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Midterm Exam Physics 505

November 5, 2010

This exam is closed book but useful equations and relations are supplied at the beginning of the exam.

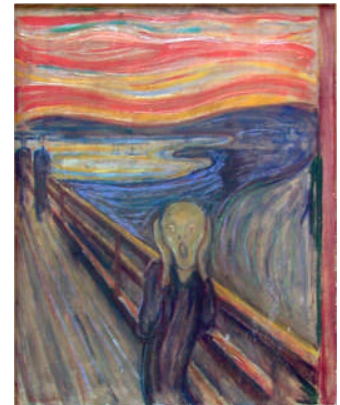
For all questions you are encouraged to (briefly) explain your reasoning (I can give full marks only when I can understand what you did). The noted point values are for reference only and are subject to (minor) fine tuning.

Exercise I _____ (14)

Exercise II _____ (12)

Exercise III _____ (24)

TOTAL _____ (50)



Possibly useful general formulae:

Newton's Law
$$\frac{d(m\dot{\vec{r}})}{dt} = \vec{F}$$

Cylindrical coordinates

$$\begin{aligned}\vec{r} &= \rho\hat{\rho} + z\hat{z}, \\ d\vec{r} &= d\rho\hat{\rho} + \rho d\phi\hat{\phi} + dz\hat{z}, \\ \dot{\vec{r}} &= \dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi} + \dot{z}\hat{z}, \\ \ddot{\vec{r}} &= \ddot{\rho}\hat{\rho} - \rho\dot{\phi}^2\hat{\rho} + 2\dot{\rho}\dot{\phi}\hat{\phi} + \rho\ddot{\phi}\hat{\phi} + \ddot{z}\hat{z}\end{aligned}$$

Spherical coordinates

$$\begin{aligned}\vec{r} &= r\hat{r}, \\ d\vec{r} &= dr\hat{r} + r d\theta\hat{\theta} + r \sin\theta d\phi\hat{\phi}, \\ \dot{\vec{r}} &= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r \sin\theta\dot{\phi}\hat{\phi}, \\ \ddot{\vec{r}} &= (\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin\theta\cos\theta\dot{\phi}^2)\hat{\theta} \\ &\quad + (2\dot{r}\sin\theta\dot{\phi} + 2r\dot{\theta}\cos\theta\dot{\phi} + r\sin\theta\ddot{\phi})\hat{\phi}\end{aligned}$$

Newton in a rotating frame (\vec{r}_0 is the vector from the origin of the inertial frame to the origin of the rotating frame)

$$\begin{aligned}m \left. \frac{d^2\vec{r}'(t)}{dt^2} \right|_{\text{acceler}} &= \vec{F}_{\text{inertial}} - m \left. \frac{d^2\vec{r}_0(t)}{dt^2} \right|_{\text{inertial}} - 2m\vec{\omega} \times \left. \frac{d\vec{r}'(t)}{dt} \right|_{\text{acceler}} \\ &\quad - m \frac{d\vec{\omega}}{dt} \times \vec{r}'(t) - m\vec{\omega} \times [\vec{\omega} \times \vec{r}'(t)]\end{aligned}$$

Lagrange's equation for a conservative system in generalized coordinates

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0$$

Lagrange with constraints specified by functions $\phi_l(q_k) = 0$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = a_{kl} \lambda_l,$$
$$a_{kl} = \frac{\partial \phi_l}{\partial q_k}$$

Hamilton's equations in canonical coordinates

$$\dot{q}_k = \frac{\partial H}{\partial p_k}$$
$$\dot{p}_k = -\frac{\partial H}{\partial q_k}$$
$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$
$$H = \sum_k p_k \dot{q}_k - L$$

The moment of inertia of disk of radius r and mass m about axis of symmetry

$$I_{disk} = \frac{mr^2}{2}$$

Moment of inertia of rod of length l and mass m about axis through CM and orthogonal to length (assuming rod is much longer than it is thick)

$$I_{rod} = \frac{ml^2}{12}$$

Moment of inertia of a hoop of radius r and mass m about the axis of symmetry

$$I_{hoop} = mr^2$$

I) (14 pts) In the following questions we address some qualitative issues. Determining the correct answer should require minimal calculation, but be certain to provide a brief description of how you arrived at your answer.

Consider the motion of a point particle of mass m moving in two dimensions described by the Cartesian coordinates x and y . For each of the listed Lagrangians ($L = T - U$) determine which of the following quantities are conserved, the x -momentum (p_x), the y -momentum (p_y), the angular momentum in the z direction (L_z) and the total energy ($E = T + U$). Note that more than one quantity may be conserved. In each case briefly note the relevant invariance(s).

NOTE: This last part of the question was to help encourage you to approach such problems in terms of the invariances displayed by the Lagrangian. Nearly everyone ignored this part of the problem. Full points were awarded only if the situation was fully analyzed using other techniques. This grading choice is the primary explanation of why there were no perfect papers.

a) (3 pts)
$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - \frac{k}{2}x^2$$

Solution: This Lagrangian is translationally invariant in the y direction (*i.e.*, $\partial L/\partial y = 0$), and without explicit time dependence (but not rotationally or x translationally invariant) $\Rightarrow p_y$ and E conserved (but not p_x or L_z).

b) (3 pts)
$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - \frac{k}{2}(x^2 + y^2)$$

Solution: This Lagrangian has no translational invariance, but is rotationally invariant and explicitly time independent $\Rightarrow L_z$ and E conserved (but not p_x or p_y).

Next we want to consider some of the properties of the scattering of a point particle (mass m) in a central potential. In particular, consider the potential described by $U(r) = +k/r^2$, $k < 0$. The particle is initially (far from the origin) traveling towards the origin with velocity $\vec{v} = -v_0\hat{x}$ along a path parallel to the x -axis but displaced from it in the $+\hat{y}$ direction by a distance b (the impact parameter).

c) (3 pts) Identify and evaluate (in terms of the parameters defined above) the constants of the motion that characterize the subsequent motion. Explain your

reasoning.

Solution: Since the interaction is defined by a (time independent) potential, this is a conservative system and the total energy is conserved: $E = T + U = E_0 = T_0 = mv_0^2/2$. Since the potential is a central potential and provides no torques, the angular momentum must be a constant: $\vec{L} = mv_0 b \hat{z}$. Note that it is the total, vector angular momentum that is conserved. The fact that $L_x = L_y = 0$ guarantees that the motion occurs in the x - y plane.

d) (5 pts) Find the distance of closest approach to the origin (*i.e.*, to the scattering center) in terms of the parameters defined above. As a check, do you expect this distance to be larger or smaller than b ? Explain your reasoning.

Solution: This should not have been a difficult problem, but often it was. You *need* to understand how to use the conserved quantities in part c) to find the simple properties of the resulting trajectory. We have discussed this issue in the Lectures and illustrated it in the HW. The simplest way to proceed is to reduce this to a 1-D problem in the radial variable and look for the point where the time derivative vanishes. This is the point where the "effective" potential equals the total energy. Since we know that the motion all occurs in the x - y plane (perpendicular to the angular momentum), I will use cylindrical coordinates (not necessary, if we are careful). We have

$$U_{\text{eff}}(\rho_{\min}) = \frac{(L)^2}{2m\rho_{\min}^2} + \frac{k}{\rho_{\min}} = \frac{(mv_0 b)^2}{2m\rho_{\min}^2} + \frac{k}{\rho_{\min}} = E_0 = \frac{mv_0^2}{2}$$
$$\Rightarrow \rho_{\min} = \sqrt{\frac{mv_0^2 b^2 / 2 + k}{mv_0^2 / 2}} = \sqrt{b^2 + \frac{2k}{mv_0^2}}.$$

Alternatively we could simply evaluate the total energy at the point where $\dot{\rho} = 0$ using the value of the angular momentum to find $\dot{\phi}$. As expected for an attractive potential ($k < 0$), the distance of closest approach is smaller than the impact parameter, which is the distance of closest approach if either the interaction is turned off, $k \rightarrow 0$, or the kinetic energy overwhelms the potential energy, $2k/mv_0^2 b^2 \rightarrow 0$. Many people did not carefully read the problem and missed the sign of the constant k .

II) (12 pts) Consider a space station in a synchronous orbit such that the station is always above the same point on the equator ($\omega_E \approx 7.3 \times 10^{-5} \text{ s}^{-1}$, *i.e.*, a rotating reference frame) at a radius R_0 from the center of the earth. Around the CM of the station is a large room in which we can perform experiments. [You may assume that the dimensions of the station are very small compared to the radius of the orbit.] The room is clearly marked with the usual right-handed set of spherical coordinate axes, $\hat{r}, \hat{\theta}, \hat{\phi}$, aligned in the usual way (\hat{r} is perpendicular to the surface of the earth, $\hat{\theta}$ is parallel to the surface of the earth and points south, $\hat{\phi}$ is parallel to the surface of the earth and points east, the direction of motion of the station). You may assume that in each experiment that we are only sensitive to the appropriate first order corrections to the motion in the rotating frame, which will themselves be small [we approximate the earth as uniform and spherical].

- a) (4 pts) Assume that we release a small point mass, mass m , at rest at the center of the room (*i.e.*, at the CM of the station). Determine the initial acceleration (magnitude and direction) of the test mass observed in the station frame, and explain your reasoning.

Solution: Since we are in orbit where the force due to gravity ($\vec{F}_g = -GM_E m \hat{r} / R_0^2$) is exactly matched by the centripetal force ($-m\vec{\omega} \times [\vec{\omega} \times \vec{R}_0(t)] = m\omega_E^2 R_0 \hat{r}$), *i.e.*, $g_{\text{eff}} = 0$, there is no apparent acceleration for an object released at rest at the CM, $\ddot{\vec{r}}' = 0$. Note that gravity at the location of the space station is not (well) described by g , the gravitational constant at the surface of the earth. The major issue for this problem was simply being able to explain the situation, *i.e.*, the cancellation of gravity and centripetal forces.

- b) (4 pts) Next we probe the response to a fixed (small) displacement h ($\ll R_0$) in the \hat{r} direction. Thus we release the test mass at rest in the station frame with initial conditions $\vec{r}'(0) = h\hat{r}$, $\dot{\vec{r}}'(0) = \vec{0}$. Determine the initial acceleration (magnitude and direction) of the test mass observed in the station frame in terms of the given quantities, and explain your reasoning.

Solution: Since there is initially no velocity in the space station frame, we are only interested in the implied mismatch between the centrifugal force and the gravitational attraction, *both* of which are changed by the displacement,

$$\begin{aligned}
m\ddot{\vec{r}}' &= -\frac{GM_E m}{(R_0 + h)^2} \hat{r} - m\vec{\omega} \times (\vec{\omega} \times (R_0 + h) \hat{r}) \\
&\approx -\frac{GM_E m}{(R_0)^2} \hat{r} + m\omega_E^2 R_0 \hat{r} + 2\frac{GM_E m}{(R_0)^2} \frac{h}{R_0} \hat{r} + m\omega_E^2 h \hat{r} \\
\Rightarrow \ddot{\vec{r}}' &\approx 3\frac{GM_E}{(R_0)^2} \frac{h}{R_0} \hat{r} = 3\omega_E^2 h \hat{r}.
\end{aligned}$$

For the final simplifications we have used the constraint that arises from being in a stable orbit, $\vec{F}_g = -GM_E m \hat{r} / R_0^2 = -\vec{F}_c = -m\omega_E^2 R_0 \hat{r}$, *i.e.*, $GM_E / R_0^3 = \omega_E^2$. Note that we need the changes in *both* the gravitational force and the centrifugal force as they are of the same order in smallness. Many simply assumed, without checking, that the perturbation in one of the two forces was too small to be included. It is essential to carry out such comparisons carefully and explicitly. In any case the test mass tends to drift outward in the radial direction, but admittedly *very* slowly! For example, if the displacement h has a magnitude of 1 cm the acceleration is

$$3\omega_E^2 h \approx 3(7.3 \times 10^{-5} \text{ s}^{-1})^2 (1 \text{ cm}) \approx 1.6 \times 10^{-8} \text{ cm/s}^2.$$

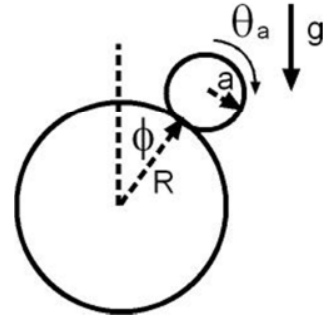
- c) (4 pts) Finally consider a test released from the center of the room but now with a nonzero initial velocity in the radial direction, *i.e.*, $\vec{r}'(0) = \vec{0}$, $\dot{\vec{r}}'(0) = v_0 \hat{r}$, $v_0 = 2 \text{ cm/s}$. Determine the initial acceleration (*numerical* magnitude and direction) of the test mass observed in the station frame in terms of the given quantities, and explain your reasoning.

Solution: The dominant extra force in this case is the Coriolis force, $-2m\vec{\omega} \times \dot{\vec{r}}' \Big|_{\text{acceler}}$, where $(-\vec{\omega} \propto -\hat{z} = \hat{\theta})$.

$$\ddot{\vec{r}}' = -2\vec{\omega} \times \dot{\vec{r}}' \Big|_{\text{acceler}} = 2\omega_E \left(\frac{2 \text{ cm}}{\text{s}} \right) (\hat{\theta} \times \hat{r}) = -2.92 \times 10^{-4} \frac{\text{cm}}{\text{s}^2} \hat{\phi}.$$

The test mass drifts slowly to the west (slows down with respect to the space station), conserving its angular momentum. Again some difficulty was created by not reading the instructions carefully and neglecting to provide a numerical result.

III) (24 pts) Consider a system composed of a uniform hoop of mass m and radius a that rolls without slipping on a fixed cylinder of radius $R > a$. The only external force is gravity. The hoop starts rolling from rest on top of the cylinder. We want to analyze the motion of the hoop and, in particular determine the point when it falls off the cylinder. [Show your work and briefly explain your reasoning.]



NOTE: This exercise, with only minor changes (the moment of inertia), is right out of Lecture 6 or HW IV (see problem 3.17b in F&W). Many problems arose from carelessness at many levels in setting-up the problem, finding the equations of motion and solving them. Points were lost both for the first careless mistake and then again when the original mistake led to a nonsensical result that went unnoticed. The willingness of many students to write down (and try to solve) non-equations whose terms displayed different units is quite troubling. It is essential that everyone develop the habit and checking results as you go along!

- a) (6 pts) Define an appropriate set of coordinates (see suggestions in the figure) and write down the Lagrangian that describes the general motion of this system before enforcing the constraints. Also write down the two constraint equations that describe the fact that the hoop is in contact with the cylinder and that it is rolling without slipping.

Solution: We need 2 variables to define the location of CM (*i.e.*, the center) of the hoop, plus an angle to describe the motion about the CM. Here I use cylindrical coordinates ρ, ϕ (the CM moves in a plane) for the former and θ_a for the latter, as suggested in the figure. The kinetic energy, potential energy and Lagrangian take the forms

$$T = \frac{m}{2} (\dot{\rho}^2 + \rho^2 \dot{\phi}^2 + a^2 \dot{\theta}_a^2), U = mg \rho \cos \phi$$

$$\Rightarrow L = \frac{m}{2} (\dot{\rho}^2 + \rho^2 \dot{\phi}^2 + a^2 \dot{\theta}_a^2) - mg \rho \cos \phi.$$

Note that the radial and angular velocities are orthogonal, meaning they are to be squared and added and NOT added and then squared! Note also that gravity is a force in the down, not up, direction!

The contact and non-slipping constraints take the form

$$\text{contact: } \varphi_1(\rho) = R + a - \rho = 0,$$

$$\text{non-slipping: } \varphi_2(\theta_a, \phi) = [R\dot{\phi} - a(\dot{\theta}_a - \dot{\phi})] = 0.$$

Note the form of the second constraint, especially the $a\dot{\phi}$ term as we discussed at length in class. A good way to remember this is that the non-slip constraint means that the instantaneous velocity of the point on the hoop that is touching the cylinder must vanish. The velocity of this point is $(R + a)\dot{\phi} - a\dot{\theta}_a$ (the CM motion plus the motion with respect to the CM) providing the required constraint.

As noted above, this problem is out of the HW (see problem 3.17b in F&W) and from Lecture 6 with only minor changes. Still, many did not seem to recognize it and were untroubled when careless setup, calculation or transcription led to overly complicated results that should have been recognized as incorrect. Note that one of the goals of the HW sets is to develop intuition with such problems so that you can recognize quickly when careless mistakes are leading you astray. The poor performance by many on this exercise right out of the HW, suggests that the HW is not being performed with this goal in mind.

- b) (7 pts) Write down the equations of motion for this system, including a Lagrange multiplier for the contact constraint. [You may use the other constraint to eliminate a variable or keep a second Lagrange multiplier.] Simplify the resulting equations using the constraints.

Solution: Since we are not interested here in finding the frictional (non-slipping) force, we can use the non-slipping constraint to eliminate θ_a , $\theta_a = \phi[(R + a)/a]$.

From Lagrange we have the following equations of motion

$$L \rightarrow \frac{m}{2}(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + (R + a)^2\dot{\phi}^2) - mg\rho\cos\phi - \lambda\varphi_1$$

$$\Rightarrow \begin{cases} \rho : m\ddot{\rho} - m\rho\dot{\phi}^2 + mg\cos\phi - \lambda = 0 \\ \phi : m[\rho^2 + (R + a)^2]\ddot{\phi} + 2m\rho\dot{\rho}\dot{\phi} - mg\rho\sin\phi = 0. \end{cases}$$

We can simplify these equations using the constraint $\rho = R + a$, $\dot{\rho} = \ddot{\rho} = 0$ and find

$$\dot{\phi}^2 = \frac{mg \cos \phi - \lambda}{m(R+a)},$$

$$\ddot{\phi} = \frac{g \sin \phi}{2(R+a)}.$$

If instead we keep all variables and Lagrange multipliers, we have

$$L \Rightarrow \frac{m}{2}(\dot{\rho}^2 + \rho^2 \dot{\phi}^2 + a^2 \dot{\theta}_a^2) - mg \rho \cos \phi - \lambda_1 \phi_1 - \lambda_a \phi_a$$

$$\Rightarrow \begin{cases} \rho : m\ddot{\rho} - m\rho \dot{\phi}^2 + mg \cos \phi - \lambda_1 = 0 \\ \phi : m\rho^2 \ddot{\phi} + 2m\rho \dot{\rho} \dot{\phi} - mg \rho \sin \phi + \lambda_a (R+a) = 0. \\ \theta_a : ma^2 \ddot{\theta}_a - \lambda_a a = 0 \end{cases}$$

Simplifying we have

$$\dot{\phi}^2 = \frac{mg \cos \phi - \lambda_1}{m(R+a)},$$

$$\ddot{\phi} = \frac{mg \sin \phi - \lambda_a}{m(R+a)}, \quad \ddot{\theta}_a = \frac{R+a}{a} \ddot{\phi} = \frac{\lambda_a}{ma} \Rightarrow \ddot{\phi} = \frac{g \sin \phi}{2(R+a)},$$

as above.

The most common problems here arose from careless mistakes that went undetected (even when they led to nonsensical results). Check as you go!!

- c) (8 pts) Find an expression for the normal force between the hoop and the cylinder as a function of the hoop's position on the cylinder and determine when the hoop separates from the cylinder, *i.e.*, the angle when separation occurs.

Solution: To solve for the constraint force we need to find an expression for $\dot{\phi}^2$, which we can obtain by using the conservation of energy,

$$E = m(R+a)^2 \dot{\phi}^2 + mg(R+a) \cos \phi = mg(R+a)$$

$$\Rightarrow \dot{\phi}^2 = \frac{g(1 - \cos \phi)}{(R+a)},$$

or by integrating the above equation using the integrating factor $\dot{\phi}$,

$$\dot{\phi}\ddot{\phi} = \frac{\dot{\phi}g \sin \phi}{2(R+a)} \Rightarrow \dot{\phi}^2 = \left| \frac{-g \cos \phi}{(R+a)} \right|_0^\phi = \frac{g(1 - \cos \phi)}{(R+a)}.$$

Thus we have the normal constraint force given by

$$\lambda = m(R+a) \left[\frac{g \cos \phi}{(R+a)} - \dot{\phi}^2 \right] = mg(2 \cos \phi - 1).$$

The normal force has magnitude mg at $\phi = 0$. The force vanishes and contact is lost at

$$\phi_{off} = \cos^{-1}\left(\frac{1}{2}\right) = 60^\circ \text{ or } \frac{\pi}{3} \text{ radians.}$$

- d) (3 pts) Find an equation involving an integral which, if you could evaluate the integral, would (implicitly) define the angle ϕ as a function of t while the hoop is still in contact with the cylinder. You do not need to evaluate the integral.

Solution: We can proceed much as we did with the central potential problem using the conservation of energy to obtain a first order differential equation, separating variables and then integrating. From the previous part of this problem we have

$$\begin{aligned} \dot{\phi} &= \sqrt{\frac{g(1 - \cos \phi)}{(R+a)}} \Rightarrow \frac{d\phi}{\sqrt{1 - \cos \phi}} = \sqrt{\frac{g}{(R+a)}} dt \\ \Rightarrow \int_{\phi_0}^{\phi} \frac{d\phi'}{\sqrt{1 - \cos \phi'}} &= \int_{t_0}^t \sqrt{\frac{g}{(R+a)}} dt' = \sqrt{\frac{g}{(R+a)}} (t - t_0). \end{aligned}$$

Note, in particular, that one must separate variables by putting all the ϕ dependence on the LHS before integration. Keeping ϕ terms inside the time integral is not a useful option.

While the following was not required on the exam, we can proceed for fun. Note the interesting fact that, if we really start the hoop at rest at the top of the cylinder, $\phi_0 \rightarrow 0$, the ϕ integral is logarithmically divergent implying that $t_0 \rightarrow -\infty$. The point at the top

of the cylinder is an (unstable) equilibrium point and the hoop will sit there forever if we start there at rest with no perturbation. Also we can perform the ϕ integral to find

$$\left[\sqrt{2} \ln \left(\tan \left(\frac{\phi'}{4} \right) \right) \right]_{\phi_0}^{\phi} = \sqrt{2} \ln \left(\frac{\tan(\phi/4)}{\tan(\phi_0/4)} \right) = \sqrt{\frac{g}{R+a}} (t - t_0)$$
$$\Rightarrow \phi(t) = 4 \arctan \left[\tan(\phi_0/4) e^{\sqrt{g/2(R+a)}(t-t_0)} \right].$$

There is a *Mathematica* notebook and animation on our web page that illustrates this result.