Lecture 3 – Inertial Reference Frames (Intro to Chapter 2 in F&W)

We have been explicitly making assumptions about the frames of reference in which we have been defining equations and now we want to make those issues more explicit. We start with Newton's second equation in either vector (*i.e.*, linear algebra) notation

$$\frac{d\left(m\dot{\vec{r}}\right)}{dt} = \vec{F}, \tag{3.1}$$

or tensor (*i.e.*, component) notation

$$\frac{d\left(m\dot{r}_j\right)}{dt} = F_j. \tag{3.2}$$

These equations implicitly define a class of reference frames, the inertial frames, simply by the fact that these equations are true in those frames with no extra contributions. We are also explicitly using Cartesian coordinates in the second equation (*i.e.*, rectilinear axes that extend to infinity). We also typically assume that these axes are embedded in a (3-dimensional) Euclidean (vector) space with a metric defined so that we know how to construct scalar and vector products. By assumption we take the Cartesian coordinates axes to be orthogonal, *i.e.*, to have vanishing scalar products, and to be specified by a (complete) set of (3) unit vectors. By definition any vector in this space can be represented by a linear combination of the unit vectors with appropriate coefficients (the components). We start by considering an abstract inertial frame in some "empty" Euclidean space with enormous symmetry, *i.e.*, invariant under (possibly time dependent) translations and rotations. In practice we have in mind a frame "at rest" with respect to the distant (low number density) stars. In this lecture we want to consider the range of other reference frames, which we are transported to by some transformation, such that the form of Newton's equation is, in some sense, unchanged.

When we say "unchanged" by some coordinate transformation, there are actually two relevant levels of constancy. The weaker level is covariance, meaning that in the new, primed coordinates the equation has the same form,

$$\frac{d\left(m\dot{r}_j\right)}{dt} = F_j\left(r_k\right) \xrightarrow[r_k \to r_k']{} \frac{d\left(m\dot{r}_j'\right)}{dt} = F_j'\left(r_k'\right), \tag{3.3}$$

although the right-hand-side of the new equation is allowed to have a new functional form in the sense that the individual components are different, $F_j'\left(r_k'\right) \neq F_j\left(r_k'\right)$. This change in detail still allows the new frame to be inertial. What we are excluding is the appearance of new terms in the equations (and we will provide examples below). (Actually certain types of new terms can appear in the component version of Newton if we transform to curvilinear coordinates still in an underlying inertial frame.) If the new frame is an inertial frame, the vector version, Eq. (3.1), of Newton remains unchanged. On the other hand, it is also possible that in the new reference frame even the details of the component version of the right-hand-side are unchanged, *i.e*., that $F_j'\left(r_k'\right) = F_j\left(r_k'\right)$. In this case we have an *in*variance of the dynamics (a stronger statement than *co*variance) and, as noted in Lecture 1, there will be an associated conserved quantity (constant of the motion) as specified by Noether's Theorem. Examples are translational invariance when $F_j'\left(r_k'\right) = F_j\left(r_k'\right) = 0$ leading to momentum conservation and rotational invariance in the central force problem leading to angular momentum conservation.

Returning to the issue of covariance we consider first the simplest inhomogeneous transformations, translations. We simply move the origin of the reference frame by a specific (vector) distance, $\vec{r}_0$. Thus in the new frame the coordinates are given in terms of the old components by

$$\begin{aligned} r_k' &= r_k - r_{0,k}, \\ r_k &= r_k' + r_{0,k}. \end{aligned} \tag{3.4}$$

We can rewrite Newton in terms of the coordinates in the new frame as

$$\frac{d\left(m\dot{r}_j\right)}{dt} = \frac{d\left(m\dot{r}_j' + m\dot{r}_{0,j}\right)}{dt} = F_j\left(r_k\right) = F_j\left(r_k' + r_{0,k}\right) \equiv F_j'\left(r_k'\right). \tag{3.5}$$

It is clear that Newton's equation is form covariant (and still valid in the new frame if it was valid in the old frame) as long as there is no new term on the left-hand-side of

Eq. (3.5). Thus the constrain of covariance for translations, *i.e.*, that the new frame is still an inertial frame, is that

$$\ddot{r}_{0,k} = 0. \tag{3.6}$$

Thus we are allowed to make constant translations and allowed to transform to a frame moving with a constant relative velocity and still be in an inertial frame where the form of Newton's equation is unchanged,

$$\frac{d\left(m\dot{r}_j'\right)}{dt} = F_j'\left(r_k'\right). \tag{3.7}$$

In such an inertial frame we cannot determine by performing a physics experiment, for example throwing a ball, whether the frame is fixed with respect to the distant stars or moving with a constant velocity (other than looking at the stars themselves).

On the other hand, if we transform to an accelerating frame, we find

$$\frac{d\left(m\dot{r}_j'\right)}{dt} + m\ddot{r}_{0,j} = F_j'\left(r_k'\right). \tag{3.8}$$

Observed in this (primed) frame force-free particles will accelerate past a primed observer (*i.e.*, an observer fixed in the frame) with acceleration $\ddot{\vec{r}}_0$ just as if there were a uniform gravitational field. Similarly the observer herself will experience a force $M\ddot{\vec{r}}_0$ in order to stay at a fixed point in the frame just as if there were gravity. Thus the acceleration of a reference frame is experimentally detectable and acts locally just like a gravitational field. The fact that the identical mass factor acts both as the coupling strength for gravity and as the inertial "resistance" in Newton's equations is know as the "principle of equivalence", and is under precise experimental study at the UW. (We don't have precision tests at length scales below about a fraction of a millimeter.)

The other side of the similarity between accelerating frames and gravity is that a frame of reference that is "freely falling" in a gravitational field, and is thus accelerating, yields the same physics as an inertial reference frame, but with no gravitational interaction.

It is important to note that, if we are comparing experiments in 2 different laboratories in two different reference frames moving with respect to each other, we should use both identical apparatus and identical initial conditions in the two frames. Thus the initial conditions are identical and not related by the transformation of Eq. (3.4).

So it is reasonable to ask – what is the *complete set* of coordinate transformations (the Galilean transformations) that preserve the inertial properties of the frame of reference. To answer this question we want consider a general functional form for the transformations, *i.e.*, the transformations are allowed to depend on the coordinates themselves. The mathematical background here includes the study of vector spaces (our Euclidean configuration space, $\vec{x}$, is such a space), linear algebra and group theory (see Lectures 4 and 5 on the web page for my Phys. 557 course).

We can express the most general transformation of this nature, including the possibility that it is nonlinear, as (using component notation)

$$\begin{aligned} r_k'\left(\vec{r}\left(t\right),t\right) &= g_k\left(r_1\left(t\right),r_2\left(t\right),r_3\left(t\right),t\right), \\ r_k\left(\vec{r}'\left(t\right),t\right) &= h_k\left(r_1'\left(t\right),r_2'\left(t\right),r_3'\left(t\right),t\right). \end{aligned} \tag{3.9}$$

The second equation expresses the inverse transformation, which we require to exist. The only other required features are that g_k and h_k are differentiable, single-valued and invertible.
Note that in the first line we are treating $\vec{r}'$ as a function of $\vec{r}\left(t\right)$ and t, while $\vec{r}\left(t\right)$ is a function of only t (*i.e.*, $dr/dt = \partial r/\partial t$). The converse is true for the inverse transformation in the second line.

Considering the time derivatives of these expressions and using the chain rule we can obtain the corresponding expressions for the transformations of the velocity and the acceleration (as usual in such an expression repeated indices are summed over),

$$\begin{aligned} \dot{r}_j' &= \frac{dr_j'}{dt} = \frac{\partial r_j'}{\partial r_k}\dot{r}_k + \frac{\partial r_j'}{\partial t} = \frac{\partial g_j}{\partial r_k}\dot{r}_k + \frac{\partial g_j}{\partial t} \\ &\equiv R_{jk}\dot{r}_k + \frac{\partial g_j}{\partial t}, \end{aligned} \tag{3.10}$$

and

$$\ddot{r}'_j = \frac{d^2 r'_j}{d^2 t} = \frac{\partial^2 r'_j}{\partial r_k \partial r_l}\dot{r}_k \dot{r}_l + 2\frac{\partial^2 r'_j}{\partial r_k \partial t}\dot{r}_k + \frac{\partial r'_j}{\partial r_k}\ddot{r}_k + \frac{\partial^2 r'_j}{\partial^2 t}$$
$$= R_{jk}\ddot{r}_k + \frac{\partial^2 r'_j}{\partial r_k \partial r_l}\dot{r}_k \dot{r}_l + 2\frac{\partial^2 r'_j}{\partial r_k \partial t}\dot{r}_k + \frac{\partial^2 r'_j}{\partial^2 t}, \quad (3.11)$$

where we have introduced the 3x3 matrix

$$R_{jk} = \frac{\partial r'_j}{\partial r_k}. \quad (3.12)$$

The inverse transformation for the velocity, for example, has the corresponding form

$$\dot{r}_j = \frac{dr_j}{dt} = \frac{\partial r_j}{\partial r'_k}\dot{r}'_k + \frac{\partial r_j}{\partial t} = \frac{\partial h_j}{\partial r_k}\dot{r}_k + \frac{\partial h_j}{\partial t}$$
$$\equiv R^{-1}_{jk}\dot{r}'_k + \frac{\partial h_j}{\partial t}, \quad (3.13)$$

with the inverse matrix

$$R^{-1}_{jk} = \frac{\partial r_j}{\partial r'_k}. \quad (3.14)$$

We can verify that this is the inverse by considering

$$R^{-1}_{jk}R_{kl} = \frac{\partial r_j}{\partial r'_k}\frac{\partial r'_k}{\partial r_l} = \frac{\partial r_j}{\partial r_l} = \delta_{jl}, \quad (3.15)$$

where δ_{jl} is the Kronecker delta function (the unit matrix) that is 1 for $j = l$ and zero otherwise.

Now we want to consider the other side of Newton's second law, the forces. For now we focus on conservative forces. Since the potential energy is described by a scalar function, we require that it transforms as

$$U'\left(\vec{r}',t\right)=U\left(\vec{r},t\right)=U\left(\vec{h}\left(\vec{r}',t\right),t\right). \tag{3.16}$$

Now we can again use the chain rule of partial derivatives to derive the transformation of the Cartesian components of the corresponding force

$$\begin{aligned} F_j'\left(\vec{r}',t\right) &= -\frac{\partial U'}{\partial r_j'} = -\frac{\partial U}{\partial r_k}\frac{\partial r_k}{\partial r_j'} \equiv -T_{jk}\,\frac{\partial U}{\partial r_k} \\ &= T_{jk}\,F_k\left(\vec{r},t\right)\left[T_{jk} \equiv \frac{\partial r_k}{\partial r_j'}\right]. \end{aligned} \tag{3.17}$$

We note that $T_{jk} = R_{kj}^{-1}$, or in the more efficient notation of linear algebra

$$R^{-1}=\tilde{T}, \tag{3.18}$$

where the right hand side is the transpose of *T*.

Our goal is to ensure that the two sides of Newton's equation transform in the same way, *i.e.*, we want

$$\frac{d\left(m\dot{r}_k\right)}{dt}=F_k \Rightarrow \frac{d\left(m\dot{r}_k'\right)}{dt}=F_k', \tag{3.19}$$

Thus, in order to be certain that we are transforming to another inertial frame (assuming we started in one), it is necessary that the acceleration transform just like the force. Comparing Eqs. (3.11) and (3.17) we first notice that the transformation of the force exhibits no explicit velocity or time dependence. Thus we require that the final 3 terms in Eq. (3.11) vanish for any value of $\dot{r}_k$, the velocity in the original frame. This requires that each term separately vanish, which is accomplished by transformations of the form

$$g_j\left(r_1,r_2,r_3,t\right)=a_j+b_j\left(r_1,r_2,r_3\right)+c_j t, \tag{3.20}$$

where a_j and c_j are constants and b_j is a linear function of the 3 components of r. The first and last terms are just the constant translation and transformation to a reference frame with constant relative velocity mentioned earlier. The fact that the middle term is a linear function ensures that the matrix R is constant, *i.e.*, not a function of t or the 3 components of r. The final step in defining the Galilean transformation is to require that

$$T_{jk}=R_{jk}. \tag{3.21}$$

In this case both the acceleration and the force transform in the same way and we have

$$\frac{d\left(m\dot{r}_j'\right)}{dt}-F_j'=R_{jk}\left(\frac{d\left(m\dot{r}_k\right)}{dt}=F_k\right)=0. \tag{3.22}$$

From Eqs. (3.18) and (3.21) it follows that

$$R_{jk}=R_{kj}^{-1}\Rightarrow R^{-1}=\tilde{R}. \tag{3.23}$$

Thus the linear transformation included in the Galilean transformation is described by a 3x3 constant matrix for which the inverse is equal to the transpose. These matrices are the orthogonal matrices and the associated transformations are the (global or rigid) rotations in 3-dimensions. (The name derives from the fact that right angles are preserved by the transformations.) The associated group is O(3), although we typically focus on the group SO(3) where the "S" means we do not include reflections. Examples of reflections are $x'=-x$ or $\left(x',y',z'\right)=\left(-x,-y,-z\right)$. Mathematically this means that we include only matrices with determinant = +1. More generally the above property for orthogonal matrices requires only that

$$\det\left(R\tilde{R}\right)=\left\{\det\left(R\right)\right\}^2=\det\left(I\right)\Rightarrow\det\left(R\right)=\pm 1. \tag{3.24}$$

ASIDE: The orthogonal 3x3 matrices with determinant +1 constitute the fundamental representation of the (abstract) group SO(3), *i.e.*, the smallest matrices whose multiplication properties provide a faithful representation of the (multiplication) properties of the elements of the group. These matrices and the group elements are parameterized by three continuous parameters (the Euler angles), which we can think of as the angles by which we rotate in the three independent planes, (xy), (yz), and (zx) (or, in the special case of 3-dimensions, as the angles of rotation about the three unique axes perpendicular to these planes). This parameterization is differentiable so that SO(3) is a so-called Lie group and we can define derivatives with respect to each of the angles infinitesimally close to the origin in parameter space, *i.e.*, close to the identity operator in the group space. The derivatives serve to define operators that generate infinitesimal transformations and are called simply the generators of the group. For SO(3) there are 3 generators, which, with appropriate normalizations, are the angular momentum operators, L_x, L_y, L_z, familiar from quantum mechanics. The generators serve to define a vector space, called the algebra of the group, in which the vector product is provided by the commutator, for example,

$$\left[L_x, L_y\right] = iL_z : \left[L_j, L_k\right] = i\varepsilon_{jkl} L_l .$$

The relevant structure of the algebra, and thus the structure of the group near the identity, is specified by the 3-index tensor on the right-hand-side of the commutator equation, which is called the structure constant of the group (the factor of *i* is a result of the choice to define the L_k as Hermitian). The transformations studied by Lie will play an important role in our discussion of flows in phase space.

In summary we have seen that the form of Newton's second law is covariant under the Galilean transformations, which include translation of the origin by a constant vector (3 parameters), transformation to a reference frame moving with *constant* relative velocity (3 parameters) and constant rigid rotations of the Cartesian coordinate axes (3 parameters). Although we have not discussed it explicitly, it should be clear that there is also covariance with respect to a constant translation in time,

$$t' = t + \tau , \tag{3.25}$$

i.e., physics is the same in different time zones. Overall the Galilean transformations are described by 10 parameters. Note that, if the form of the force exhibits *in*variance with respect to time translations (no explicit time dependence), then we expect the energy to be conserved.