Name ____________________

Final Exam Solutions Physics 505

December 14, 2010

This exam is closed book but useful equations and relations are supplied as an attachment.

The first question requires only brief answers with little calculation but some explanation. The last four require calculations. In all cases you are encouraged to (briefly) explain your reasoning (I can give full marks only when I can understand what you did). The noted point values are for reference only and are subject to (minor) fine tuning. I note below the average score on each question.

Exercise 1	15.1	(17)
Exercise 2	8.9	(18)
Exercise 3	16.0	(23)
Exercise 4	12.6	(18)
Exercise 5	16.4	(24)
TOTAL	69.0	(100)

Possibly useful general formulae:

Newton's Law

$$\frac{d\left(m\dot{\vec{r}}\right)}{dt}=\vec{F}$$

Cylindrical coordinates

$$\begin{aligned}
&\vec{r}=\rho\hat{\rho}+z\hat{z},\\
&d\vec{r}=d\rho\hat{\rho}+\rho d\phi\hat{\phi}+dz\hat{z},\\
&\dot{\vec{r}}=\dot{\rho}\hat{\rho}+\rho\dot{\phi}\hat{\phi}+\dot{z}\hat{z},\\
&\ddot{\vec{r}}=\ddot{\rho}\hat{\rho}-\rho\dot{\phi}^{2}\hat{\rho}+2\dot{\rho}\dot{\phi}\hat{\phi}+\rho\ddot{\phi}\hat{\phi}+\ddot{z}\hat{z}
\end{aligned}$$

Spherical coordinates

$$\begin{aligned}
&\vec{r}=r\hat{r},\\
&d\vec{r}=dr\hat{r}+rd\theta\hat{\theta}+r\sin\theta d\phi\hat{\phi},\\
&\dot{\vec{r}}=\dot{r}\hat{r}+r\dot{\theta}\hat{\theta}+r\sin\theta\dot{\phi}\hat{\phi},\\
&\ddot{\vec{r}}=\left(\ddot{r}-r\dot{\theta}^{2}-r\sin^{2}\theta\dot{\phi}^{2}\right)\hat{r}+\left(2\dot{r}\dot{\theta}+r\ddot{\theta}-r\sin\theta\cos\theta\dot{\phi}^{2}\right)\hat{\theta}\\
&\qquad+\left(2\dot{r}\sin\theta\dot{\phi}+2r\dot{\theta}\cos\theta\dot{\phi}+r\sin\theta\ddot{\phi}\right)\hat{\phi}
\end{aligned}$$

Newton in a rotating frame ($\vec{r}_0$ is the vector from the origin of the inertial frame to the origin of the rotating frame)

$$\begin{aligned}
m\left.\frac{d^{2}\vec{r}'(t)}{dt^{2}}\right|_{\text{acceler}}=\vec{F}_{\text{inertial}}-m\left.\frac{d^{2}\vec{r}_{0}(t)}{dt^{2}}\right|_{\text{inertial}}-2m\vec{\omega}\times\left.\frac{d\vec{r}'(t)}{dt}\right|_{\text{acceler}}\\
-m\frac{d\vec{\omega}}{dt}\times\vec{r}'(t)-m\vec{\omega}\times\left[\vec{\omega}\times\vec{r}'(t)\right]
\end{aligned}$$

Lagrange's equation for a conservative system in generalized coordinates

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{q}_{k}}-\frac{\partial L}{\partial q_{k}}=0$$

Lagrange with constraints specified by functions $\phi_l\left(q_k\right)=0$

$$\begin{aligned}
&\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k}-\frac{\partial L}{\partial q_k}=a_{kl}\lambda_l, \\
&a_{kl}=\frac{\partial \phi_l}{\partial q_k}
\end{aligned}$$

Hamilton's equations in canonical coordinates $\left(p_k=\partial L/\partial \dot{q}_k\right.$ for U independent of $\left.\dot{q}\right)$

$$\begin{aligned}
\dot{q}_k &= \frac{\partial H}{\partial p_k} \\
\dot{p}_k &= -\frac{\partial H}{\partial q_k} \\
\frac{\partial H}{\partial t} &= -\frac{\partial L}{\partial t} \\
H &= \sum_k p_k \dot{q}_k - L
\end{aligned}$$

We defined the small oscillation problem around an equilibrium point in terms of

$$\begin{aligned}
U\left(q\right) &\simeq U\left(\overline{q}\right)+\frac{1}{2}\sum_{k,l}\left.\frac{\partial^2 U}{\partial q_k \partial q_l}\right|_{q=\overline{q}}\delta q_k \delta q_l \\
&\equiv U\left(\overline{q}\right)+\frac{1}{2}\sum_{k,l} v_{kl}\delta q_k \delta q_l, \\
T\left(p\right) &\simeq \frac{1}{2}\sum_{kl} M_{kl}\left(\overline{q}_1,\ldots,\overline{q}_f\right)\delta \dot{q}_k \delta \dot{q}_l
\end{aligned}$$

where the eigenvalues are defined by

$$\det\left[v-\omega^2 M\right]=0$$

and the normal modes by (η is a vector of the perturbations δq about the equilibrium point)

$$\omega^2 M\eta = v\eta$$

Moment of inertia tensor

$$\begin{aligned}
\overleftrightarrow{I} &= \sum_{j=1}^{N} m_j \left(r_j^2 \overleftrightarrow{\mathbf{1}} - \overleftarrow{r}_j \vec{r}_j \right) \left[I = \sum_{j=1}^{N} m_j \left(\tilde{r}_j\, r_j \mathbf{1} - r_j \tilde{r}_j \right) \right], \\
I_{kl} &= \sum_{j=1}^{N} m_j \left(r_j^2 \delta_{kl} - r_{j,k} r_{j,l} \right), \\
I_{kl} &= \int d^3 r \rho\left(\vec{r}\right)\left(r^2 \delta_{kl} - r_k r_l \right),
\end{aligned}$$

Moment of inertia of disk of radius *r* and mass *m* about axis of symmetry

$$I_{disk} = \frac{mr^2}{2}$$

Moment of inertia of rod of length *l* and mass *m* about axis through CM and orthogonal to length (assuming rod is much longer than it is thick)

$$I_{rod} = \frac{ml^2}{12}$$

Moment of inertia of a hoop of radius *r* and mass *m* about the axis of symmetry

$$I_{hoop} = mr^2$$

Euler's equations in body-fixed frame

$$\begin{aligned}
I_1 \dot{\omega}_1' - \omega_2' \omega_3' \left(I_2 - I_3 \right) &= N_1', \\
I_2 \dot{\omega}_2' - \omega_3' \omega_1' \left(I_3 - I_1 \right) &= N_2', \\
I_3 \dot{\omega}_3' - \omega_1' \omega_2' \left(I_1 - I_2 \right) &= N_3'
\end{aligned}$$

Body-fixed angular frequency in terms of Euler angles

$$\begin{aligned}
\omega_1 &= \hat{e}_1 \cdot \vec{\omega} = \dot{\alpha}\hat{e}_1 \cdot \hat{e}_3^0 + \dot{\beta}\hat{e}_1 \cdot \hat{f}_2 + \dot{\gamma}\hat{e}_1 \cdot \hat{g}_3 \\
&= -\dot{\alpha}\sin\beta\cos\gamma + \dot{\beta}\sin\gamma \\
\omega_2 &= \hat{e}_2 \cdot \vec{\omega} = \dot{\alpha}\hat{e}_2 \cdot \hat{e}_3^0 + \dot{\beta}\hat{e}_2 \cdot \hat{f}_2 + \dot{\gamma}\hat{e}_2 \cdot \hat{g}_3 \\
&= \dot{\alpha}\sin\beta\sin\gamma + \dot{\beta}\cos\gamma \\
\omega_3 &= \hat{e}_3 \cdot \vec{\omega} = \dot{\alpha}\hat{e}_3 \cdot \hat{e}_3^0 + \dot{\beta}\hat{e}_3 \cdot \hat{f}_2 + \dot{\gamma}\hat{e}_3 \cdot \hat{g}_3 \\
&= \dot{\alpha}\cos\beta + \dot{\gamma}.
\end{aligned}$$

For central force problems and the motion of a top we defined the quantity Φ to be the change in the azimuthal angle ϕ that occurs in ½ cycle of the radial motion (central force problem) or of the polar angle (θ) motion (top). Then the winding number is defined to be

$$W = \frac{\Phi}{2\pi}$$

Liapunov exponent for the map $x_n = F\left(x_{n-1}\right)$

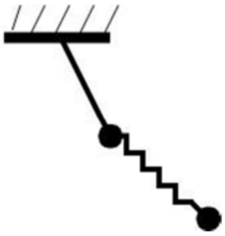
$$\lambda = \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \ln\left(\left|F'\left(x_j\right)\right|\right)$$

(Capacity) Dimension for “object” composed of $N\left(\varepsilon\right)$ elements of “linear” size ε

$$d_C = \lim_{\varepsilon\to 0} \frac{\ln\left(N\left(\varepsilon\right)\right)}{\ln L - \ln\varepsilon} = \lim_{\varepsilon\to 0} \frac{\ln\left(N\left(\varepsilon\right)\right)}{\ln\left(1/\varepsilon\right)}$$

1) (13 pts) The various parts to the first question require only brief answers. In each case provide a brief explanation of your reasoning.

a) (3 pts) Consider the compound pendulum in the figure composed of a rigid, massless rod and a massless spring suspended at one end of the rod in a uniform gravitational field. There are two masses, one at the point where the rod and spring are connected and one at the end of the spring. The joints at the suspension point and where the rod meets the spring are free to rotate in all directions. How many generalized coordinates are required to describe the motion of the system? Explain.



Solution: 5 - To describe the location of 2 masses we need 6 generalized coordinates. However, the rod supplies one constraint (the upper mass moves on the surface of a sphere). The spring does not give a constraint. (Note that the pendulum is not confined to a plane.)

b) (3 pts) Consider a flow in phase space defined by the flow field

$$\dot{x} = V_x\left(x, y\right) = y,$$
$$\dot{y} = V_y\left(x, y\right) = x.$$

Is this flow conservative or non-conservative? Why?

Solution: The question posed is answered by considering the divergence of the flow field, where, for conservative flow, we require

$$\vec{\nabla} \cdot \vec{V} = 0.$$

Hence we have

$$\vec{\nabla} \cdot \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0,$$

And the flow is conservative.

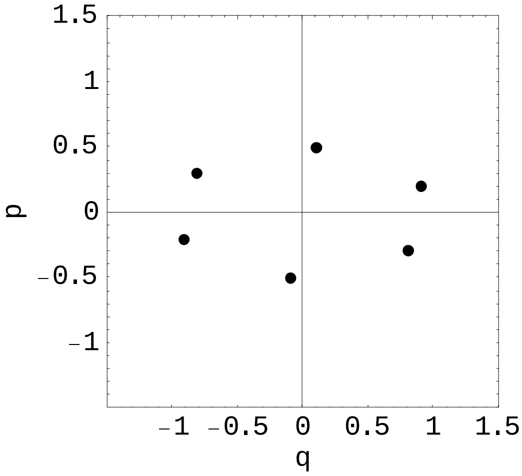
c) (8 pts) It is claimed that each of the following mechanical systems are displaying chaotic motion. For each system indicate whether this claim it is true, false or ambiguous (*i.e.*, not enough information to determine if true or false), and provide a reason for your conclusion?

A) A damped, driven simple harmonic oscillator (SHO).

B) A damped, driven pendulum.

C) A mechanical system all of whose Liapunov exponents are negative.

D) A mechanical system whose Poincaré sections look as in the figure to the right.



Solution: For chaotic behavior we only know that we require > 2-dimensional phase space. We also need nonlinear dynamics, at least one positive Liapunov exponent and aperiodic motion. So we have A) false (no nonlinearity), B) ambiguous (can exhibit both non-chaotic and chaotic depending on the specific parameters, C) false (no positive exponents) and D) false (motion is periodic).

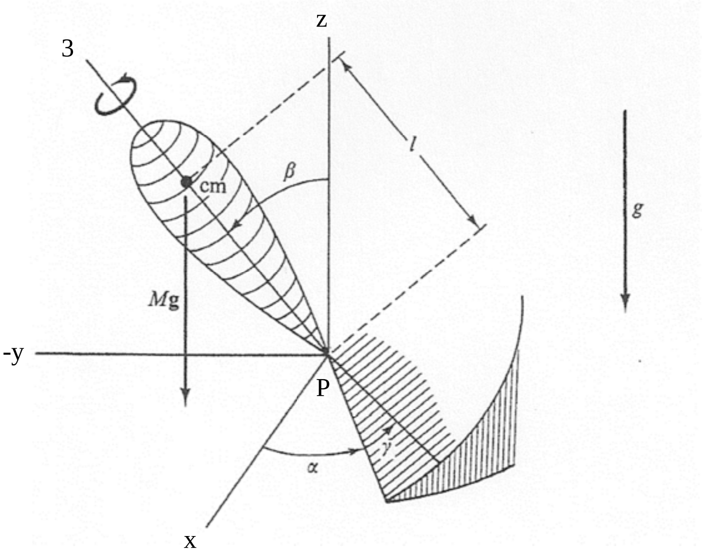
d) (3 pts) Consider the new, improved Cantor set illustrated in the figure, *i.e.*, cut out the middle 2/3 of each segment (leaving 1/6 at each end) at each step. Find its (capacity) dimension. Explain your work.

Solution: By inspection we have $N(\varepsilon) = N(n) = 2^{n-1}$, $\varepsilon = \left(1/6\right)^{n-1}$. $\vdots$

Thus we find

$$d = \lim_{n\to\infty} \frac{\ln\left(2^{n-1}\right)}{-\ln\left(\left(1/6\right)^{n-1}\right)} = \frac{\ln 2}{\ln 6} = 0.38685\ldots.$$

2) (18 pts) Consider the symmetric top indicated in the figure. It is supported in a uniform gravitational field at the bottom point P and spins without friction. The principle moments of inertia, evaluated with respect to axes through the point P (not the CM), are given by $I_1 = I_2 > I_3$ where the 3 axis is the long axis of symmetry indicated in the figure. At time $t = 0$ we have initial conditions



$$\alpha(0) = 0, \frac{d\alpha}{dt}(0) = \dot{\alpha}_0 \neq 0,$$

$$\gamma(0) = 0, \frac{d\gamma}{dt}(0) = \dot{\gamma}_0 \neq 0,$$

$$\beta(0) = \beta_0 \neq 0, \frac{d\beta}{dt}(0) = 0.$$

Comments: This exercise was intended to test our command of the use of Euler angles. The structure is taken directly from the Lagrange's top example in the text (where the figure above appears) and Lecture 10 with the definition of the angles exactly as in the "helpful" equations. Further it represents the nearly the simplest possible motion of such a top – precession but no nutation. However, the second and third parts of the question, which involved understanding the relation between the body fixed frame and the inertial frame (arguably the whole point of using Euler angles) was too large a challenge for most students.

a) (6 pts) Determine the total angular velocity $(\vec{\omega})$ and the total angular momentum vector $(\vec{L})$ of the top at $t = 0$. You may express your answer as Cartesian components (with the x-y-z unit vectors in the figure, *i.e.*, the inertial frame) or in terms of some other unit vectors of your choice (not necessarily orthogonal). The result should be expressed in terms of the parameters and angles defined above. Explain your work (including indicating your unit vectors in the figure).

Solution: We can proceed by noticing that the Euler angles in the help section are just the angles above. Thus, at $t = 0$, the angular velocity components (in non-orthogonal coordinates) are seen to be composed of the rotations about the z and 3 axes,

$$\vec{\omega} = \dot{\alpha}\hat{e}_z + \dot{\gamma}\hat{e}_3.$$

Thus in the body fixed 1-2-3 frame we have

$$\begin{aligned}
\omega_1 &= \hat{e}_1 \cdot \vec{\omega} = \dot{\alpha}\hat{e}_1 \cdot \hat{e}_3^0 + \dot{\beta}\hat{e}_1 \cdot \hat{f}_2 + \dot{\gamma}\hat{e}_1 \cdot \hat{g}_3 \\
&= -\dot{\alpha}_0 \sin\beta_0 \\
\omega_2 &= \hat{e}_2 \cdot \vec{\omega} = \dot{\alpha}\hat{e}_2 \cdot \hat{e}_3^0 + \dot{\beta}\hat{e}_2 \cdot \hat{f}_2 + \dot{\gamma}\hat{e}_2 \cdot \hat{g}_3 \\
&= 0 \\
\omega_3 &= \hat{e}_3 \cdot \vec{\omega} = \dot{\alpha}\hat{e}_3 \cdot \hat{e}_3^0 + \dot{\beta}\hat{e}_3 \cdot \hat{f}_2 + \dot{\gamma}\hat{e}_3 \cdot \hat{g}_3 \\
&= \dot{\alpha}_0 \cos\beta_0 + \dot{\gamma}_0.
\end{aligned}$$

In this notation the unit vector $\hat{e}_1$ (at $t = 0$) is the unit vector perpendicular to $\hat{e}_3$ and lying in the x-z plane (with a negative z component, while $\hat{e}_2 \parallel \hat{e}_y$). Thus the angular momentum is given most easily in the body-frame where the moment of inertia is diagonal and we have (note that to use non-orthogonal unit vectors here you must know the form of the moment of inertia in those coordinates)

$$\vec{L}\left(t=0\right) = \vec{\vec{I}} \cdot \vec{\omega} = -I_1\dot{\alpha}_0 \sin\beta_0\hat{e}_1 + I_3\left(\dot{\alpha}_0\cos\beta_0 + \dot{\gamma}_0\right)\hat{e}_3.$$

In terms of the inertial frame unit vectors (at $t = 0$) we have

$$\begin{aligned}
&\hat{e}_1 = \cos\beta_0\hat{e}_x - \sin\beta_0\hat{e}_z,\ \hat{e}_3 = \cos\beta_0\hat{e}_z + \sin\beta_0\hat{e}_x \\
&\Rightarrow \vec{L}\left(t=0\right) = \left(\left(I_3 - I_1\right)\dot{\alpha}_0 \sin\beta_0\cos\beta_0 + I_3\dot{\gamma}_0\sin\beta_0\right)\hat{e}_x \\
&\qquad + \left(\dot{\alpha}_0\left(I_1\sin^2\beta_0 + I_3\cos^2\beta_0\right) + I_3\dot{\gamma}_0\cos\beta_0\right)\hat{e}_z.
\end{aligned}$$

b) (6 pts) Assuming that in the subsequent motion $\beta_0, \dot{\alpha}_0$ and $\dot{\gamma}_0$ (and gravity) all remain constant (in time), write down an expression for $\vec{L}\left(t\right)$ in terms of the (fixed) x-y-z unit vectors in the inertial frame. Explain your work. HINT: It may be useful to consider what quantities are constants of the motion.

<u>Solution</u>: This was meant to lead us to think in the inertia frame where the torque due to gravity is easiest to evaluate. Given the uniform motion the only time variation is in the x (and y) component(s) above as the top precesses. Recall from the discussion

in class that the magnitudes of the components of $\vec{L}(t)$ along both $\hat{e}_3$ and $\hat{e}_z$ are constant when only gravity acts. Here the only time variation is the precession, *i.e.*, no nutation,

$$\hat{e}_3 = \hat{e}_z \cos\beta_0 + \sin\beta_0 \left(\hat{e}_x \cos\dot{\alpha}_0 t + \hat{e}_y \sin\dot{\alpha}_0 t\right).$$

So the *z* component is constant and the (orthogonal) component in the *x*-*y* plane precesses,

$$\begin{aligned}\vec{L}(t) = {} & \left(\left(I_3 - I_1\right)\dot{\alpha}_0 \cos\beta_0 + I_3\dot{\gamma}_0\right)\sin\beta_0 \left(\hat{e}_x \cos\dot{\alpha}_0 t + \hat{e}_y \sin\dot{\alpha}_0 t\right) \\ & + \left(\dot{\alpha}_0\left(I_1 \sin^2\beta_0 + I_3 \cos^2\beta_0\right) + I_3\dot{\gamma}_0 \cos\beta_0\right)\hat{e}_z.\end{aligned}$$

For future reference the time dependence of the angular momentum (in the setup of this problem) is

$$\dot{\vec{L}}(t) = \left(\left(I_3 - I_1\right)\dot{\alpha}_0 \cos\beta_0 + I_3\dot{\gamma}_0\right)\sin\beta_0\dot{\alpha}_0 \left(-\hat{e}_x \sin\dot{\alpha}_0 t + \hat{e}_y \cos\dot{\alpha}_0 t\right).$$

Note that we can return to the Euler angle equations and write down the time dependent angular velocity in the 1-2-3 frame

$$\begin{aligned}\omega_1(t) &= \hat{e}_1 \cdot \vec{\omega} = \dot{\alpha}\hat{e}_1 \cdot \hat{e}_3^0 + \dot{\beta}\hat{e}_1 \cdot \hat{f}_2 + \dot{\gamma}\hat{e}_1 \cdot \hat{g}_3 \\ &= -\dot{\alpha}_0 \sin\beta_0 \cos\dot{\gamma}_0 t \\ \omega_2(t) &= \hat{e}_2 \cdot \vec{\omega} = \dot{\alpha}\hat{e}_2 \cdot \hat{e}_3^0 + \dot{\beta}\hat{e}_2 \cdot \hat{f}_2 + \dot{\gamma}\hat{e}_2 \cdot \hat{g}_3 \\ &= \dot{\alpha}_0 \sin\beta_0 \sin\dot{\gamma}_0 t \\ \omega_3(t) &= \hat{e}_3 \cdot \vec{\omega} = \dot{\alpha}\hat{e}_3 \cdot \hat{e}_3^0 + \dot{\beta}\hat{e}_3 \cdot \hat{f}_2 + \dot{\gamma}\hat{e}_3 \cdot \hat{g}_3 \\ &= \dot{\alpha}_0 \cos\beta_0 + \dot{\gamma}_0 = \text{ constant.}\end{aligned}$$

Thus we can also write

$$\begin{aligned}\vec{L}(t) = {} & \dot{\alpha}_0 \sin\beta_0 I_1 \left(-\hat{e}_1 \cos\dot{\gamma}_0 t + \hat{e}_2 \sin\dot{\gamma}_0 t\right) \\ & + I_3\left(\dot{\alpha}_0 \cos\beta_0 + \dot{\gamma}_0\right)\hat{e}_3.\end{aligned}$$

c) (6 pts) Determine the equation relating β_0, $\dot{\alpha}_0$ and $\dot{\gamma}_0$ (and gravity) that must be satisfied (for all *t*) in order for the angle β and the 2 angular velocities ($\dot{\alpha}_0$ and $\dot{\gamma}_0$) to remain constant in magnitude (*i.e.*, independent of time). You do **not** need to solve this equation. Explain your work.

Solution: The important issue is the torque due to gravity, which has magnitude $Mgl\sin\beta_0$ and, as a vector, is parallel to the vector $\hat{e}_3 \times -\hat{e}_z$, which, at $t = 0$, is just $\hat{e}_y$. If the motion is uniform as required, we can express the torque in the inertial frame as

$$\vec{N}_g = Mgl\sin\beta_0\left(\cos\dot{\alpha}_0 t\ \hat{e}_y - \sin\dot{\alpha}_0 t\ \hat{e}_x\right),$$

and thus, using the time derivative in the previous section, the required result

$$\begin{aligned}
&\dot{\vec{L}} = \vec{N}_g \\
&\Rightarrow \left(\left(I_3 - I_1\right)\dot{\alpha}_0\cos\beta_0 + \dot{\gamma}_0\right)\sin\beta_0\dot{\alpha}_0\left(\cos\dot{\alpha}_0 t\ \hat{e}_y - \sin\dot{\alpha}_0 t\ \hat{e}_x\right) \\
&\qquad = Mgl\sin\beta_0\left(\cos\dot{\alpha}_0 t\ \hat{e}_y - \sin\dot{\alpha}_0 t\ \hat{e}_x\right) \\
&\Rightarrow \left(I_3 - I_1\right)\dot{\alpha}_0^2\cos\beta_0 + I_3\dot{\gamma}_0\dot{\alpha}_0 = Mgl.
\end{aligned}$$

It may seem natural to use Euler's equations in the body-frame. In general, the difficulty with this approach is expressing the torque in that frame without already solving for the motion. Here, since the motion is so simple, we can proceed that way. The essential point is to express the torque in the 1-2-3 frame using the knowledge that the torque must be orthogonal to both the 3 and the *z* directions (and also to the direction of the angular momentum). Thus we have

$$\vec{N}_g = Mgl\sin\beta_0\left(\sin\dot{\gamma}_0 t\ \hat{e}_1 + \cos\dot{\gamma}_0 t\ \hat{e}_2\right).$$

Next we note that with a fixed value of the angle β we have the derivatives

$$\begin{aligned}
\dot{\omega}_1 &= \dot{\alpha}_0\dot{\gamma}_0\sin\beta_0\sin\dot{\gamma}_0 t, \\
\dot{\omega}_2 &= \dot{\alpha}_0\dot{\gamma}_0\sin\beta_0\cos\dot{\gamma}_0 t, \\
\dot{\omega}_3 &= 0.
\end{aligned}$$

So Euler's equations become (recall $I_1 = I_2$)

$$
\begin{aligned}
&I_1\dot{\omega}_1-\omega_2\omega_3\left(I_1-I_3\right)=N_1=Mgl\sin\beta_0\sin\dot{\gamma}_0 t\\
&\quad=I_1\dot{\alpha}_0\dot{\gamma}_0\sin\beta_0\sin\dot{\gamma}_0 t-\dot{\alpha}_0\sin\beta_0\sin\dot{\gamma}_0 t\left(\dot{\alpha}_0\cos\beta_0+\dot{\gamma}_0\right)\left(I_1-I_3\right)\\
&\Rightarrow I_3\dot{\alpha}_0\dot{\gamma}_0+\left(I_3-I_1\right)\dot{\alpha}_0^2\cos\beta_0=Mgl.
\end{aligned}
$$

This is just the required result above. The second equation of Euler produces the same result,

$$
\begin{aligned}
&I_1\dot{\omega}_2-\omega_3\omega_1\left(I_3-I_1\right)=N_2=Mgl\sin\beta_0\cos\dot{\gamma}_0 t\\
&\quad=I_1\dot{\alpha}_0\dot{\gamma}_0\sin\beta_0\cos\dot{\gamma}_0 t+\dot{\alpha}_0\sin\beta_0\cos\dot{\gamma}_0 t\left(\dot{\alpha}_0\cos\beta_0+\dot{\gamma}_0\right)\left(I_3-I_1\right)\\
&\Rightarrow I_3\dot{\alpha}_0\dot{\gamma}_0+\left(I_3-I_1\right)\dot{\alpha}_0^2\cos\beta_0=Mgl.
\end{aligned}
$$

Finally the third equation is trivial, 0 = 0.

An alternative approach is to think in terms of the effective potential, as in Lecture 10 (recall Eq, (10.36))

$$
\begin{aligned}
V_{eff}\left(\beta\right)&=\frac{\left(L_3^0-L_3\cos\beta\right)^2}{2I_1\sin^2\beta}+\frac{L_3^2}{2I_3}+Mgl\cos\beta,\\
&=\frac{\left(p_\alpha-p_\gamma\cos\beta\right)^2}{2I_1\sin^2\beta}+\frac{p_\gamma^2}{2I_3}+Mgl\cos\beta,
\end{aligned}
$$

with conserved angular momenta

$$
\begin{aligned}
p_\alpha&=I_1\dot{\alpha}_0\sin^2\beta_0+I_3\cos\beta_0\left(\dot{\alpha}_0\cos\beta_0+\dot{\gamma}_0\right)\\
&=L_3^0=\text{ constant},\\
p_\gamma&=I_3\left(\dot{\alpha}_0\cos\beta_0+\dot{\gamma}_0\right)=I_3\omega_3=L_3=\text{ constant}.
\end{aligned}
$$

The constraint of a constant β is then

$$\left.\frac{\partial V_{eff}}{\partial\beta}\right|_{\beta=\beta_0}=0$$

$$\Rightarrow Mgl\sin\beta = p_\gamma \sin\beta\,\frac{\left(p_\alpha - p_\gamma\cos\beta\right)}{I_1\sin^2\beta} - \cos\beta\,\frac{\left(p_\alpha - p_\gamma\cos\beta\right)^2}{I_1\sin^3\beta}$$

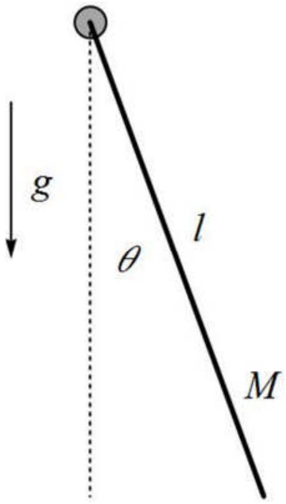
$$\Rightarrow p_\alpha - p_\gamma\cos\beta_0 = I_1\dot\alpha_0\sin^2\beta_0$$

$$\Rightarrow Mgl = I_3\left(\dot\alpha_0\cos\beta_0+\dot\gamma_0\right)\dot\alpha_0 - I_1\cos\beta_0\dot\alpha_0^2.$$

For fun we note that in the usual simplifying limit of the top spinning much faster than it is precessing $|\dot\gamma_0| \gg |\dot\alpha_0|$, we have

$$\dot\alpha_0 \simeq \frac{Mgl}{I_3\dot\gamma_0}.$$

3) (23 pts) We want to analyze a pendulum composed of a uniform rigid rod of mass M and length l as suggested in the figure. The pendulum is suspended from a frictionless support in a uniform gravitational field $\vec{g} = g\hat{z}$, *i.e.*, the $\hat{z}$ direction is "down". In this question assume that support restricts the motion of the pendulum to a fixed vertical plane ($\phi =$constant), but there is no restriction on the θ motion. (You may also assume that the radius of the rod is very small compared to its length, but note that the rod is NOT massless!)



Aside: An important concept in this problem is that the kinetic energy receives contributions from both the motion of the CM of the rod and the motion of the rod about the CM (the basic tenet of rigid body motion). Many people missed one of these.

a) (8 pts) Write down the Lagrangian, the canonical momentum, the Hamiltonian and Hamilton's equations of motion for this system using the polar angle θ to describe the motion of the system ($\theta = 0$ is the down direction). Explain your work. HINT: Don't forget to find ALL of the requested quantities.

Solution: With the constraint that there is motion only in θ we have CM motion and rotation for the rod about the CM $\left(I = Ml^2/12\right)$. Thus the total kinetic energy is (note you could also use the moment of inertia of a rod around one end, $Ml^2/3$)

$$\begin{aligned} T &= \frac{M}{2}\left(\frac{l\dot{\theta}}{2}\right)^2 + \frac{1}{2}\frac{Ml^2}{12}\dot{\theta}^2 \\ &= \frac{M}{6}\left(l\dot{\theta}\right)^2, \end{aligned}$$

while the potential energy is

$$U = -Mg\frac{l}{2}\cos\theta.$$

Thus we have

$$L = \frac{Ml^2}{6}\dot{\theta}^2 + \frac{Mgl}{2}\cos\theta.$$

Using the standard definitions we find the canonical momentum and the Hamiltonian

$$\begin{aligned}
&p_\theta \equiv \frac{\partial L}{\partial \dot{\theta}} = \frac{Ml^2}{3}\dot{\theta}, \\
&H\left(\theta, p_\theta\right) = p_\theta \dot{\theta} - L = \frac{3}{2Ml^2}p_\theta^2 - \frac{Mgl}{2}\cos\theta, \\
&\Rightarrow \begin{cases} \dot{\theta} = \dfrac{\partial H}{\partial p_\theta} = \dfrac{3}{Ml^2}p_\theta \\ \dot{p}_\theta = -\dfrac{\partial H}{\partial \theta} = -\dfrac{Mgl}{2}\sin\theta \end{cases}.
\end{aligned}$$

b) (2 pts) What quantities, if any, remain constant during the motion of the pendulum? Explain.

<u>Solution</u>: The total energy, *H*, is a constant of the motion (*H* is free of explicit time independence), but nothing else (*i.e.*, not p_θ). (Of course, p_ϕ is also constant but it has been trivially defined to vanish.)

c) (5 pts) Find the period for small oscillations about $\theta = 0$ and evaluate it (in seconds) for the case *M* = 1 kg and *l* = 50 cm ($g = 9.8$ m/s^2). Explain your work.

<u>Solution</u>: We can use either Lagrange's EoM or Hamilton's equations expanded about $\theta = 0$. Using the latter we have

$$\begin{aligned}
&\theta \to \delta\theta \\
&\Rightarrow \frac{Ml^2}{3}\delta\ddot{\theta} + \frac{Mgl}{2}\delta\theta = 0, \\
&\delta\ddot{\theta} + \frac{3g}{2l}\delta\theta = 0.
\end{aligned}$$

Thus assuming oscillatory behavior, $\delta\theta\left(t\right) \propto e^{i\omega_\delta t}$, we have

$$\omega_{\delta\theta} = \sqrt{\frac{3g}{2l}} \Rightarrow \tau_{\delta\theta} = 2\pi\sqrt{\frac{2l}{3g}} = 1.16 \text{ s.}$$

d) (8 pts) If the initial conditions are such that the motion is not confined to small angles but extends to a large angle θ_0 $(-\theta_0 \le \theta \le \theta_0)$, write down an integral expression for the period of this motion. You do **not** need to perform the integral! On physical grounds, do you expect the small oscillation period you found in part c) to be larger or smaller than the large angle result here? Explain your reasoning.

Solution: In this case it is difficult to solve the EoM directly but, as in the central force problem, we can use the conservation of energy to proceed. If the (fixed) energy is E_0, at the extrema of the motion (the turning points) we have zero kinetic energy and only potential energy,

$$E_0 = -\frac{Mgl}{2}\cos\theta_0.$$

Then, from the form of the energy in terms of $\dot{\theta}^2$, we have

$$\begin{aligned} E_0 &= \frac{Ml^2}{6}\dot{\theta}^2 - \frac{Mgl}{2}\cos\theta \\ \Rightarrow \dot{\theta} &= \sqrt{\frac{E_0 + \frac{Mgl}{2}\cos\theta}{\frac{Ml^2}{6}}} = \sqrt{\frac{3g\left(\cos\theta - \cos\theta_0\right)}{l}} \\ &= \omega_{\delta\theta}\sqrt{2\left(\cos\theta - \cos\theta_0\right)} \\ \Rightarrow \tau_0 &= 4\sqrt{\frac{l}{3g}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = \frac{\tau_{\delta\theta}\sqrt{2}}{\pi}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} \ge \tau_{\delta\theta}. \end{aligned}$$

The large angle result is sensitive to the fact that $\sin\theta \ne \theta$ for large angles. In fact, $\sin\theta \le \theta$, so that at larger angles the restoring force is smaller in the true finite angle expression than in the linear, small angle (HO) form extended to large angles. With a smaller restoring force we expect slower oscillation and

$$\tau_0 \ge \tau_{\delta\theta}.$$

On the other hand, the numerical agreement with the approximate (small angle) result is quite good ($\sim 10\%$) until the angle is quite large $\left(\sim \pi/2\right)$.

An alternative approach is to use an integrating factor ($\dot{\theta}$ in this case) and simply integrate the EoM, but we must be careful about the constants of integration. Note in particular that θ_0 is the point where $\dot{\theta} = 0$. If we choose that $\theta\left(t = 0\right) = -\theta_0$ we have

$$\ddot{\theta} = -\frac{3g}{2l}\sin\theta \Rightarrow \ddot{\theta}\dot{\theta} = -\frac{3g}{2l}\sin\theta\dot{\theta}$$

$$\Rightarrow \int_0^t \ddot{\theta}\dot{\theta}dt' = \frac{\dot{\theta}^2}{2}\bigg|_0^t = \frac{\dot{\theta}\left(t\right)^2}{2} = -\frac{3g}{2l}\int_0^t \sin\theta\dot{\theta}dt' = -\frac{3g}{2l}\int_{-\theta_0}^{\theta(t)} \sin\theta' d\theta'$$

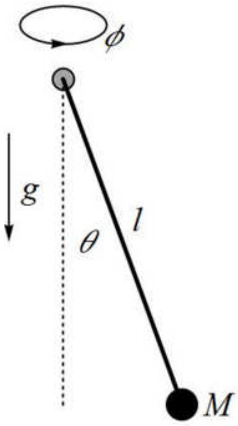
$$= \frac{3g}{2l}\cos\theta'\Big|_{-\theta_0}^{\theta} = \frac{3g}{2l}\left(\cos\theta - \cos\theta_0\right)$$

$$\Rightarrow \frac{d\theta}{dt} = \sqrt{\frac{3g}{l}\left(\cos\theta - \cos\theta_0\right)} = \omega_{\delta\theta}\sqrt{2\left(\cos\theta - \cos\theta_0\right)}$$

$$\tau_0 = \int_0^{\tau_0} dt = \frac{4}{\omega_{\delta\theta}}\int_0^{\theta_0} \frac{d\theta}{\sqrt{2\left(\cos\theta - \cos\theta_0\right)}} = \frac{\tau_{\delta\theta}\sqrt{2}}{\pi}\int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}},$$

just as above. The most common problem in using this technique was to simply ignore the issues of constants of integration and limits of integration. That approach may be lazy but it is certainly not smart. Note also that $\omega_{\delta\theta}$ is the time derivative of the phase of the θ motion and is NOT the time derivative of θ itself.

4) (18 pts) In this problem we want to analyze a system (a spherical pendulum) similar to that of the previous problem. However, now we simplify by putting all of the mass M at the end of the rigid (but now massless) rod of length l and we include motion in the azimuthal angle ϕ as suggested in the figure. In this question assume that support restricts the motion of the pendulum to exhibit a constant angular velocity about the z axis $\left(\dot{\phi} = \Omega = \text{constant}\right)$, but again there is no restriction on the θ motion.



ASIDE: This is just a disguised version of problem 3 in HW V (F&W #4.4)

a) (6 pts) Write down the Lagrangian and equation of motion describing this system (in terms of θ and $\dot{\theta}$). Explain your reasoning.

Solution: The new Lagrangian and equation of motion are given by

$$\begin{aligned}
&T = \frac{M}{2}\left(l\dot{\theta}\right)^2 + \frac{M}{2}\left(l\sin\theta\Omega\right)^2, \\
&U = -Mgl\cos\theta \\
&\Rightarrow L = \frac{M}{2}\left(l\dot{\theta}\right)^2 + \frac{M}{2}\left(l\sin\theta\Omega\right)^2 + Mgl\cos\theta \\
&\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial L} = Ml^2\ddot{\theta} - Ml^2\Omega^2\sin\theta\cos\theta + Mgl\sin\theta = 0 \\
&\Rightarrow \ddot{\theta} = \left(\Omega^2\cos\theta - \frac{g}{l}\right)\sin\theta.
\end{aligned}$$

b) (12 pts) Find the values of θ that correspond to equilibria and determine which of the equilibria are stable (to small perturbations) and which are unstable. Find the frequencies of oscillation at any stable equilibrium. Explain your reasoning. HINT: Be explicit about the dependence on the parameter Ω.

Solution: To find the equilibrium points we want solutions where $\ddot{\theta} = \dot{\theta} = 0$. Thus we want solutions of

$$\ddot{\theta} = 0 \Rightarrow Ml^2\Omega^2 \sin\theta\cos\theta = Mgl\sin\theta$$
$$\Rightarrow \begin{cases} \sin\theta_0 = 0 \\ \cos\theta_0 = \dfrac{g}{l\Omega^2} \end{cases}.$$

The first equation gives the expected values $\theta_0 = 0, \pi$ with the pendulum hanging straight down or balancing straight up. The non-trivial equilibrium is at $\theta_0 = \cos^{-1} g/l\Omega^2$. In order for this to be at a physical value of the polar angle we must require

$$\Omega^2 \geq \frac{g}{l} \Rightarrow \Omega \geq \sqrt{\frac{g}{l}} = \omega_p,$$

which means for this extra equilibrium point to be physical the azimuthal angular velocity must be larger than the natural frequency of the pendulum. To determine the stability of the equilibria we can expand the equation motion for small perturbations near the equilibrium point, $\theta \to \theta_0 + \delta\theta$, yielding

$$\delta\ddot{\theta} = \left[\Omega^2\left(\cos^2\theta_0 - \sin^2\theta_0\right) - \frac{g}{l}\cos\theta_0\right]\delta\theta.$$

The stability question is answered by whether the quantity in the square brackets is (strictly) less than zero, corresponding to a restoring force. Thus we conclude that

$$\theta_0 = 0 : \Omega^2 - \frac{g}{l} = \Omega^2\left(1 - \frac{\omega_p^2}{\Omega^2}\right) \Rightarrow \begin{cases} \text{stable} : \Omega < \omega_p, \omega_{\text{osc}} = \omega_p\sqrt{1 - \dfrac{\Omega^2}{\omega_p^2}} \\ \text{unstable} : \Omega > \omega_p \end{cases},$$

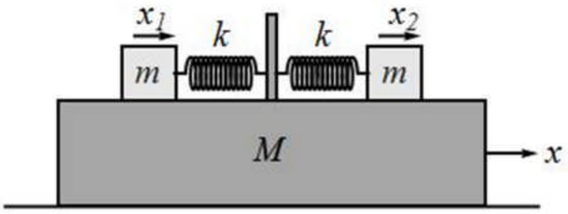
$$\theta_0 = \pi : \Omega^2 + \frac{g}{l} = \Omega^2 + \omega_p^2 \Rightarrow \{\text{unstable all } \Omega,$$

$$\theta_0 = \cos^{-1}\frac{\omega_p^2}{\Omega^2} : \Omega^2\left(2\left(\frac{\omega_p^2}{\Omega^2}\right)^2 - 1 - \left(\frac{\omega_p^2}{\Omega^2}\right)^2\right) = \Omega^2\left(\left(\frac{\omega_p^2}{\Omega^2}\right)^2 - 1\right)$$

$$\Rightarrow \begin{cases} \text{stable} : \Omega > \omega_p, \omega_{\text{osc}} = \Omega\sqrt{1 - \dfrac{\omega_p^4}{\Omega^4}} \\ \text{unphysical} : \Omega < \omega_p \end{cases}.$$

Thus for $\Omega \ll \omega_p$ only small oscillations around $\theta = 0$ (the usual pendulum) are stable at the usual pendulum frequency ω_p; for increasing values Ω there are still stable oscillations about zero but with decreasing frequency; for $\Omega > \omega_p$ the stable equilibrium moves to larger values of θ ($\cos^{-1} \omega_p^2 \big/ \Omega^2$) and the oscillation frequency increases towards Ω; finally as Ω diverges the stable equilibrium point approaches $\pi/2$ ($\cos^{-1} 0$).

Note that it is essential that you be able to expand expressing like $\cos(\theta+\delta\theta)$ for small $\delta\theta$ quickly and accurately (so memorize the double angle formula).



5) (24 pts) A platform of mass M sits on a frictionless table. Two identical blocks of mass m are attached with identical springs (spring constant k) to a central post fixed to the platform and move without friction on top of the platform as indicated in the figure. The platform and the two blocks are constrained to move horizontally (without friction) along the same direction with this motion measured by the coordinates x, x_1, x_2. The coordinates x_1 and x_2 measure the extension/compression of the springs, *i.e.*, are measured with respect to the platform.

Comment: This is a Qualifying Exam style problem.

a) (7 pts) Write down the Lagrangian and (all) equations of motion describing this system in terms of the coordinates defined in the figure. Explain your reasoning.

<u>Solution</u>: The important point here is to recognize that the coordinates x_1 and x_2 are measured with respect to the platform and thus the blocks move with the platform even when coordinates x_1 and x_2 are constant. In terms of the coordinates in the figure the Lagrangian is given by

$$\begin{aligned}
T &= \frac{M}{2}\dot{x}^2 + \frac{m}{2}\left[\left(\dot{x}_1+\dot{x}\right)^2 + \left(\dot{x}_2+\dot{x}\right)^2\right], \\
U &= \frac{k}{2}\left[x_1^2 + x_2^2\right], \\
L &= \frac{M}{2}\dot{x}^2 + \frac{m}{2}\left[\left(\dot{x}_1+\dot{x}\right)^2 + \left(\dot{x}_2+\dot{x}\right)^2\right] - \frac{k}{2}\left[x_1^2 + x_2^2\right] \\
&= \frac{(M+2m)}{2}\dot{x}^2 + m\dot{x}\left(\dot{x}_1+\dot{x}_2\right) + \frac{m}{2}\left[\dot{x}_1^2+\dot{x}_2^2\right] - \frac{k}{2}\left[x_1^2+x_2^2\right].
\end{aligned}$$

Hence the equations of motion are

$$\begin{aligned}
&x : \left(M+2m\right)\ddot{x} + m\left(\ddot{x}_1+\ddot{x}_2\right) = 0, \\
&x_1 : m\left(\ddot{x}+\ddot{x}_1\right) + kx_1 = 0, \\
&x_2 : m\left(\ddot{x}+\ddot{x}_2\right) + kx_2 = 0.
\end{aligned}$$

These equations correspond to coupled oscillators, where the coupling arises from the kinetic energy not the potential energy. Note there is no explicit x dependence and we can use the first equation to eliminate $\ddot{x}$. The first equation is just the statement that, with no external forces in the *x* direction, the CM will not accelerate.

b) (7 pts) This system has a trivial normal mode where the CM moves with constant speed, $\dot{x} = \text{constant}$ with $\ddot{x} = \ddot{x}_1 = \ddot{x}_2 = 0$ and $x_1 = x_2 = 0$. Eliminate this mode and set up the desired 2-D (matrix form) coupled oscillator equation for the 2 non-trivial normal modes. Explain your reasoning. HINT: Combine the equations of motion to eliminate the *x* coordinate. [If you have difficulty eliminating the trivial mode, you will receive full credit for solving the full 3D problem directly.]

Solution: People who incorrectly find no coupling of the oscillators in part a) should be troubled at this point. Using the first equation to eliminate $\ddot{x}$ from the second and third equations and going to matrix notation we have

$$\begin{aligned}
&\ddot{x} = -\frac{m}{M+2m}\left(\ddot{x}_1 + \ddot{x}_2\right) \Rightarrow \\
&m\left(\ddot{x}_1 - \frac{m}{M+2m}\left(\ddot{x}_1 + \ddot{x}_2\right)\right) = m\left(\frac{M+m}{M+2m}\ddot{x}_1 - \frac{m}{M+2m}\ddot{x}_2\right) = -kx_1, \\
&m\left(\ddot{x}_2 - \frac{m}{M+2m}\left(\ddot{x}_1 + \ddot{x}_2\right)\right) = m\left(\frac{M+m}{M+2m}\ddot{x}_2 - \frac{m}{M+2m}\ddot{x}_1\right) = -kx_2 \\
&\Rightarrow \begin{pmatrix} \dfrac{M+m}{M+2m} & -\dfrac{m}{M+2m} \\ -\dfrac{m}{M+2m} & \dfrac{M+m}{M+2m} \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\frac{k}{m}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.
\end{aligned}$$

c) (10 pts) From now on consider the special case *M* = 2*m*. Assuming harmonic behavior solve for the (eigen-) frequencies of the two normal modes and the form of the modes themselves in terms of the initial coordinates, *i.e.*, find the relative magnitude and relative phase of x, x_1 and x_2 for each mode. Explain your reasoning.

Solution: With the usual Ansatz of complex exponential behavior, $x_k \propto e^{i\omega t}, \ddot{x}_k = -\omega^2 x_k$, and the eigenvalues defined by $\lambda\, k/m = \omega^2$ and $\omega_0^2 = k/m$, the corresponding matrix equation is

$$\omega_0^2\begin{pmatrix} \frac{3}{4}\lambda-1 & -\frac{1}{4}\lambda \\ -\frac{1}{4}\lambda & \frac{3}{4}\lambda-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=0$$
$$\Rightarrow \det\begin{bmatrix} \frac{3}{4}\lambda-1 & -\frac{1}{4}\lambda \\ -\frac{1}{4}\lambda & \frac{3}{4}\lambda-1 \end{bmatrix}=0.$$

So the eigenvalue (characteristic) equation is

$$\begin{aligned}
&\left(\frac{3}{4}\right)^2\lambda^2-2\left(\frac{3}{4}\right)\lambda+1-\left(\frac{1}{4}\right)^2\lambda^2=0\\
&\Rightarrow 8\lambda^2-2\left(12\right)\lambda+\left(16\right)=0\\
&\Rightarrow \lambda^2-3\lambda+2=0\\
&\Rightarrow \lambda_\pm=\frac{3\pm\sqrt{9-8}}{2}=\begin{cases}2\\1\end{cases}.
\end{aligned}$$

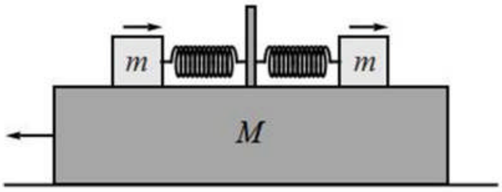
Thus the frequencies of the normal mode motion are

$$\omega_\mp=\begin{cases}\sqrt{2}\sqrt{k/m}=\sqrt{2}\omega_0\\ \sqrt{k/m}=\omega_0\end{cases}.$$

The modes are solutions of

$$\begin{pmatrix} \frac{3}{4}\lambda_+-1 & -\frac{1}{4}\lambda_+ \\ -\frac{1}{4}\lambda_+ & \frac{3}{4}\lambda_+-1 \end{pmatrix}\begin{pmatrix} x_{1+} \\ x_{2+} \end{pmatrix}=\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x_{1+} \\ x_{2+} \end{pmatrix}=0\Rightarrow\begin{pmatrix} x_{1+} \\ x_{2+} \end{pmatrix}\propto\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$
$$\begin{pmatrix} \frac{3}{4}\lambda_--1 & -\frac{1}{4}\lambda_- \\ -\frac{1}{4}\lambda_- & \frac{3}{4}\lambda_--1 \end{pmatrix}\begin{pmatrix} x_{1-} \\ x_{2-} \end{pmatrix}=\begin{pmatrix} -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} \end{pmatrix}\begin{pmatrix} x_{1-} \\ x_{2-} \end{pmatrix}=0\Rightarrow\begin{pmatrix} x_{1-} \\ x_{2-} \end{pmatrix}\propto\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

In both cases, as already noted, the amplitudes of motion are such that the CM does not move. Thus the larger frequency corresponds to the 2 blocks moving in phase with the same amplitude but with the platform moving in the opposite direction with ½ of the amplitude ($x = -x_1/2 = -x_2/2$), as suggested in the first figure.



The mode with the smaller frequency, which is just the “spring” frequency, corresponds to the 2 blocks moving in opposite directions with the same amplitude (exactly out of phase) and the platform not moving $\left(x = 0, x_1 = -x_2\right)$, as suggested in the second figure.