Propagation of Errors—Basic Rules

See Chapter 3 in Taylor, *An Introduction to Error Analysis.*

1. If x and y have independent random errors δx and δy, then the error in $z = x + y$ is

$$\delta z = \sqrt{\delta x^2 + \delta y^2}.$$

2. If x and y have independent random errors δx and δy, then the error in $z = x \times y$ is

$$\frac{\delta z}{z} = \sqrt{\left(\frac{\delta x}{x}\right)^2 + \left(\frac{\delta y}{y}\right)^2}.$$

3. If $z = f(x)$ for some function $f()$, then

$$\delta z = |f'(x)|\delta x.$$

We will justify rule 1 later. The justification is easy as soon as we decide on a mathematical definition of δx, etc.

Rule 2 follows from rule 1 by taking logarithms:

$$\begin{aligned}
z &= x \times y \\
\log z &= \log x + \log y \\
\delta \log z &= \sqrt{(\delta \log x)^2 + (\delta \log y)^2} \\
\frac{\delta z}{z} &= \sqrt{\left(\frac{\delta x}{x}\right)^2 + \left(\frac{\delta y}{y}\right)^2}
\end{aligned}$$

where we used

$$\delta \log X = \frac{\delta X}{X},$$

the calculus formula for the derivative of the logarithm.

Rule 3 is just the definition of derivative of a function f.

Quick Check 3.4

Problem: To find the volume of a certain cube, you measure its side as $2.00 \pm 0.02\,\text{cm}$. Convert this uncertainty to a percent and then find the volume with its uncertainty.

Solution: The volume V is given in terms of the side s by

$$V = s^3,$$

so the uncertainty in the volume is, by rule 3,

$$\delta V = 3s^2\,\delta s = 0.24,$$

and the volume is $8.0 \pm 0.2\,\text{cm}^3$.

Quick Check 3.8

Problem: If you measure x as 100 ± 6, what should you report for $\sqrt{x}$, with its uncertainty?

Solution: Use rule 3 with $f(x) = \sqrt{x}$, $f'(x) = 1/(2\sqrt{x})$, so the uncertainty in $\sqrt{x}$ is

$$\frac{\delta x}{2\sqrt{x}} = \frac{6}{2 \times 10} = 0.3$$

and we would report $\sqrt{x} = 10.0 \pm 0.3$.

We cannot solve this problem by indirect use of rule 2. You might have thought of using $x = \sqrt{x} \times \sqrt{x}$, so

$$\frac{\delta x}{x} = \sqrt{2}\,\frac{\delta\sqrt{x}}{\sqrt{x}}$$

and

$$\delta\sqrt{x} = \frac{\delta x}{\sqrt{2x}},$$

which leads to $\sqrt{x} = 10 \pm 0.4$. The fallacy here is that the two factors $\sqrt{x}$ have the *same* errors, and the addition in quadrature rule requires that the various errors be *independent*.

Quick Check 3.9

Problem: Suppose you measure three numbers as follows:

$$x = 200 \pm 2, \quad y = 50 \pm 2, \quad z = 40 \pm 2,$$

where the three uncertainties are independent and random. Use step-by-step propagation to find the quantity $q = x/(y - z)$ with its uncertainty.

Solution: Let $D = y - z = 10 \pm 2\sqrt{2} = 10 \pm 3$. Then

$$q = \frac{x}{D} = 20 \pm 20\sqrt{0.01^2 + 0.3^2} = 20 \pm 6.$$

General Formula for Error Propagation

We measure $x_1, x_2 \ldots x_n$ with uncertainties $\delta x_1, \delta x_2 \ldots \delta x_n$. The purpose of these measurements is to determine q, which is a function of $x_1, \ldots, x_n$:

$$q = f(x_1, \ldots, x_n).$$

The uncertainty in q is then

$$\delta q = \sqrt{\left(\frac{\partial q}{\partial x_1}\delta x_1\right)^2 + \ldots + \left(\frac{\partial q}{\partial x_n}\delta x_n\right)^2}$$

If

$$q = x_1 + x_2,$$

we recover rule 1:

$$\begin{aligned} \frac{\partial q}{\partial x_1} &= 1, \\ \frac{\partial q}{\partial x_2} &= 1, \\ \delta q &= \sqrt{\delta x_1^2 + \delta x_2^2} \end{aligned}$$

If

$$q = x_1 \times x_2,$$

we recover rule 2:

$$\begin{aligned}
\frac{\partial q}{\partial x_1} &= x_2, \\
\frac{\partial q}{\partial x_2} &= x_1, \\
\delta q &= \sqrt{x_2^2\,\delta x_1^2 + x_1^2\,\delta x_2^2} \\
&= \sqrt{q^2\left[\left(\frac{\delta x_1}{x_1}\right)^2 + \left(\frac{\delta x_2}{x_2}\right)^2\right]} \\
\frac{\delta q}{q} &= \sqrt{\left(\frac{\delta x_1}{x_1}\right)^2 + \left(\frac{\delta x_2}{x_2}\right)^2}
\end{aligned}$$

Problem 3.47

The Atwood machine consists of two masses M and m (with $M > m$) attached to the ends of a light string that passes over a light, frictionless pulley. When the masses are released, the mass M is easily shown to accelerate down with an acceleration

$$a = g\,\frac{M - m}{M + m}.$$

Suppose that M and m are measured as $M = 100 \pm 1$ and $m = 50 \pm 1$, both in grams. Find the uncertainty δa.

Solution: The partial derivatives are

$$\begin{aligned}\frac{\partial a}{\partial M} &= g\,\frac{(M+m)-(M-m)}{(M+m)^2} \\ &= \frac{2mg}{(M+m)^2}, \\ \frac{\partial a}{\partial m} &= g\,\frac{-(M+m)-(M-m)}{(M+m)^2} \\ &= -\frac{2Mg}{(M+m)^2}\end{aligned}$$

so we have

$$\delta a = \frac{2g}{(M+m)^2}\sqrt{m^2\delta M^2 + M^2\delta m^2}.$$

Now put the numbers in to get

$$a = \frac{9.8\times 50}{150} = 3.27\,\mathrm{m/s^2}$$

and

$$\delta a = \frac{2 \times 9.8}{150^2} \sqrt{50^2 + 100^2} = 0.097$$

so our answer is

$$a = 3.3 \pm 0.1 \, \mathsf{m/s}^2.$$

Problem 3.49

If an object is placed at a distance p from a lens and an image is formed at a distance q from the lens, the lens's focal length can be found as

$$f = \frac{pq}{p+q}. \tag{1}$$

(a) Use the general rule to derive a formula for the uncertainty δf in terms of p, q, and their uncertainties.

(b) Starting from (1) directly, you cannot find δf in steps because p and q both appear in numerator and denominator. Show, however, that f can be rewritten as

$$f = \frac{1}{(1/p) + (1/q)}.$$

Starting from this form, you *can* evaluate δf in steps. Do so, and verify that you get the same answer as in part (a).

Solution:

(a) The partial derivatives are

$$\begin{aligned} \frac{\partial f}{\partial p} &= \frac{q(p+q) - pq}{(p+q)^2} = \frac{q^2}{(p+q)^2}, \\ \frac{\partial f}{\partial q} &= \frac{p(p+q) - pq}{(p+q)^2} = \frac{p^2}{(p+q)^2}. \end{aligned}$$

Therefore

$$\delta f = \frac{\sqrt{q^4 \delta p^2 + p^4 \delta q^2}}{(p+q)^2}.$$

(b) The uncertainty in $1/p$ is $\delta p/p^2$, and the uncertainty in $1/q$ is $\delta q/q^2$. The uncertainty in

$$\frac{1}{p}+\frac{1}{q}$$

is

$$\sqrt{\left(\frac{\delta p}{p^2}\right)^2+\left(\frac{\delta q}{q^2}\right)^2},$$

which is a relative uncertainty of

$$\frac{1}{\frac{1}{p}+\frac{1}{q}}\sqrt{\left(\frac{\delta p}{p^2}\right)^2+\left(\frac{\delta q}{q^2}\right)^2}.$$

The relative uncertainty in f, as given by (1), is the same, so the absolute uncertainty in f is

$$\begin{aligned} \delta f &= f^2 \sqrt{\left(\frac{\delta p}{p^2}\right)^2 + \left(\frac{\delta q}{q^2}\right)^2} \\ &= \frac{1}{(p+q)^2} \sqrt{q^4 \delta p^2 + p^4 \delta q^2}, \end{aligned}$$

exactly as in part (a).

Problem 3.50

Suppose that you measure three independent variables as

$$x = 10 \pm 2, \quad y = 7 \pm 1, \quad \theta = 40 \pm 3^\circ$$

and use these values to compute

$$q = \frac{x+2}{x + y\cos 4\theta}. \tag{2}$$

What should be your answer for q and its uncertainty?

Solution: Find the partial derivatives, using θ in radians:

$$\frac{\partial q}{\partial x} = \frac{x + y\cos 4\theta - (x+2)}{(x + y\cos 4\theta)^2}$$

$$
\begin{aligned}
&= \frac{y\cos 4\theta - 2}{(x + y\cos 4\theta)^2} = -0.732 \\
\frac{\partial q}{\partial y} &= -\frac{(x+2)\cos 4\theta}{(x + y\cos 4\theta)^2} = 0.963 \\
\frac{\partial q}{\partial \theta} &= \frac{4(x+2)y\sin 4\theta}{(x + y\cos 4\theta)^2} = 9.813.
\end{aligned}
$$

So we have $q = 3.507$ and

$$
\begin{aligned}
\delta q^2 &= (0.732 \times 2)^2 \\
&\quad +(0.963 \times 1)^2 \\
&\quad +(9.813 \times 3 \times \pi/180)^2 \\
&= 3.3.
\end{aligned}
$$

Our answer is $q = 3.5 \pm 2$.

Alternate Solution

Rewrite (2) in the form

$$\frac{1}{q} = 1 + \frac{y\cos 4\theta - 2}{x+2}.$$

We can find the uncertainty in $1/q$, and therefore in q by the simple step-by-step procedure.

1. The relative uncertainty in $y\cos 4\theta$ is

$$\Delta_1 = \sqrt{\left(\frac{\delta y}{y}\right)^2 + \left(\frac{4\delta\theta \sin 4\theta}{\cos 4\theta}\right)^2} = 0.16.$$

2. The absolute uncertainty in $y\cos 4\theta$ is

$$\Delta_2 = |y\cos 4\theta| \times \Delta_1 = 1.1.$$

3. The relative uncertainty in $1/q - 1$ is

$$\begin{aligned}\Delta_3 &= \sqrt{\left(\frac{\Delta_2}{y\cos 4\theta - 2}\right)^2 + \left(\frac{\delta x}{x+2}\right)^2} \\ &= 0.21.\end{aligned}$$

4. The absolute uncertainty in $1/q - 1$ is

$$\Delta_4 = |1/q - 1| \times \Delta_3 = 0.15,$$

which is also the absolute uncertainty in $1/q$.

5. The relative uncertainty in $1/q$ is $q \times \Delta_4$, which is also the relative uncertainty in q. Therefore the absolute uncertainty in q is

$$\delta q = q^2 \times \Delta_4 = 2.$$