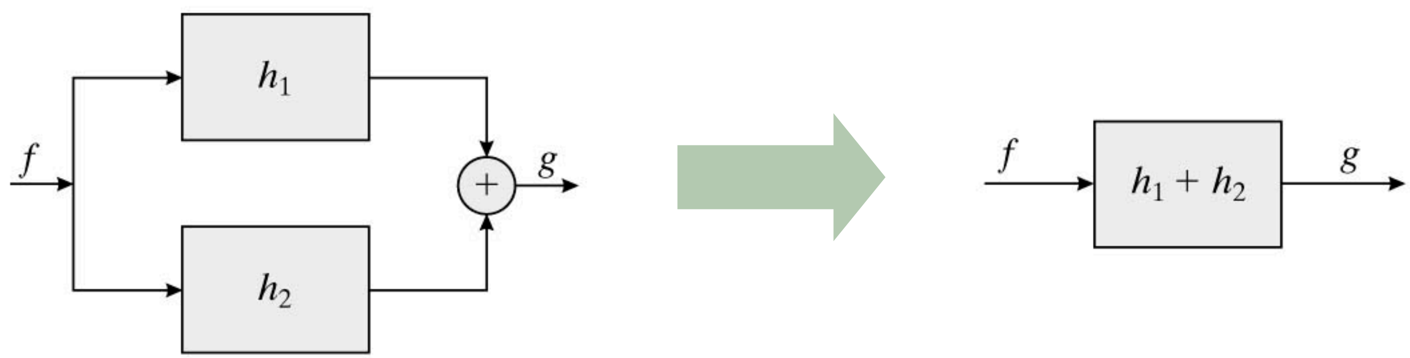
Equivalent arrangements



Combined LSI Systems

- Parallel systems have property of
  5. Distributivity

$$g(x,y) = h_1(x,y) * f(x,y) + h_2(x,y) * f(x,y)$$
$$= \left[h_1(x,y) + h_2(x,y)\right] * f(x,y)$$



Summary of advantages of Linear Shift Invariant Systems

- For LSI systems we have

$$f(x,y) \longrightarrow \boxed{h(x,y)} \longrightarrow g(x,y)$$

object *system* *image*

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\, h(\xi - x, \eta - y)\, d\xi\, d\eta$$
$$= f(x,y) ** h(x,y)$$

- Treating imaging systems as LSI significantly simplifies analysis
- In many cases of practical value, non-LSI systems can be approximated as LSI
- Allows use of Fourier transform methods that accelerate computation

2D Fourier Transforms

Fourier Transforms

- Recall from the sifting property (with a change of variables)

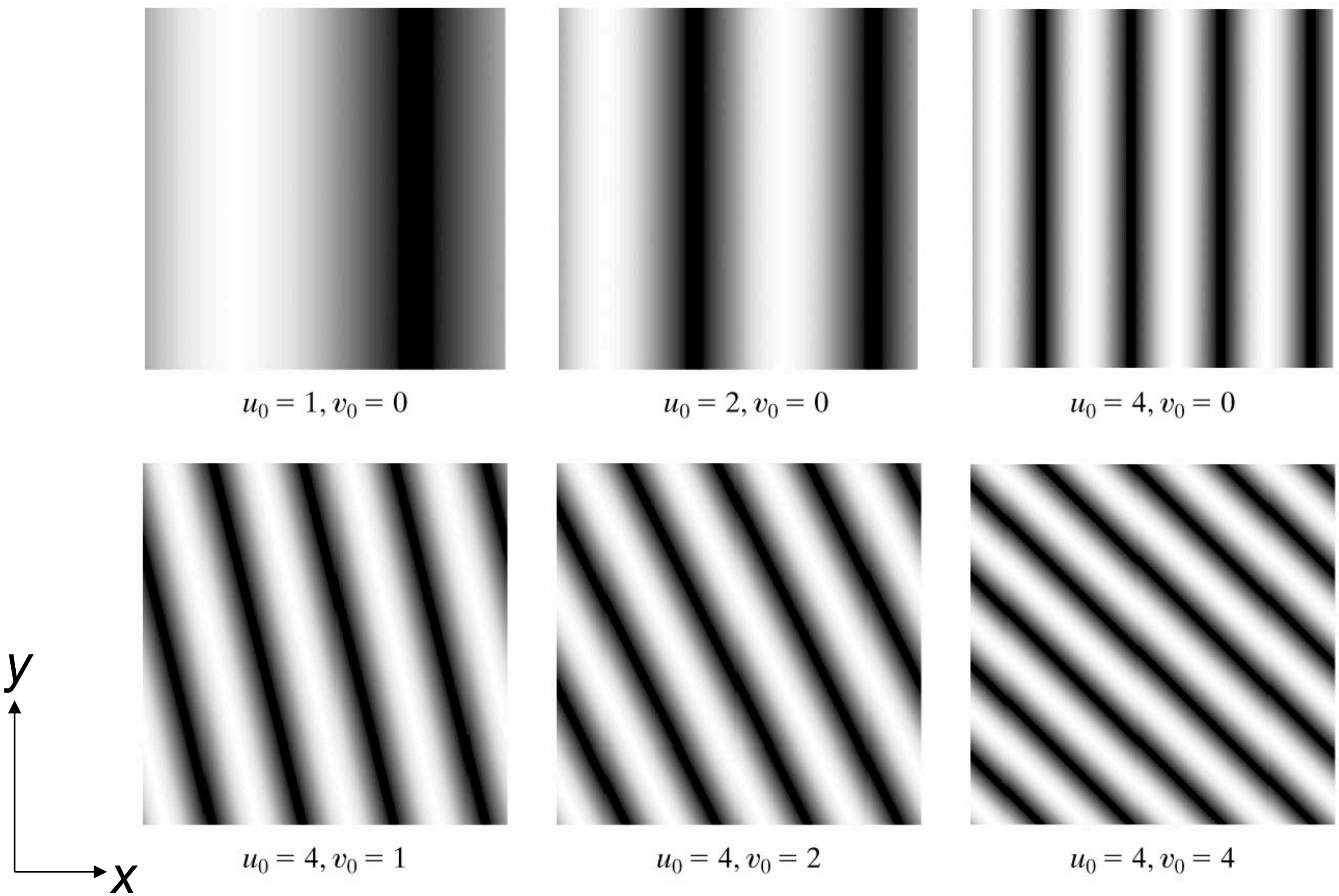
$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\delta(\xi - x, \eta - y)\,d\xi\,d\eta$$

- Expresses *f(x,y)* as a weighted combination of shifted basis functions, $\delta(x,y)$, also called the superposition principle
- An alternative and convenient set of basis functions are sinusoids, which bring in the concept of frequency
- Using the complex exponential function allows for compact notation, with *u* and *v* as the frequency variables

$$e^{j2\pi(ux+vy)} = \cos\left[2\pi\left(ux+vy\right)\right] + j\sin\left[2\pi\left(ux+vy\right)\right]$$

Exponential and sinusoidal signals as basis functions

- Intensity images for $s(x,y) = \sin\left[2\pi\left(u_0 x + v_0 y\right)\right]$



Fourier Transforms

- Using this approach we write

$$f(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(u,v)e^{j2\pi(ux+vy)}\,du\,dv$$

- F(u,v) are the weights for each frequency, exp{ $j2\pi(ux+vy)$} are the basis functions
- It can be shown that using exp{ $j2\pi(ux+vy)$} we can readily calculate the needed weights by

$$F(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)e^{-j2\pi(ux+vy)}\,dx\,dy$$

- This is the 2D Fourier Transform of *f(x,y),* and the first equation is the inverse 2D Fourier Transform

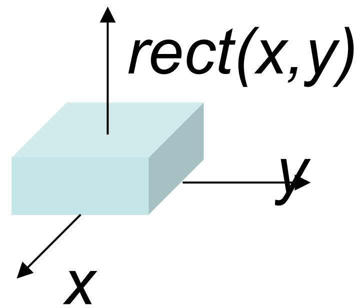
Fourier Transforms

- For even more compact notation we use

$$F(u,v) = \mathscr{F}_{\text{2D}}\left\{f(x,y)\right\}, \text{ and } f(x,y) = \mathscr{F}_{\text{2D}}^{-1}\left\{F(u,v)\right\}$$

- Notes on the Fourier transform
  - *F(u,v)* can be calculated if *f(x,y)* is continuous, or has a finite number of discontinuities, and is absolutely integrable
  - (u,v) are the spatial frequencies
  - *F(u,v)* is in general complex-valued, and is called the spectrum of *f(x,y)*
- As we will see, the Fourier transform allows consideration of an LSI system for each separate sinusoidal frequency

Fourier Transform Example



- What is the Fourier transform of

$$\text{rect}(x,y) = \begin{cases} 1, & \text{for } |x| < 1/2 \text{ and } |y| < 1/2 \\ 0, & \text{otherwise} \end{cases}$$

- First note that it is separable $\quad \text{rect}(x,y) = \text{rect}(x)\,\text{rect}(y)$
- So we compute

$$\mathcal{F}_{\text{1D}}\left\{\text{rect}(x)\right\} = \int_{-\infty}^{\infty} \text{rect}(x)\, e^{-j2\pi ux}\, dx$$

$$= \int_{-1/2}^{1/2} e^{-j2\pi ux}\, dx = \frac{1}{j2\pi u} e^{-j2\pi ux}\Big|_{-1/2}^{1/2}$$

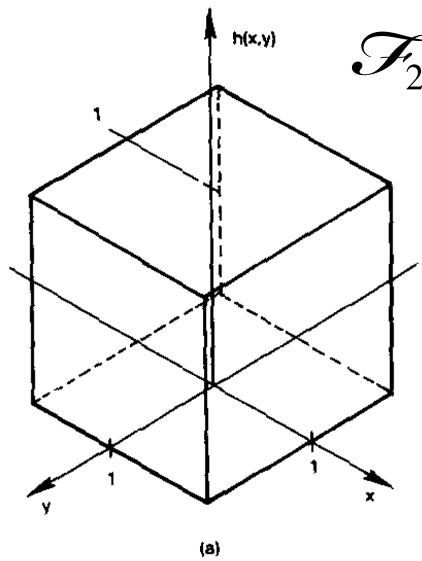
$$= \frac{1}{\pi u}\,\frac{e^{j\pi u} - e^{-j\pi u}}{j2} = \frac{\sin(\pi u)}{\pi u}$$

$$= \text{sinc}(u)$$

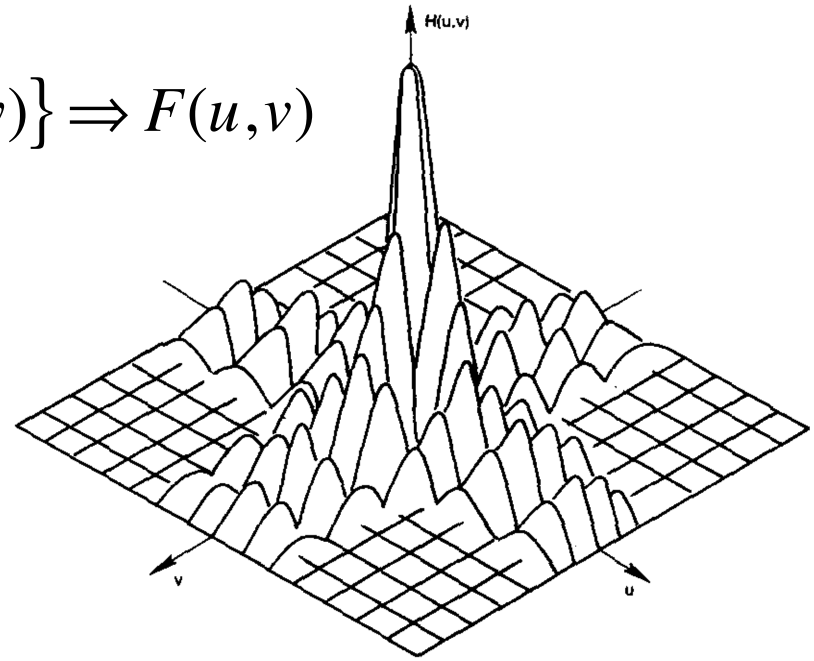
Thus $\quad \mathcal{F}_{\text{2D}}\left\{\text{rect}(x,y)\right\} = \text{sinc}(u,v)$

Fourier Transform Example

$$\mathscr{F}_{2D}\{f(x,y)\} \Rightarrow F(u,v)$$



$\mathrm{rect}(x,y)$

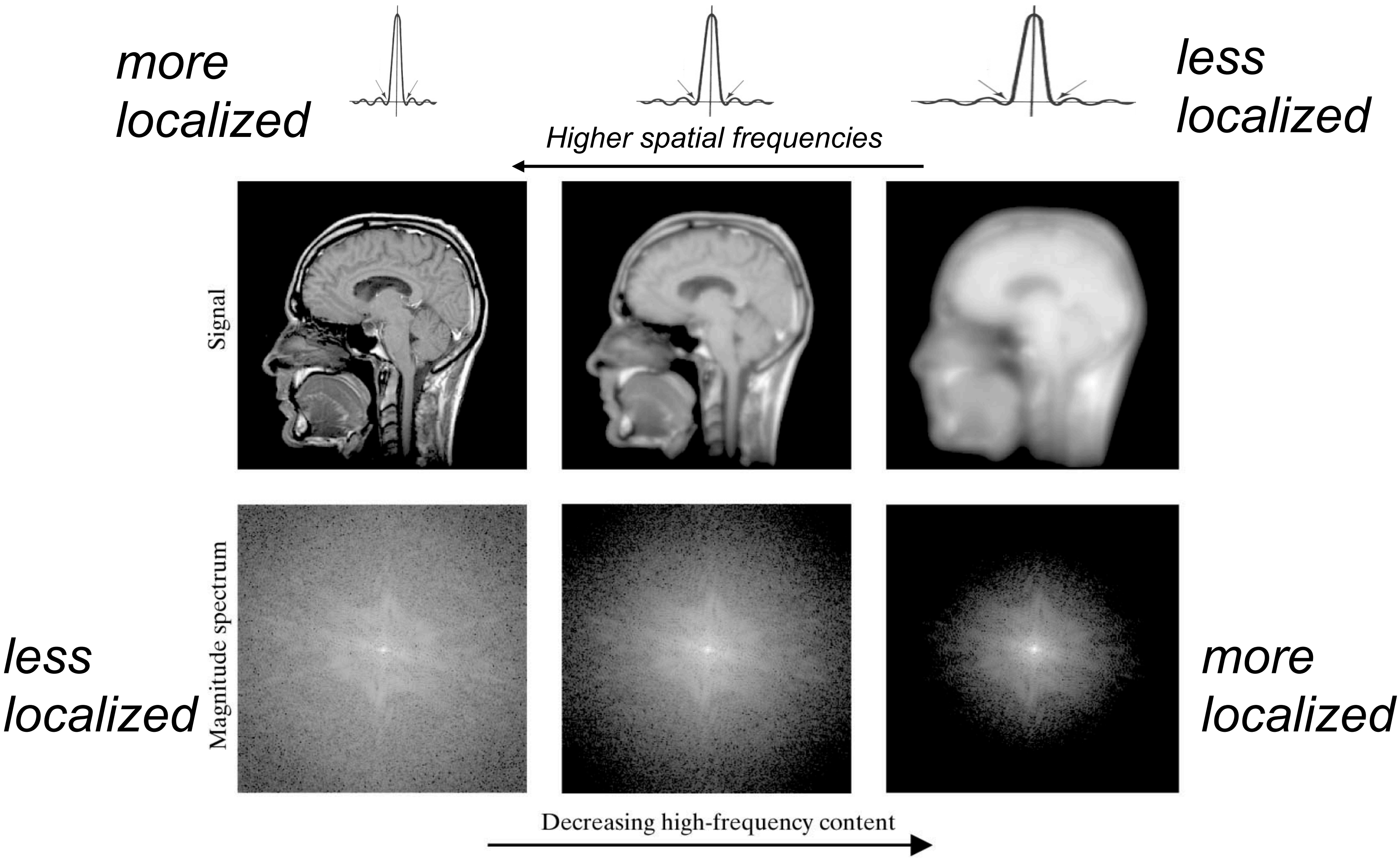


$\mathrm{sinc}(u,v)$

Two Key Properties of the 2D Fourier Transform

- Linearity $$\mathcal{F}_{\mathrm{2D}}\left\{a_1 f(x,y) + a_2 g(x,y)\right\} = a_1 F(u,v) + a_2 G(u,v)$$
- Scaling $$\mathcal{F}_{\mathrm{2D}}\left\{f(ax,by)\right\} = \frac{1}{|ab|} F\left(\frac{u}{a},\frac{v}{b}\right)$$

Signal localization in image versus frequency space



Fourier Transforms and Convolution

- Very useful! $$\mathcal{F}_{\text{2D}}\left\{f(x,y)*g(x,y)\right\}=F(u,v)G(u,v)$$
- Proof (1-D)

$$\mathcal{F}\left\{f(x)*g(x)\right\}=\int_{-\infty}^{\infty}\left(f(x)*g(x)\right)e^{-j2\pi ux}\,dx$$

$$=\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}f(\xi)g(x-\xi)d\xi\right)e^{-j2\pi ux}\,dx=\int_{-\infty}^{\infty}f(\xi)\left(\int_{-\infty}^{\infty}g(x-\xi)\,e^{-j2\pi ux}\,dx\right)d\xi$$

$$=\int_{-\infty}^{\infty}f(\xi)\left(\int_{-\infty}^{\infty}\mathcal{F}\left\{g(x-\xi)\right\}\right)d\xi=\int_{-\infty}^{\infty}f(\xi)\left(e^{-j2\pi u\xi}\,G(u)\right)d\xi$$

$$=G(u)\int_{-\infty}^{\infty}f(\xi)e^{-j2\pi u\xi}\,d\xi=F(u)G(u)$$

Fourier transform pairs

Signal	Fourier Transform
1	$\delta(u, v)$
$\delta(x, y)$	1
$\delta(x - x_0, y - y_0)$	$e^{-j2\pi(ux_0 + vy_0)}$
$\delta_s(x, y; \Delta x, \Delta y)$	$\text{comb}(u\Delta x, v\Delta y)$
$e^{j2\pi(u_0 x + v_0 y)}$	$\delta(u - u_0, v - v_0)$
$\sin[2\pi(u_0 x + v_0 y)]$	$\frac{1}{2j}[\delta(u - u_0, v - v_0) - \delta(u + u_0, v + v_0)]$
$\cos[2\pi(u_0 x + v_0 y)]$	$\frac{1}{2}[\delta(u - u_0, v - v_0) + \delta(u + u_0, v + v_0)]$
$\text{rect}(x, y)$	$\text{sinc}(u, v)$
$\text{sinc}(x, y)$	$\text{rect}(u, v)$
$\text{comb}(x, y)$	$\text{comb}(u, v)$
$e^{-\pi(x^2 + y^2)}$	$e^{-\pi(u^2 + v^2)}$

- Note the reciprocal symmetry in Fourier transform pairs
  - often 2-D versions can be calculated from 1-D versions by seperability
  - In general: a broad extent in one domain corresponds to a narrow extent in the other domain

Summary of key properties of the Fourier Transform

Theorem	$f(x,y)$	$F(u,v)$
Similarity	$f(ax,by)$	$\frac{1}{\lvert ab \rvert} F\left(\frac{u}{a}, \frac{v}{b}\right)$
Addition	$f(x,y) + g(x,y)$	$F(u,v) + G(u,v)$
Shift	$f(x - a, y - b)$	$e^{-2\pi i(au+bv)}F(u,v)$
Modulation	$f(x,y) \cos \omega x$	$\frac{1}{2} F\left(u + \frac{\omega}{2\pi}, v\right) + \frac{1}{2} F\left(u - \frac{\omega}{2\pi}, v\right)$
Convolution	$f(x,y) * g(x,y)$	$F(u,v)G(u,v)$
Autocorrelation	$f(x,y) * f^*(-x,-y)$	$\lvert F(u,v) \rvert^2$
Rayleigh	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \lvert f(x,y) \rvert^2 \, dx \, dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \lvert F(u,v) \rvert^2 \, du \, dv$	
Power	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) g^*(x,y) \, dx \, dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u,v) G^*(u,v) \, du \, dv$	
Parseval	$\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \lvert f(x,y) \rvert^2 = \sum \sum a_{mn}{}^2,$ where $F(u,v) = \sum \sum a_{mn} [{}^2\delta(u - m, v - n)]$	
Differentiation	$\left(\frac{\partial}{\partial x}\right)^m \left(\frac{\partial}{\partial y}\right)^n f(x,y)$	$(2\pi i u)^m (2\pi i v)^n F(u,v)$

Transfer Functions

Transfer Function for an LSI System

- Recall that for an LSI system $\quad f(x,y) \longrightarrow \boxed{\mathscr{S}} \longrightarrow g(x,y)$

$$g(x,y) = f(x,y) * h(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\, h(\xi - x, \eta - y)\, d\xi\, d\eta$$

- We can define the Transfer Function as the 2D Fourier transform of the PSF

$$H(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\xi,\eta) e^{j2\pi(u\xi + v\eta)} d\xi\, d\eta = \mathscr{F}_{\mathrm{2D}}\left\{h(x,y)\right\}$$

- In this case the LSI imaging system can be simply described by:

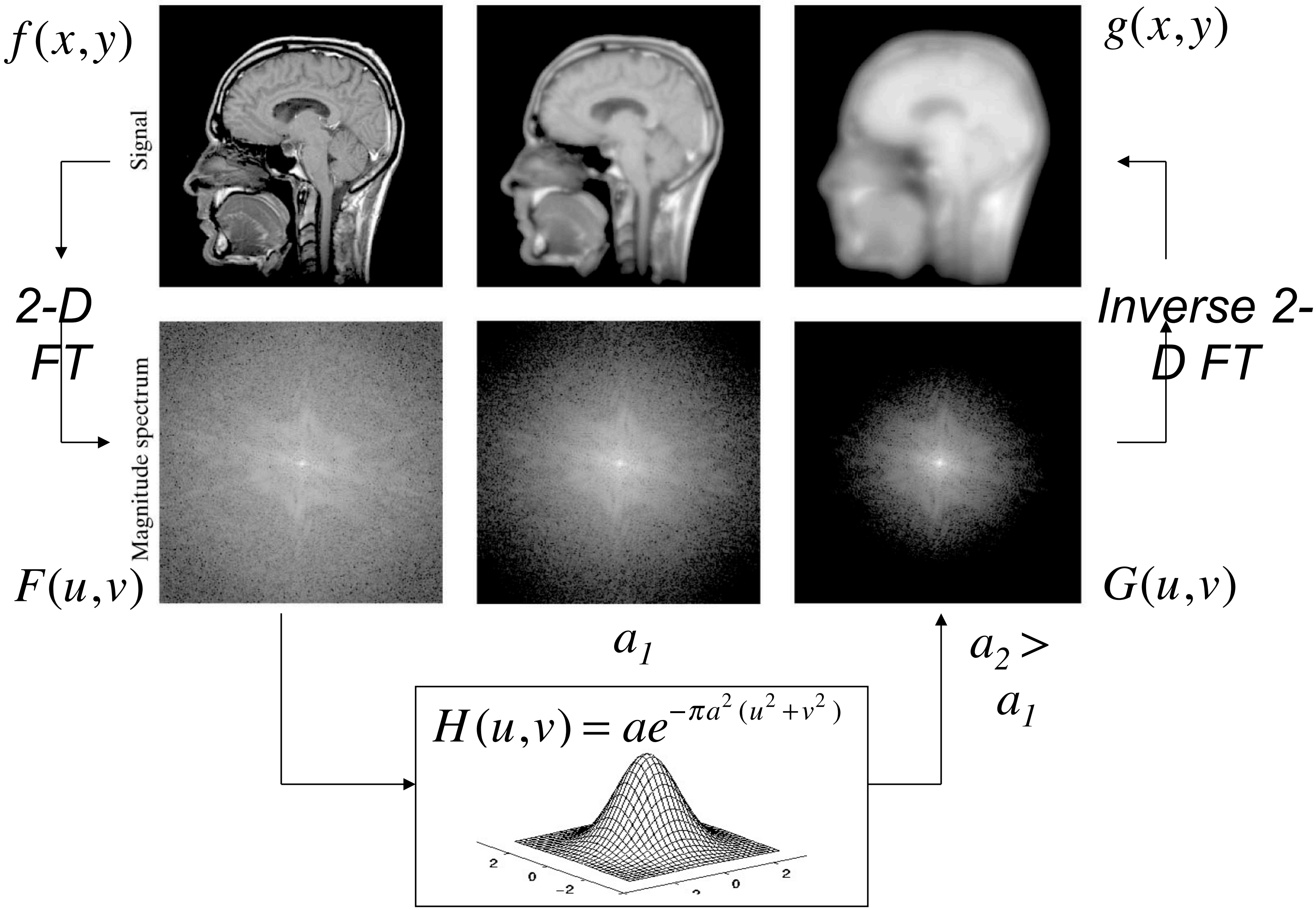
$$g(x,y) = f(x,y) * h(x,y) = \mathscr{F}_{2D}^{-1}\left\{F(u,v)H(u,v)\right\}$$

- or $\quad G(u,v) = F(u,v)H(u,v)$

- which provides a very powerful tool for understanding systems

Illustration of transfer function

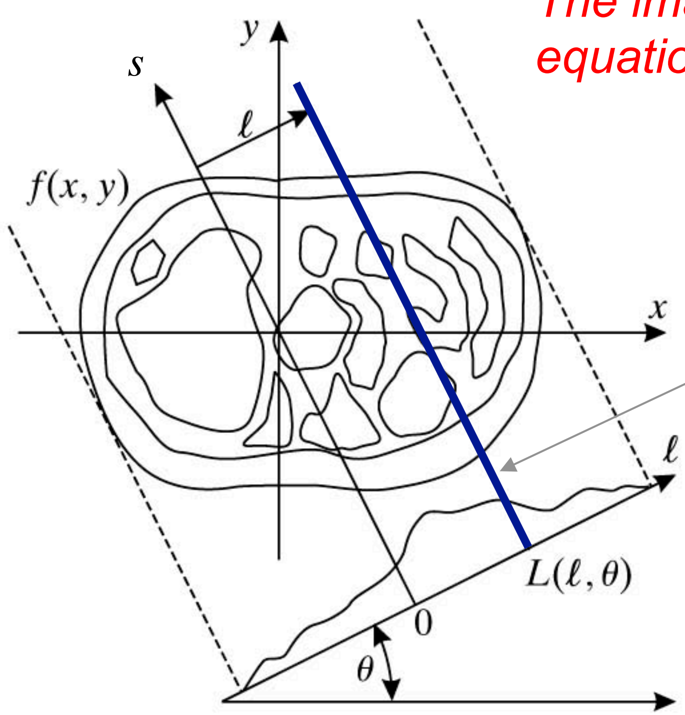
$$f(x,y) \rightarrow \boxed{h(x,y)} \rightarrow g(x,y)$$



2D Image Reconstruction from X-ray Transforms

Mathematical Model

- Many imaging systems acquire *line-integral* data of the object being scanned (or data that can be approximated as line-integrals) often called a line of response



The imaging equation

$$g(l,\theta) = \int_{-\infty}^{\infty} f(x(s), y(s))\, ds$$

The integral is along a line

$$L(l,\theta) = \{(x,y) \mid x\cos\theta + y\sin\theta = l\}$$

With rotated coordinates (l,s)

$$x(s) = l\cos\theta - s\sin\theta$$

$$y(s) = l\sin\theta + s\cos\theta$$

Example

- Consider the unit disk with radius R $f(x,y)=\begin{cases} 1 & x^2+y^2\le R \\ 0 & \text{otherwise} \end{cases}$

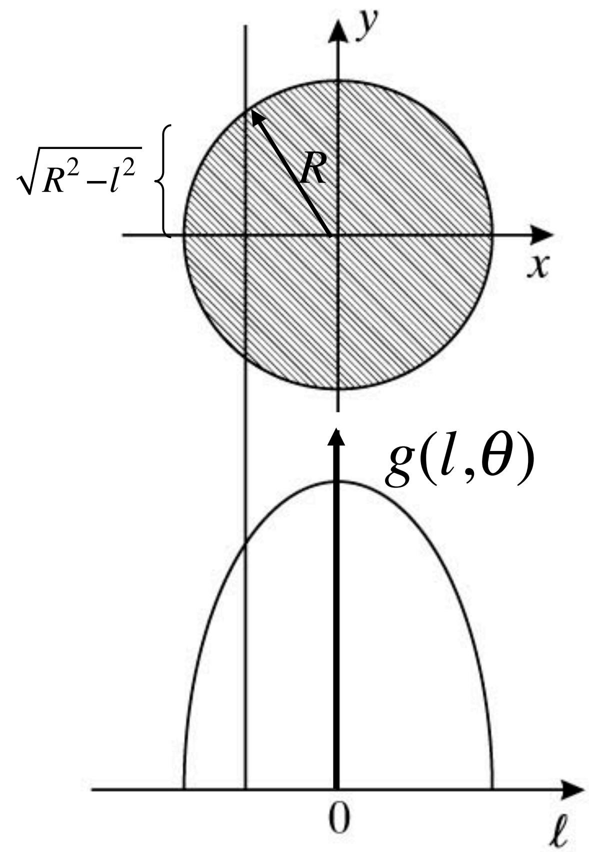
- By geometry

$$g(l,\theta)=\int_{-\infty}^{\infty} f(x(s),y(s))\,ds$$

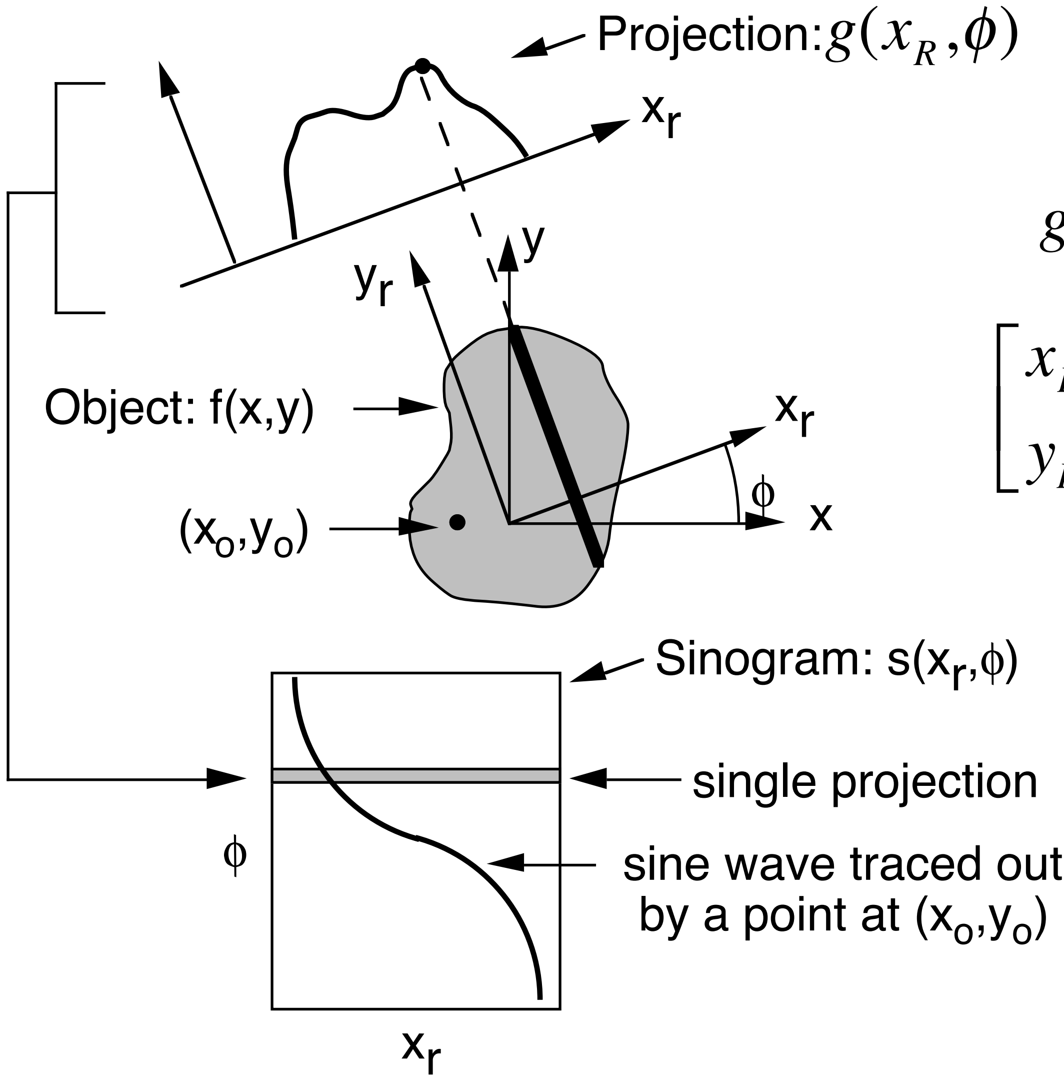
$$=\int_{-\sqrt{R^2-l^2}}^{\sqrt{R^2-l^2}} 1\,ds = 2\int_{0}^{\sqrt{R^2-l^2}} ds$$

$$=\begin{cases} 2\sqrt{R^2-l^2} & |l|\le R \\ 0 & \text{otherwise} \end{cases}$$

Check: $g(l=0,\theta)=2R,\quad \forall\theta$



One-dimensional projections



$$g(x_R,\phi) = \int_{-\infty}^{\infty} dy_R\, f(x,y)$$

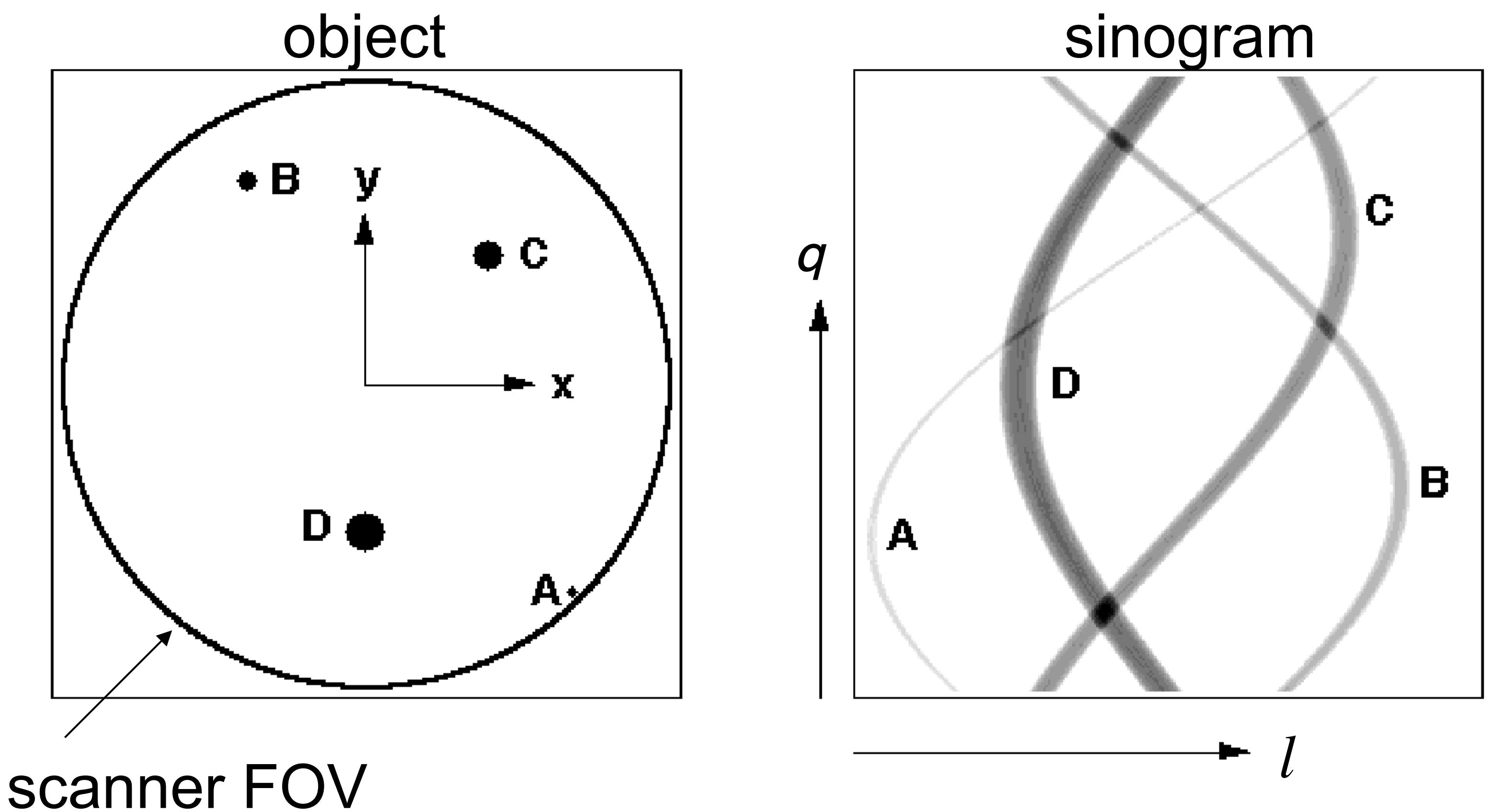
$$\begin{bmatrix} x_R \\ y_R \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

To specify the orientation of the line integrals, two parameters are needed, and sets of parallel lines are grouped into projections.

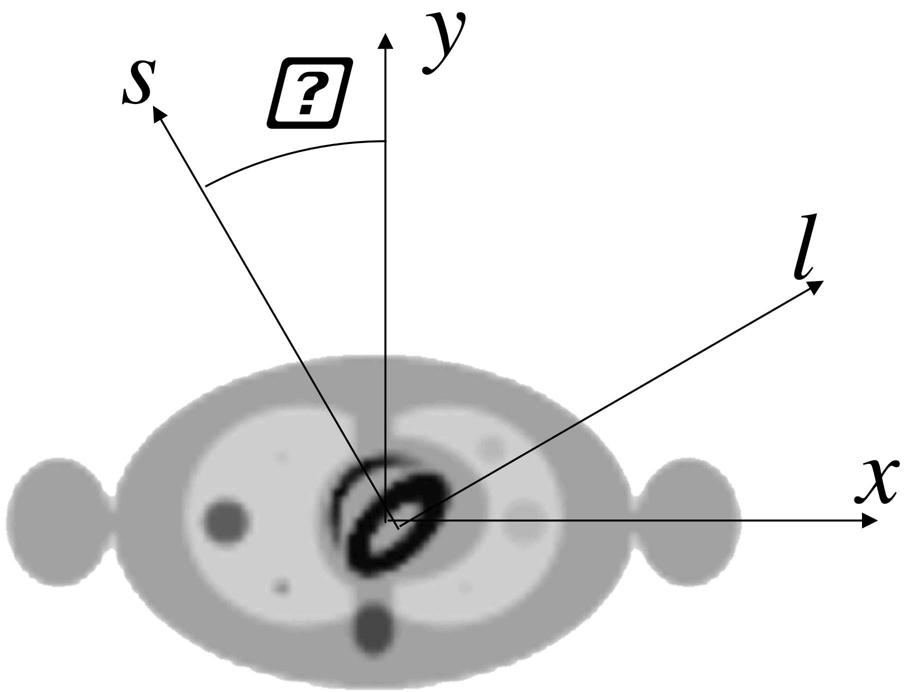
The projections are typically further grouped into sinograms.

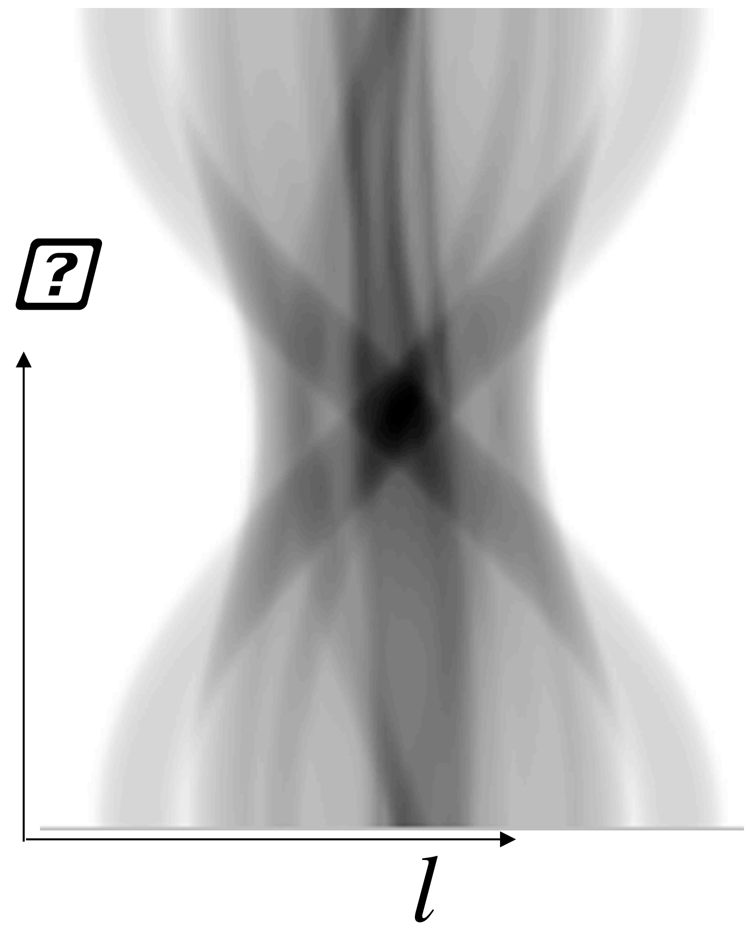
Sinograms

- We can represent the projection data $g(l,?)$, as a 2-D image, which is called a <u>sinogram</u>
- Each row is a projection at a fixed angle ?, with an intensity of $g(l,??$
- A point in the object projects to a sine wave in the sinogram



More complex sinogram example





Imaging equation, Inverse Problem, and Image reconstruction

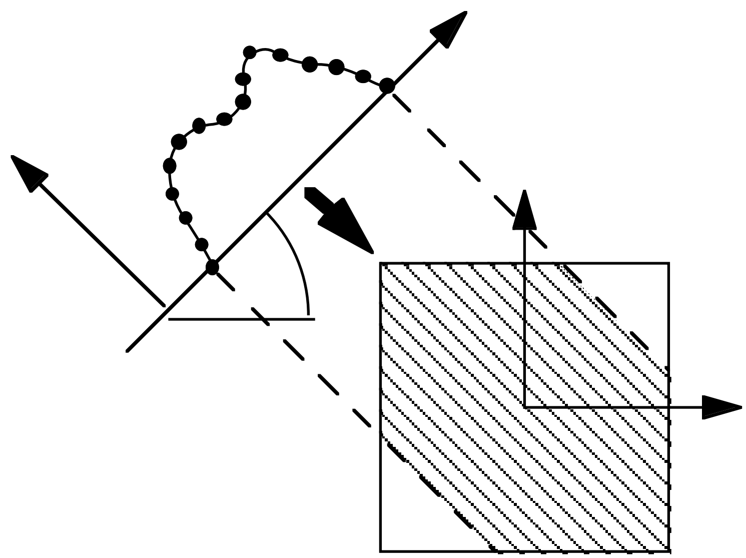
- Our generic imaging system acquires projections, which can be grouped into a sinogram

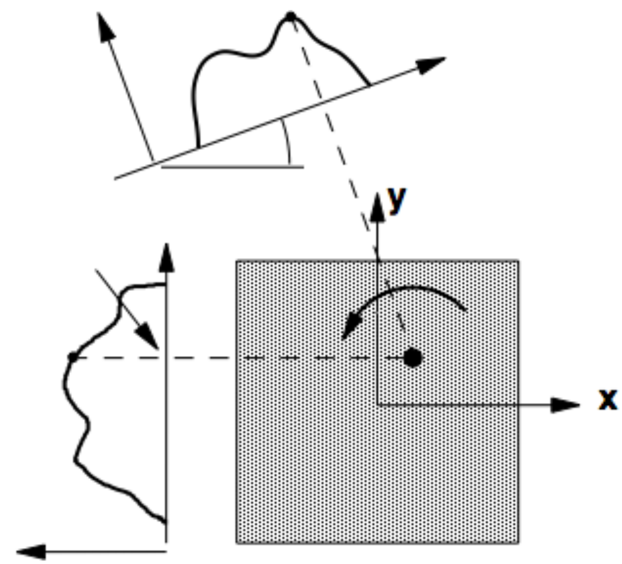
$$g(l,\theta) = \int_{-\infty}^{\infty} f(x(s), y(s))\, ds$$

- The above is an imaging equation
- This is an inverse problem: given $g(l,\theta)$, what is $f(x,y)$?
- In medical imaging this is called image reconstruction

Back-projection (or Backprojection)

- First idea - try the *adjoint* operation to the x-ray transform to see if it gives us the inverse operation (adjoint ~ reverse)
- If the initial operation is integration along a line (2-D to 1-D), then the 'opposite' operation is to spread values back along a line (1-D to 2-D)
- This is called backprojection

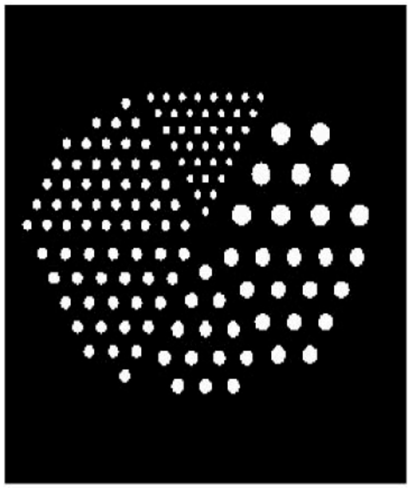




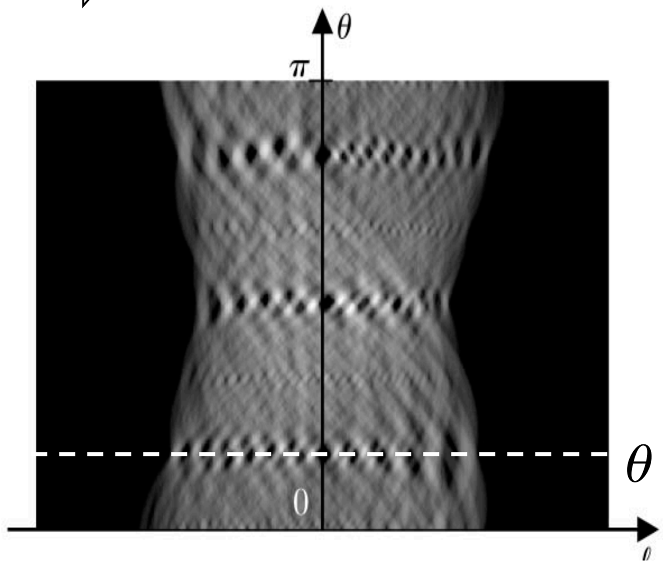
alternative mode of calculation

Backprojection does not work

Original object

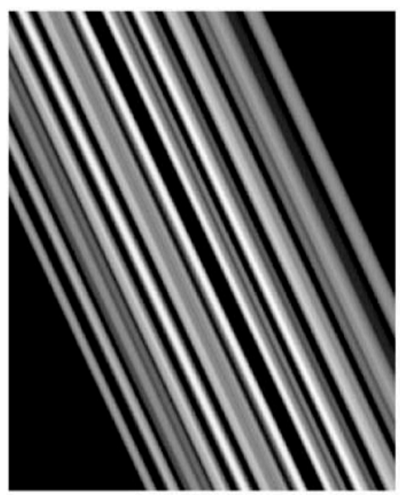


sinogram

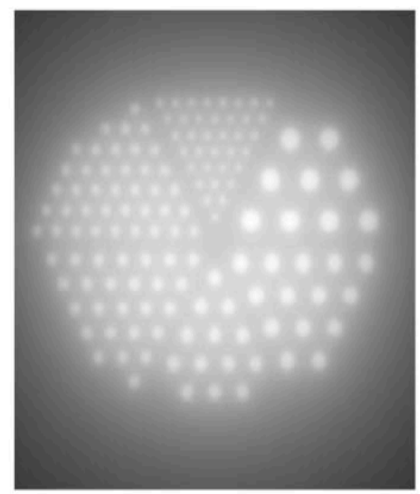


backprojection of $g(l,\theta)$ along angle θ

image matrix

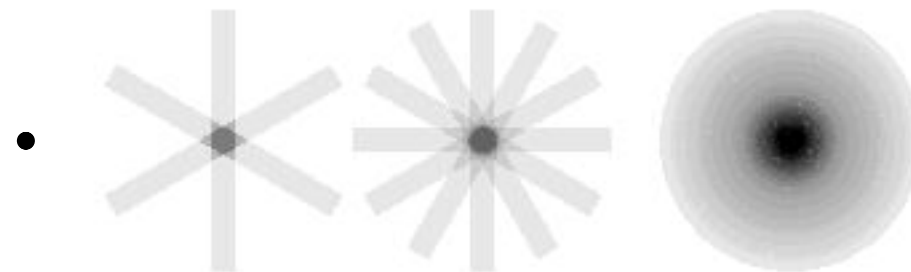


backprojection for all θ



Backprojection Reconstruction

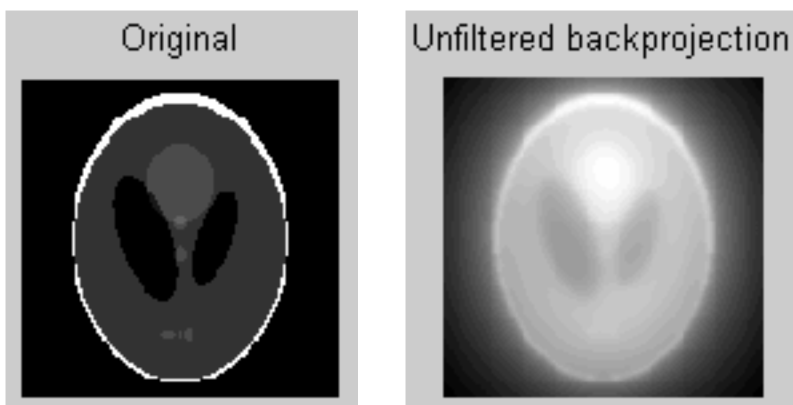
- Backprojection leads to a $1/r$ low-pass filter, so backprojected images are very blurry, and are typically unusable
- Examples
  - illustration for a small source



object 3 6 many

# of projections

  - for a more realistic object

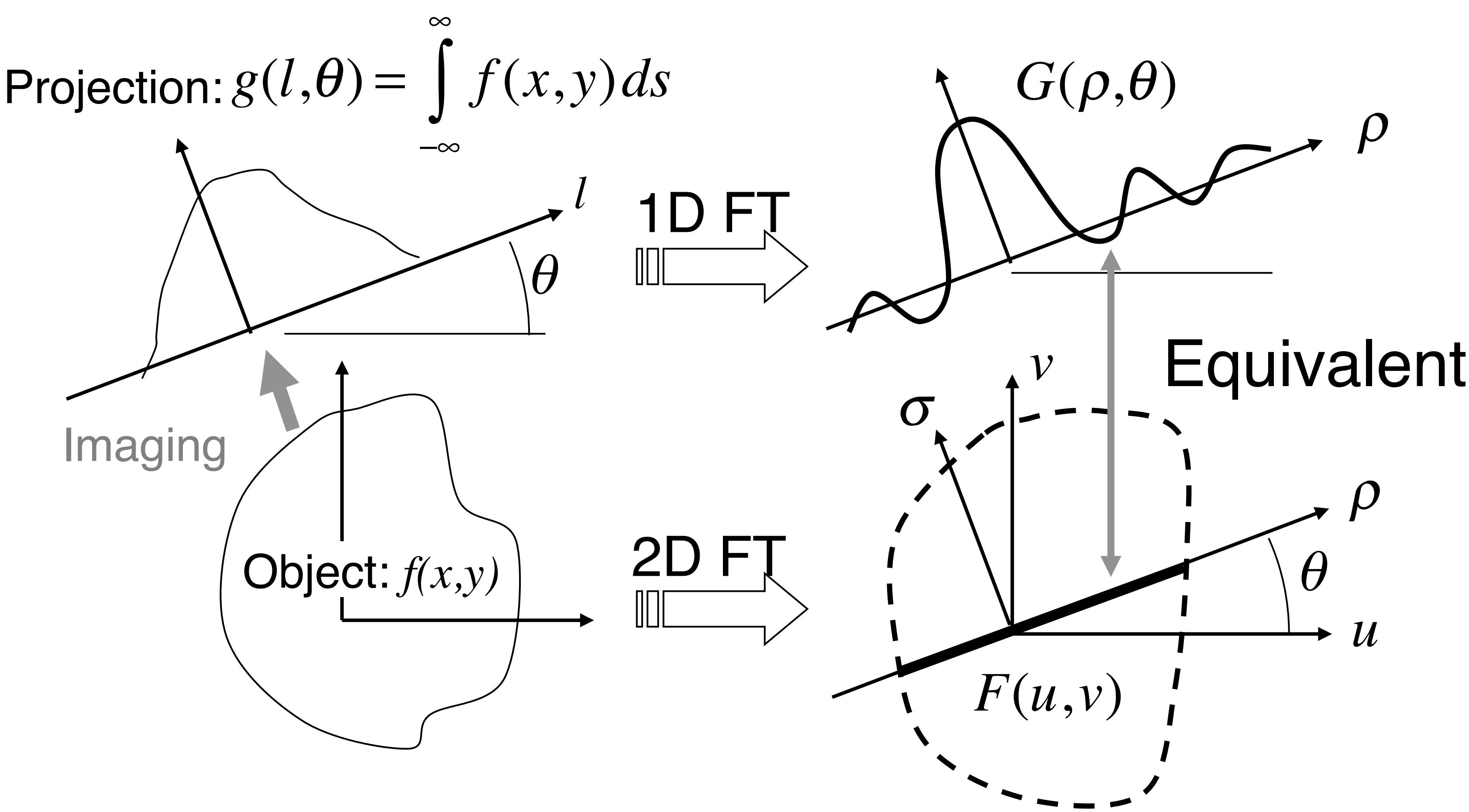


Shepp-Logan head phantom

Projection-Slice Theorem

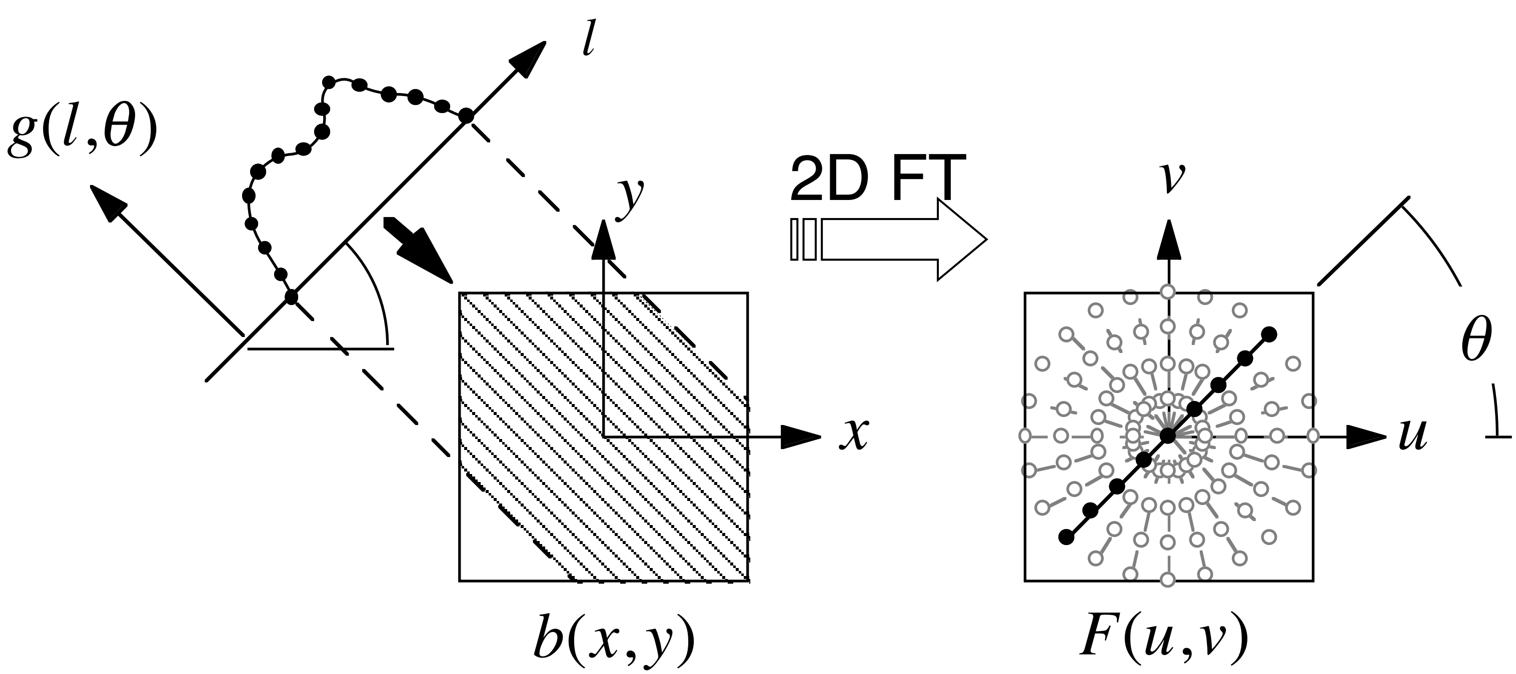
Projection-Slice Theorem

- The simplest way to understand 2-D image reconstruction, and a good way to start understanding 3-D image reconstruction.



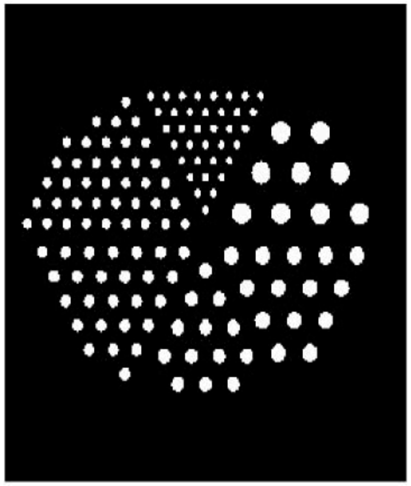
Backprojection Revisited

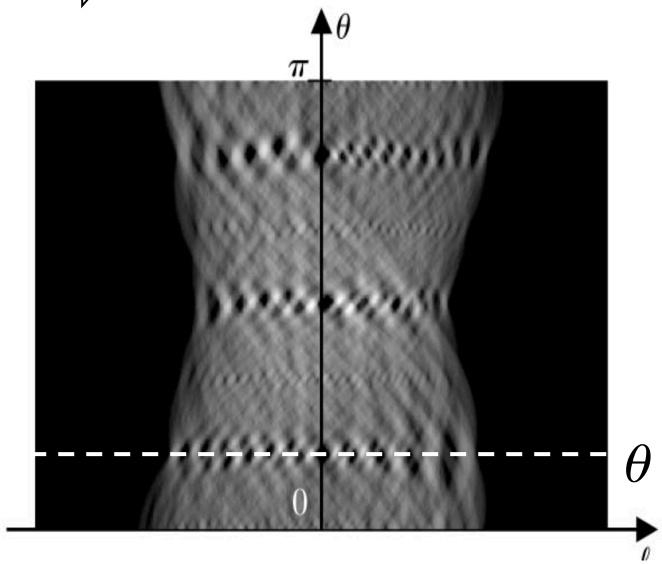
- By a corollary of the projection-slice theorem, backprojection is equivalent to placing the Fourier transformed values into an array representing $F(u,v)$, as shown



This is why backprojection does not work

Original object → sinogram



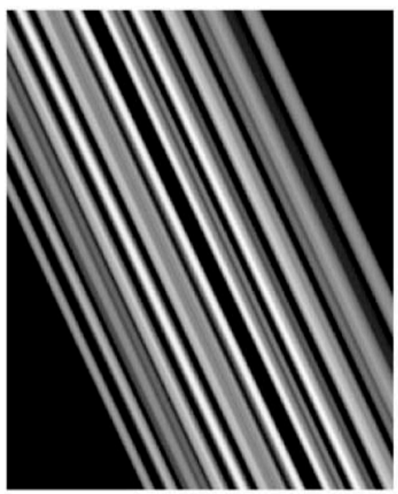


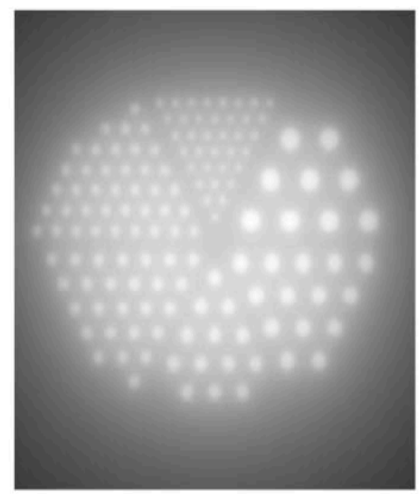
$$b(x,y) = f(x,y) * h(x,y)$$
$$= f(x,y) * \frac{1}{r}$$
$$= f(x,y) * \frac{1}{\sqrt{x^2 + y^2}}$$

backprojection of $g(l,\theta)$ along angle θ

backprojection for all θ

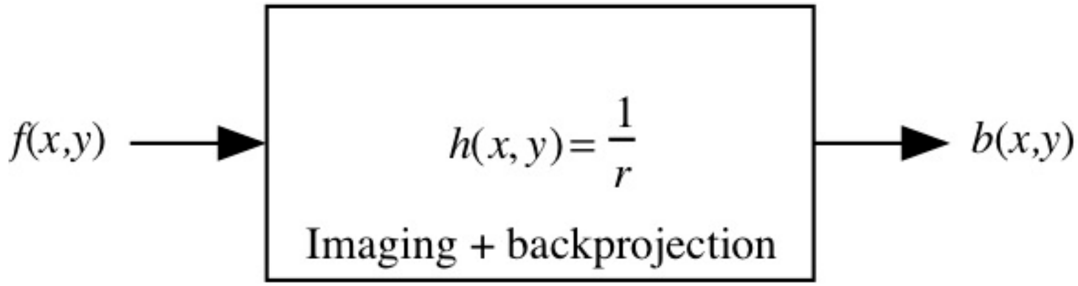
image matrix



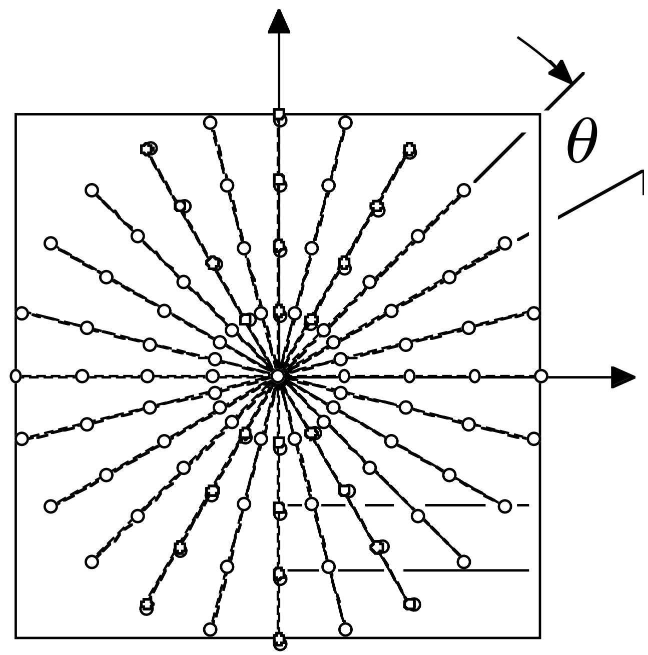


Backprojection Reconstruction

- Thus the backprojection of X-ray transform data comprises a shift-invariant imaging system blurred with a $1/r$ function



- This can also be seen intuitively by considering the sampling of the Fourier transform of the backprojected image
- In the limiting case the sampling density in frequency space is proportional to $1/\theta$



Backprojection Filtering

- We can fix this! Recall that $B(u,v) = F(u,v)/q$
  so very simply $F(u,v) = qB(u,v)$
- *Backprojection Filtering Algorithm*
  - for each θ, backproject measured data $g(l, \theta)$ into image array $b(x,y)$
  - compute the 2-D Fourier transform $B(u,v)$
  - multiply by 2-D 'cone' filter $q = \sqrt{u^2 + v^2}$ to get $F(u,v)$
  - compute the inverse 2-D Fourier transform to get $f(x,y)$

Challenges with Backprojection Filtering

- The low-pass blurring operation of $1/q$ has very long tails, so backprojection must be done on a much larger array than is needed for just the image

- Backprojection filtering is computationally very expensive
  - CT images are typically 512 x 512, and a typical factor of 4 needed will bring backprojection image size to 2048 x 2048, and another factor of 2 for zero padding for FFTs gets us to 4096 x 4096, per image

- An alternative solution is to <u>interchange</u> order of filtering and backprojection
  - the proof that we can do this is a bit complex