2D Signals and Systems
Signals

- A signal can be either continuous $f(x), f(x, y), f(x, y, z), f(x)$
- or discrete $f_{i, j, k}$ etc. where $i, j, k$ index specific coordinates

- Digital images on computers are necessarily discrete sets of data
- Each element, or bin, or voxel, represents some value, either measured or calculated
Digital Images

- Real objects are continuous (at least above the quantum level), but we represent them digitally as an approximation of the true continuous process (pixels or voxels).
- For image representation this is usually fine (we can just use smaller voxels as necessary).

For data measurements the element size is critical (e.g. Shannon's sampling theorem).

For most of our work we will use continuous function theory for convenience, but sometimes the discrete theory will be required.
Important signals - rect() and sinc() functions

• 1D rect() and sinc() functions
  – both have unit area

(a) \( \text{rect}(x) = \begin{cases} 
1, & \text{for } |x| < 1/2 \\
0, & \text{for } |x| > 1/2 
\end{cases} \)

(b) \( \text{sinc}(x) = \frac{\sin(\pi x)}{\pi x} \)

what is sinc(0)?
Important signals - 2D rect() and sinc() functions

• 2D rect() and sinc() functions are straightforward generalizations

\[
\text{rect}(x, y) = \begin{cases} 
1, & \text{for } |x| < 1/2 \text{ and } |y| < 1/2 \\
0, & \text{otherwise}
\end{cases}
\]

\[
\text{sinc}(x, y) = \frac{\sin(\pi x)\sin(\pi y)}{\pi^2 xy}
\]

• Try to sketch these
• 3D versions exist and are sometimes used
• Fundamental connection between rect() and sinc() functions and very useful in signal and image processing
Important signals - Impulse function

- 1D Impulse (delta) function
- A 'generalized function'
  - operates through integration
  - has zero width and unit area
  - has important 'sifting' property
  - can be understood by considering:

\[
\delta(x) = 0, \quad x \neq 0, \\
\int_{-\infty}^{\infty} \delta(x) \, dx = 1 \\
\int_{-\infty}^{\infty} f(x) \delta(x) \, dx = f(0) \\
\int_{-\infty}^{\infty} f(x) \delta(x-t) \, dx = f(t)
\]

- Ways to approach the delta function

\[
\delta(t) = \lim_{a \to \infty} a \text{rect}(at) \quad \delta(t) = \lim_{a \to \infty} a \text{sinc}(at) \\
\delta(t) = \lim_{a \to \infty} ae^{-\pi a^2 t^2}
\]
Exponential and sinusoidal signals

- Recall Euler's formula, which connects trigonometric and complex exponential functions
  \[ e^{j\theta} = \cos(\theta) + j\sin(\theta) \] (not i)

- The exponential signal is defined as:
  \[ e^{j2\pi x} = \cos(2\pi x) + j\sin(2\pi x), \quad \text{where } j^2 = -1 \]

- \( u_0 \) and \( v_0 \) are the fundamental frequencies in x- and y-directions, with units of 1/distance
  \[ e(x, y) = e^{j2\pi(u_0 x + v_0 y)} \]

- We can write
  \[ e(x, y) = e^{j2\pi(u_0 x + v_0 y)} \]
  \[ = \cos[2\pi(u_0 x + v_0 y)] + j\sin[2\pi(u_0 x + v_0 y)] \]

  real and even          imaginary and odd
Exponential and sinusoidal signals

- Recall that
  \[
  \sin(2\pi x) = \frac{1}{2j}(e^{j2\pi x} - e^{-j2\pi x})
  \]
  \[
  \cos(2\pi x) = \frac{1}{2}(e^{j2\pi x} + e^{-j2\pi x})
  \]

- so we have
  \[
  \sin[2\pi(u_0x + v_0y)] = \frac{1}{2j}(e^{j2\pi(u_0x + v_0y)} - e^{-j2\pi(u_0x + v_0y)})
  \]
  \[
  \cos[2\pi(u_0x + v_0y)] = \frac{1}{2}(e^{j2\pi(u_0x + v_0y)} + e^{-j2\pi(u_0x + v_0y)})
  \]

- Fundamental frequencies \(u_0, v_0\) affect the oscillations in \(x\) and \(y\) directions, E.g. small values of \(u_0\) result in slow oscillations in the \(x\)-direction

- These are complex-valued and directional plane waves
Exponential and sinusoidal signals

- Intensity images for \( s(x, y) = \sin\left[2\pi(u_0x + v_0y)\right] \)

\( u_0 = 1, v_0 = 0 \)
\( u_0 = 2, v_0 = 0 \)
\( u_0 = 4, v_0 = 0 \)
\( u_0 = 4, v_0 = 1 \)
\( u_0 = 4, v_0 = 2 \)
\( u_0 = 4, v_0 = 4 \)
System models

- Systems analysis is a powerful tool to characterize and control the behavior of biomedical imaging devices.
- We will focus on the special class of continuous, linear, shift-invariant (LSI) systems.
- Many (all) biomedical imaging systems are not really any of the three, but it can be useful tool, as long as we understand the errors in our approximation.
- "all models are wrong, but some are useful" - George E. P. Box.
- Continuous systems convert a continuous input to a continuous output.

\[ g(x) = \mathcal{F}[f(x)] \quad (g(t) = \mathcal{F}[f(t)]) \]

\[ f(x) \rightarrow \mathcal{F} \rightarrow g(x) \]
Linear Systems

• A system $\mathcal{S}$ is a linear system if: we have $\mathcal{S}[f(x)] = g(x)$ then

$$\mathcal{S}[a_1 f_1(x) + a_2 f_2(x)] = a_1 g_1(x) + a_2 g_2(x)$$

or in general

$$\mathcal{S} \left[ \sum_{k=1}^{K} w_k f_k(x) \right] = \sum_{k=1}^{K} w_k \mathcal{S}[f_k(x)] = \sum_{k=1}^{K} w_k g_k(x)$$

• Which are linear systems?

$$g(x) = e^x f(x)$$
$$g(x) = f(x) + 1$$
$$g(x) = x f(x)$$
$$g(x) = (f(x))^2$$
2D Linear Systems

- Now use 2D notation
- Example: sharpening filter

\[ S_wf_k(x, y) = \sum_{k=1}^{K} w_k g_k(x, y) \]

In general
Shift-Invariant Systems

• Start by shifting the input 
  \[ f_{x_0y_0}(x,y) \overset{\Delta}{=} f(x-x_0,y-y_0) \]
  then if
  \[ g_{x_0y_0}(x,y) = \mathcal{S} \left[ f_{x_0y_0}(x,y) \right] = g(x-x_0,y-y_0) \]
  the system is *shift-invariant*, i.e. response does not depend on location

• Shift-invariance is separate from linearity, a system can be
  – shift-invariant and linear
  – shift-invariant and non-linear
  – shift-variant and linear
  – shift-variant and non-linear
  – (what else have we forgotten?)
Shift invariant and shift-variant system response

Scanner

Object

\[ f(x,y) \]

Image

\[ g(x,y) \]

FOV

Unshifted response

Shift invariant

Shift variant

(shape, location)
Shift invariant and shift-variant system response

Scanner

Object

\( f(x,y) \)

Image

\( g(x,y) \)

FOV

Unshifted response

Shift invariant

Shift

Shift variant (value)
Impulse Response

- Linear, shift-invariant (LSI) systems are the most useful
- First we start by looking at the response of a system using a point source at location \((\xi, \eta)\) as an input

The output \( h() \) depends on location of the point source \((\xi, \eta)\) and location in the image \((x,y)\), so it is a 4-D function

- Since the input is an impulse, the output is called the *impulse response function*, or the *point spread function (PSF)* - why?
Impulse Response of Linear Shift Invariant Systems

- For LSI systems
  \[ \mathcal{F} \left[ f(x - x_0, y - y_0) \right] = g(x - x_0, y - y_0) \]

- So the PSF is
  \[ \mathcal{F} \left[ \delta(x - x_0, y - y_0) \right] = h(x - x_0, y - y_0) \]

- Through something called the superposition integral, we can show that
  \[
g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) h(x, y; \xi, \eta) \, d\xi \, d\eta
\]

- And for LSI systems, this simplifies to:
  \[
g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) h(\xi - x, \eta - y) \, d\xi \, d\eta
\]

- The last integral is a convolution integral, and can be written as
  \[
g(x, y) = f(x, y) * h(x, y) \quad \text{(or } f(x, y) ** h(x, y))
\]
Review of convolution

- Illustration of \( h(x) = f(x) * g(x) = \int_{-\infty}^{\infty} f(u)g(x-u)du \)

original functions

\( g(x-u) \), reversed and shifted to \( x \)

curve = product of \( f(u)g(x-u) \)

area = integral of \( f(u)g(x-u) \)

= value of \( h() \) at \( x \)
Properties of LSI Systems

• The convolution integral has the basic properties of
  1. Linearity (definition of a LSI system)
  2. Shift invariance (ditto)

3. Associativity
   \[ g(x,y) = h_2(x,y) * \left[ h_1(x,y) * f(x,y) \right] \]
   \[ = [h_2(x,y) * h_1(x,y)] * f(x,y) \]

4. Commutativity
   \[ h_1(x,y) * h_2(x,y) = h_2(x,y) * h_1(x,y) \]
Combined LSI Systems

- Parallel systems have property of
  5. Distributivity

\[ g(x,y) = h_1(x,y) \ast f(x,y) + h_2(x,y) \ast f(x,y) \]
\[ = [h_1(x,y) + h_2(x,y)] \ast f(x,y) \]
Summary of advantages of Linear Shift Invariant Systems

• For LSI systems we have

\[ f(x,y) \rightarrow h(x,y) \rightarrow g(x,y) \]

\( g(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi,\eta) h(\xi - x, \eta - y) \, d\xi \, d\eta \)

\[ = f(x,y) \ast h(x,y) \]

• Treating imaging systems as LSI significantly simplifies analysis
• In many cases of practical value, non-LSI systems can be approximated as LSI
• Allows use of Fourier transform methods that accelerate computation
2D Fourier Transforms
Fourier Transforms

• Recall from the sifting property (with a change of variables)

\[ f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) \delta(\xi - x, \eta - y) \, d\xi \, d\eta \]

• Expresses \( f(x, y) \) as a weighted combination of shifted basis functions, \( \delta(x, y) \), also called the superposition principle

• An alternative and convenient set of basis functions are sinusoids, which bring in the concept of frequency

• Using the complex exponential function allows for compact notation, with \( u \) and \( v \) as the frequency variables

\[ e^{j2\pi(ux+vy)} = \cos[2\pi(ux+vy)] + j\sin[2\pi(ux+vy)] \]
Exponential and sinusoidal signals as basis functions

- Intensity images for  \( s(x, y) = \sin\left[2\pi\left(u_0x + v_0y\right)\right] \)
Fourier Transforms

• Using this approach we write

\[ f(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u,v) e^{j2\pi(ux+vy)} \, du \, dv \]

• \( F(u,v) \) are the weights for each frequency, \( \exp\{ j2\pi(ux+vy) \} \) are the basis functions

• It can be shown that using \( \exp\{ j2\pi(ux+vy) \} \) we can readily calculate the needed weights by

\[ F(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} \, dx \, dy \]

• This is the 2D Fourier Transform of \( f(x,y) \), and the first equation is the inverse 2D Fourier Transform
Fourier Transforms

- For even more compact notation we use
  \[ F(u,v) = \mathcal{F}_{2D} \{ f(x,y) \}, \text{ and } f(x,y) = \mathcal{F}_{2D}^{-1} \{ F(u,v) \} \]

- Notes on the Fourier transform
  - \( F(u,v) \) can be calculated if \( f(x,y) \) is continuous, or has a finite number of discontinuities, and is absolutely integrable
  - \( (u,v) \) are the spatial frequencies
  - \( F(u,v) \) is in general complex-valued, and is called the spectrum of \( f(x,y) \)

- As we will see, the Fourier transform allows consideration of an LSI system for each separate sinusoidal frequency
Fourier Transform Example

- What is the Fourier transform of

\[ \text{rect}(x, y) = \begin{cases} 
1, & \text{for } |x| < 1/2 \text{ and } |y| < 1/2 \\
0, & \text{otherwise} 
\end{cases} \]

- First note that it is separable

\[ \text{rect}(x, y) = \text{rect}(x)\text{rect}(y) \]

- So we compute

\[
\mathcal{F}_{1D}\{\text{rect}(x)\} = \int_{-\infty}^{\infty} \text{rect}(x) e^{-j2\pi ux} \, dx
\]

\[
= \int_{-1/2}^{1/2} e^{-j2\pi ux} \, dx = \frac{1}{j2\pi u} e^{-j2\pi ux}\bigg|_{-1/2}^{1/2}
\]

\[
= \frac{1}{j2\pi u} \left( e^{j\pi u} - e^{-j\pi u} \right) = \frac{\sin(\pi u)}{\pi u}
\]

\[= \text{sinc}(u) \]

Thus

\[
\mathcal{F}_{2D}\{\text{rect}(x, y)\} = \text{sinc}(u, v)
\]
Fourier Transform Example

\[ \mathcal{F}_{2D} \{ f(x,y) \} \Rightarrow F(u,v) \]

rect\((x,y)\)  \hspace{1cm} sinc\((u,v)\)
Two Key Properties of the 2D Fourier Transform

- **Linearity**
  \[
  \mathcal{F}_{2D} \left\{ a_1 f(x,y) + a_2 g(x,y) \right\} = a_1 F(u,v) + a_2 G(u,v)
  \]

- **Scaling**
  \[
  \mathcal{F}_{2D} \left\{ f(ax,by) \right\} = \frac{1}{|ab|} F\left( \frac{u}{a}, \frac{v}{b} \right)
  \]
Signal localization in image versus frequency space

Higher spatial frequencies

more localized

less localized

Magnitude spectrum

Decreasing high-frequency content
Fourier Transforms and Convolution

- Very useful!

\[ \mathcal{F}_{2D} \{ f(x,y) \ast g(x,y) \} = F(u,v)G(u,v) \]
- Proof (1-D)

\[
\mathcal{F} \{ f(x) \ast g(x) \} = \int_{-\infty}^{\infty} (f(x) \ast g(x)) e^{-j2\pi ux} \, dx \\
= \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} f(\xi)g(x-\xi) \, d\xi \right) e^{-j2\pi ux} \, dx = \int_{-\infty}^{\infty} f(\xi) \left( \int_{-\infty}^{\infty} g(x-\xi) e^{-j2\pi ux} \, dx \right) d\xi \\
= \int_{-\infty}^{\infty} f(\xi) \left( \int_{-\infty}^{\infty} \mathcal{F} \{ g(x-\xi) \} \, d\xi \right) d\xi = \int_{-\infty}^{\infty} f(\xi) \left( e^{-j2\pi u\xi} G(u) \right) d\xi \\
= G(u) \int_{-\infty}^{\infty} f(\xi)e^{-j2\pi u\xi} \, d\xi = F(u)G(u)
\]
Fourier transform pairs

<table>
<thead>
<tr>
<th>Signal</th>
<th>Fourier Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\delta(u, v)$</td>
</tr>
<tr>
<td>$\delta(x, y)$</td>
<td>1</td>
</tr>
<tr>
<td>$\delta(x - x_0, y - y_0)$</td>
<td>$e^{-j2\pi(ux_0 + vy_0)}$</td>
</tr>
<tr>
<td>$\delta_s(x, y; \Delta x, \Delta y)$</td>
<td>comb($u\Delta x, v\Delta y$)</td>
</tr>
<tr>
<td>$e^{j2\pi(u_0x + v_0y)}$</td>
<td>$\delta(u - u_0, v - v_0)$</td>
</tr>
<tr>
<td>sin [$2\pi (u_0x + v_0y)$]</td>
<td>$\frac{1}{2j} [\delta(u - u_0, v - v_0) - \delta(u + u_0, v + v_0)]$</td>
</tr>
<tr>
<td>cos [$2\pi (u_0x + v_0y)$]</td>
<td>$\frac{1}{2} [\delta(u - u_0, v - v_0) + \delta(u + u_0, v + v_0)]$</td>
</tr>
<tr>
<td>rect($x, y$)</td>
<td>sinc($u, v$)</td>
</tr>
<tr>
<td>sinc($x, y$)</td>
<td>rect($u, v$)</td>
</tr>
<tr>
<td>comb($x, y$)</td>
<td>comb($u, v$)</td>
</tr>
<tr>
<td>$e^{-\pi(x^2+y^2)}$</td>
<td>$e^{-\pi(u^2+v^2)}$</td>
</tr>
</tbody>
</table>

- Note the reciprocal symmetry in Fourier transform pairs
  - often 2-D versions can be calculated from 1-D versions by separability
  - In general: a broad extent in one domain corresponds to a narrow extent in the other domain
## Summary of key properties of the Fourier Transform

<table>
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<tr>
<th>Theorem</th>
<th>$f(x,y)$</th>
<th>$F(u,v)$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Similarity</strong></td>
<td>$f(ax,by)$</td>
<td>$\frac{1}{</td>
</tr>
<tr>
<td><strong>Addition</strong></td>
<td>$f(x,y) + g(x,y)$</td>
<td>$F(u,v) + G(u,v)$</td>
</tr>
<tr>
<td><strong>Shift</strong></td>
<td>$f(x-a, y-b)$</td>
<td>$e^{-2\pi i (au + bv)} F(u,v)$</td>
</tr>
<tr>
<td><strong>Modulation</strong></td>
<td>$f(x,y) \cos \omega x$</td>
<td>$\frac{1}{2} F\left(u + \frac{\omega}{2\pi}, v\right) + \frac{1}{2} F\left(u - \frac{\omega}{2\pi}, v\right)$</td>
</tr>
<tr>
<td><strong>Convolution</strong></td>
<td>$f(x,y) * g(x,y)$</td>
<td>$F(u,v)G(u,v)$</td>
</tr>
<tr>
<td><strong>Autocorrelation</strong></td>
<td>$f(x,y) * f^*(-x,-y)$</td>
<td>$</td>
</tr>
</tbody>
</table>

### Rayleigh
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x,y)|^2 \, dx \, dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(u,v)|^2 \, du \, dv
\]

### Power
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)g^*(x,y) \, dx \, dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u,v)G^*(u,v) \, du \, dv
\]

### Parseval
\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} |f(x,y)|^2 = \sum \sum a_{mn}^2,
\]
where $F(u,v) = \sum \sum a_{mn} \left[2\delta(u - m, v - n)\right]$

### Differentiation
\[
\left(\frac{\partial}{\partial x}\right)^m \left(\frac{\partial}{\partial y}\right)^n f(x,y) = (2\pi i u)^m (2\pi i v)^n F(u,v)
\]
Transfer Functions
Transfer Function for an LSI System

• Recall that for an LSI system

\[
f(x, y) \xrightarrow{\mathcal{F}} g(x, y)
\]

\[
g(x, y) = f(x, y) * h(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) h(\xi - x, \eta - y) \, d\xi \, d\eta
\]

• We can define the Transfer Function as the 2D Fourier transform of the PSF

\[
H(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\xi, \eta) e^{j2\pi(u\xi + v\eta)} \, d\xi \, d\eta = \mathcal{F}_{2D} \{h(x, y)\}
\]

• In this case the LSI imaging system can be simply described by:

\[
g(x, y) = f(x, y) * h(x, y) = \mathcal{F}_{2D}^{-1} \{F(u, v)H(u, v)\}
\]

• or

\[
G(u, v) = F(u, v)H(u, v)
\]

• which provides a very powerful tool for understanding systems
Illustration of transfer function $f(x,y) \rightarrow h(x,y) \rightarrow g(x,y)$

$f(x,y)$

Signal

2-D FT

$F(u,v)$

Magnitude spectrum

$a_1$

$a_2 > a_1$

$H(u,v) = ae^{-\pi a^2 (u^2 + v^2)}$

Inverse 2-D FT

$g(x,y)$

$G(u,v)$
X-ray Radiography
Definitions

• Ion: an atom or molecule in which the total number of electrons is not equal to the total number of protons, giving it a net positive or negative electrical charge

• Radiation: a process in which energetic particles or energetic waves travel through a medium or space
Ionizing Radiation

- Radiation (such as high energy electromagnetic photons behaving like particles) that is capable of ejecting orbital elections from atoms
- Can also be particles (e.g. electrons)
- Ionizing energy required is the binding energy for that electron's shell
- Energy units are electron volts (eV or keV), the energy of an electron accelerated by 1 volt
- For Hydrogen K orbital electrons, $E=13.6$ eV
- For Tungsten K orbital electrons, $E=69.5$ keV
- In medical imaging we need photons with enough energy to transmit through tissue so are in range of 25 keV to 511 keV and is thus ionizing
Electrons as Ionizing Radiation

- Electron kinetic energy \( E = \frac{(mv^2)}{2} \)
- Three main modes of interaction in the energy range we are considering
  a) Collision with other electrons and possible creation of delta-rays (high-energy electrons)
     - This is the most common mode and excited atoms lose energy by IR radiation (heat)
  b) Ejection of an inner orbital electron
     - This orbit is filled by an outer electron and the difference in energy is released as a 'characteristic x-ray'
  c) Bending of trajectory by nucleus
     - Since acceleration of a charged particle causes radiation, this causes 'braking radiation' or \textit{bremsstrahlung}
X-ray Spectrum from Electron Bombardment

When high energy electrons hit tungsten (symbol W), three effects occur:

1. Heat (> 99.9% of the energy)
2. Characteristic x-rays
3. Bremsstrahlung x-rays

Energies for Tungsten (W)

- 59.321 keV
- 69.081 keV

Energies for Tungsten (W)

- 57.984 keV
- 66.950 keV
- 67.244 keV
- 69.081 keV

Energies for Tungsten (W)

- 2 P
- 12 O
- 32 N
- 18 M
- 8 L
- 2 K

Occupancy & Shell

- 0.02
- 0.06
- 0.50
- 2.5
- 10.2
- 69.5
- 59.321 keV