Lecture 7: The Methods of Lagrange III – Symmetries and Hamiltonians

We want to discuss again the question of the connections between the symmetry properties of a mechanical system, *i.e.*, the invariance of the Lagrangian (and the equations of motion) under a change of variables, and the existence of conserved quantities (*i.e.*, constants of the motion). As a starting point we return to the Lagrange equation, written in generalized coordinates for a conservative system,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0. \tag{7.1}$$

For a system of generalized coordinates such that the Lagrangian is independent of (at least) one of the coordinates, say q_n, it follows that the corresponding canonical momentum is independent of time, *i.e.*, is a constant of the motion,

$$\begin{aligned} &\frac{\partial L}{\partial q_k} = 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{d}{dt}\left(p_k\right) = 0 \\ &\Rightarrow p_k = \text{constant.} \end{aligned} \tag{7.2}$$

This is just the familiar statement in Cartesian coordinates that translational invariance in $\vec{r}_k$ implies conservation of $\vec{p}_k$.

Next we want to consider this result in the somewhat more general language of the general coordinate transformations of the last lecture. In particular, consider coordinate transformations (of our *f* unconstrained degrees of freedom) that are parameterized in terms of continuous, differentiable parameters (*e.g.*, rotations in terms of the rotation angles, *i.e.*, just the Lie groups we discussed earlier),

$$q'_j\left(\alpha\right) = F_j\left(q_1, \ldots, q_f, \alpha\right). \tag{7.3}$$

We represent the inverse transformation by $q_j = \tilde{F}_j\left(q'_1, \ldots, q'_f, \alpha\right)$ and the identity transformation by

$$q'_j(0) = F_j(q_1, \ldots, q_f, 0) \equiv q_j. \tag{7.4}$$

In general such a transformation will yield a new form for the Lagrangian in the sense that the new Lagrangian (where α may be a "vector" of parameters)

$$L'(q', \dot{q}', t) = L(q, \dot{q}, t) \equiv L_\alpha(q'(\alpha), \dot{q}'(\alpha), t) \tag{7.5}$$

is a different function of the new coordinates than the old Lagrangian was of the old coordinates. Now connect this transformation to our previous studies of infinitesimal variations by considering an infinitesimal transformation near the origin in α,

$$dq_k = q'_k - q_k \simeq \left[\frac{\partial q'_k}{\partial \alpha}\right]_{\alpha=0} d\alpha, \tag{7.6}$$

where it is important to recognize that this is a "real", not virtual transformation (hence the notation of $d's$ instead of $\delta's$) and there is no constraint that it vanish at the endpoints in time. [Recall from our brief introduction to group theory that the quantity in the square brackets is an element of the algebra, a linear combination of generators, times the untransformed coordinates.] The corresponding change in the action (to first order in the parameter change) is

$$\begin{aligned} dA &\simeq \int_{t_1}^{t_2} \left(\left.\frac{dL}{d\alpha}\right|_{\alpha=0} d\alpha \right) dt \\ &\simeq \int_{t_1}^{t_2} \left(\left[\frac{\partial L}{\partial q'_k(\alpha)} \frac{\partial q'_k(\alpha)}{\partial \alpha} + \frac{\partial L}{\partial \dot{q}'_k(\alpha)} \frac{\partial \dot{q}'_k(\alpha)}{\partial \alpha} \right]_{\alpha=0} d\alpha \right) dt. \end{aligned} \tag{7.7}$$

If we now use the fact that $q'_k(\alpha)$ is taken to be a solution of Lagrange's equation for any α value, we have

$$
\begin{aligned}
dA &\simeq \int_{t_1}^{t_2}\left(\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k'(\alpha)}\right)\frac{\partial q_k'(\alpha)}{\partial\alpha}+\frac{\partial L}{\partial \dot{q}_k'(\alpha)}\frac{\partial \dot{q}_k'(\alpha)}{\partial\alpha}\right]_{\alpha=0} d\alpha\right)dt \\
&\simeq \left[\left(\frac{\partial L}{\partial \dot{q}_k'(\alpha)}\frac{\partial q_k'(\alpha)}{\partial\alpha}\right)_{\alpha=0} d\alpha\right]_{t_1}^{t_2} \\
&+\int_{t_1}^{t_2}\left(\left[-\frac{\partial L}{\partial \dot{q}_k'(\alpha)}\frac{d}{dt}\left(\frac{\partial q_k'(\alpha)}{\partial\alpha}\right)+\frac{\partial L}{\partial \dot{q}_k'(\alpha)}\frac{\partial \dot{q}_k'(\alpha)}{\partial\alpha}\right]_{\alpha=0} d\alpha\right)dt \\
&\simeq \left[\left(\frac{\partial L}{\partial \dot{q}_k'(\alpha)}\frac{\partial q_k'(\alpha)}{\partial\alpha}\right)_{\alpha=0} d\alpha\right]_{t_1}^{t_2} \\
&\simeq \left[\left(p_k'(\alpha)\frac{\partial q_k'(\alpha)}{\partial\alpha}\right)_{\alpha=0} d\alpha\right]_{t_1}^{t_2} = \left[\left(p_k'(\alpha)\right)_{\alpha=0} dq_k\right]_{t_1}^{t_2}.
\end{aligned}
\tag{7.8}
$$

The change in the action is given in terms of the (difference of the) quantity

$$
G \equiv \left(p_k'(\alpha)\frac{\partial q_k'(\alpha)}{\partial\alpha}\right)_{\alpha=0}
\tag{7.9}
$$

evaluated at the end points in time (times $d\alpha$). In our previous discussion of virtual variations (used to obtain the Lagrange equations) the variation dq_k would be constrained to vanish at the endpoints in time, *i.e.*, the last expression in Eq. (7.8) would vanish. Here there is no such requirement. However, it is still possible that the action is invariant under the coordinate transformation parameterized by α, *i.e.*, $dA = 0$ for arbitrary $d\alpha$. This can be true only in the quantity in Eq. (7.9) is independent of the time,

$$
G = \left(p_k'(\alpha)\frac{\partial q_k'(\alpha)}{\partial\alpha}\right)_{\alpha=0} = \text{constant.}
\tag{7.10}
$$

Thus G is a constant of the motion, *i.e.*, it is conserved.

So far we have only assumed that the action is invariant under these coordinate transformations. If the Lagrangian itself is invariant, *i.e*., the functional dependence on the coordinates and velocities is the same before and after the transformations, $L'(q',\dot{q}',t) = L(q',\dot{q}',t)$, we can make a connection between this discussion and the earlier discussion associated with Eq. (7.2). We can now perform a further change of variables such that one of the new variables is equal to the transformation parameter, $\tilde{q}_j = \alpha$. The new Lagrangian is necessarily independent of $\tilde{q}_j$, $\partial\tilde{L}/\partial\tilde{q}_j = 0$, and the corresponding canonical momentum is conserved. It is easy to verify that this canonical momentum is just the quantity G. This is how we can understand the connection between translational invariance and momentum conservation; rotational invariance and angular momentum conservation.

In fact, the invariance of the action under transformations, which is all we really need here, does not require the full invariance of the Lagrangian. If a coordinate transformation described by a continuous parameter leaves the Lagrangian invariant except for a total time derivative, $L' = L + d\Lambda/dt$, the action still is invariant (under variations that vanish at the temporal endpoints of Eq. (7.7)) and the corresponding G is conserved. This connection between an invariance of the Lagrangian (up to a total derivative) under coordinate transformations described by continuous, differentiable parameters (Lie groups) and the existence of conserved quantities (Eq. (7.10)) is usually called Noether's Theorem (after Emmy Noether). Further the invariance of the Lagrangian is typically related to some geometrical symmetry, *e.g*., rotational symmetry, translational symmetry, *etc.* of the physical system. The concepts of symmetry, invariance and conservation laws are unavoidably connected and a major component of the physics advances of the last 100 years. It is also possible that the underlying symmetry is associated with a space other than the usual 3-dimensional configuration space, *i.e*., some "internal" space.

A well known example of both an internal space symmetry and a Lagrangian that changes by a total time derivative is provided by electromagnetism. Here we focus on the motion of a charged particle in an "external" field (*i.e.*, ignore the back reaction of the charged particle on the sources of the fields). Recall that Maxwell's equations look like

$$
\begin{aligned}
\vec{\nabla}\cdot\vec{E} &= 4\pi\rho, \\
\vec{\nabla}\times\vec{E} &= -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}, \\
\vec{\nabla}\cdot\vec{B} &= 0, \\
\vec{\nabla}\times\vec{B} &= \frac{1}{c}\frac{\partial\vec{E}}{\partial t}+\frac{4\pi}{c}\vec{j},
\end{aligned}
\tag{7.11}
$$

with ρ the external change density, $\vec{j}$ the external changed current and no magnetic monopoles. The content of these equations (in free space) is most efficiently expressed by writing the electric and magnetic fields in terms of a vector and a scalar potential, $\vec{A}$ and ϕ. We have

$$
\begin{aligned}
\vec{E} &= -\vec{\nabla}\phi-\frac{1}{c}\frac{\partial\vec{A}}{\partial t}, \\
\vec{B} &= \vec{\nabla}\times\vec{A}.
\end{aligned}
\tag{7.12}
$$

The corresponding Lagrangian for a particle of electric charge Q and mass m has the following form

$$
L_{EM} = \frac{m}{2}\dot{\vec{r}}^{\,2}-Q\phi\left(\vec{r},t\right)+\frac{Q}{c}\dot{\vec{r}}\cdot\vec{A}\left(\vec{r},t\right), \tag{7.13}
$$

where it is important to note that the potential (the second and third terms) is dependent on the velocity of the particle. Applying the Lagrange equations to this Lagrangian we find

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial L_{EM}}{\partial \dot{\vec{r}}}\right)-\frac{\partial L_{EM}}{\partial \vec{r}}=0 \\
&\qquad =\frac{d}{dt}\left(m\dot{\vec{r}}+\frac{Q}{c}\vec{A}\right)+Q\vec{\nabla}\phi-\frac{Q}{c}\vec{\nabla}\left(\dot{\vec{r}}\cdot\vec{A}\right) \\
&\Rightarrow m\ddot{\vec{r}}=Q\left(-\vec{\nabla}\phi-\frac{1}{c}\dot{\vec{r}}\cdot\vec{\nabla}\vec{A}-\frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right)+\frac{Q}{c}\vec{\nabla}\left(\dot{\vec{r}}\cdot\vec{A}\right).
\end{aligned} \tag{7.14}$$

We note in passing that the canonical momentum includes a dependence on the vector potential $\vec{p}=m\dot{\vec{r}}+Q\vec{A}/c$, a feature which plays a central role in the field theory version of E&M. To evaluate the final term we need to recall an identity from vector calculus. In particular, we need that

$$\vec{\nabla}\left(\vec{a}\cdot\vec{b}\right)=\left(\vec{a}\cdot\vec{\nabla}\right)\vec{b}+\left(\vec{b}\cdot\vec{\nabla}\right)\vec{a}+\vec{a}\times\left(\vec{\nabla}\times\vec{b}\right)+\vec{b}\times\left(\vec{\nabla}\times\vec{a}\right), \tag{7.15}$$

and the fact that, with our definitions of what the independent variables are, we have

$$\vec{\nabla}\left(\dot{\vec{r}}\right)\equiv\left.\frac{\partial}{\partial\vec{r}}\right|_{\dot{\vec{r}}\text{ fixed}}\left(\dot{\vec{r}}\right)=0. \tag{7.16}$$

Thus, from Eq. (7.14), we have

$$\vec{\nabla}\left(\dot{\vec{r}}\cdot\vec{A}\right)=\left(\dot{\vec{r}}\cdot\vec{\nabla}\right)\vec{A}+\dot{\vec{r}}\times\left(\vec{\nabla}\times\vec{A}\right). \tag{7.17}$$

Substituting into Eq. (7.14) we finally have

$$\begin{aligned}
m\ddot{\vec{r}}&=Q\left(-\vec{\nabla}\phi-\frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right)-\frac{Q}{c}\dot{\vec{r}}\cdot\vec{\nabla}\vec{A}+\frac{Q}{c}\left(\dot{\vec{r}}\cdot\vec{\nabla}\vec{A}+\dot{\vec{r}}\times\left(\vec{\nabla}\times\vec{A}\right)\right) \\
&=Q\vec{E}+\frac{Q}{c}\dot{\vec{r}}\times\vec{B}.
\end{aligned} \tag{7.18}$$

We recognize the right-hand-side of the last equation as the desired Lorentz force of electromagnetism, confirming that we have the correct Lagrangian. We know that the electric and magnetic fields (the physical quantities) are invariant under gauge transformations defined by a single scalar function Λ of the form

$$\begin{aligned} \vec{A}' &= \vec{A} + \vec{\nabla}\Lambda, \\ \phi' &= \phi - \frac{1}{c}\frac{\partial \Lambda}{\partial t}. \end{aligned} \tag{7.19}$$

Thus the equations of motion, Eq. (7.18), are also invariant under such a transformation. Is the Lagrangian? Let us check. The only component that changes is the potential

$$\begin{aligned} U' &= Q\left(\phi - \frac{1}{c}\frac{\partial \Lambda}{\partial t}\right) - \frac{Q}{c}\dot{\vec{r}}\cdot\left(\vec{A} + \vec{\nabla}\Lambda\right) \\ &= U - \frac{Q}{c}\left(\dot{\vec{r}}\cdot\vec{\nabla}\Lambda + \frac{\partial \Lambda}{\partial t}\right) = U - \frac{Q}{c}\frac{d\Lambda}{dt}. \end{aligned} \tag{7.20}$$

Thus the Lagrangian changes by a total time derivative

$$L'_{\mathrm{EM}} = L_{\mathrm{EM}} + \frac{Q}{c}\frac{d\Lambda}{dt}, \tag{7.21}$$

under a gauge transformation, which we have already noted does not change the physics. (Since the equations of motion derive from the study of virtual displacements that vanish at the end points in time, a change in the action of the form $\Lambda\left(\vec{r},t\right)\Big|_{t_1}^{t_2}$ does not contribute to the virtual variation of the action and hence to the equations of motion.) As you may know this gauge transformation (corresponding to a change of phase for the electrically charged fields) is described by the group U(1) and invariance leads, via Noether, to conversed electric charge and currents.

The last topics to be discussed in this lecture are Hamilton's canonical equations. The goal is to switch from second order differential equations *a la* Newton and Lagrange to first order differential equations. The subtext is that we will be refocusing our attention from configuration space alone (the q_k) to phase space involving both the generalized coordinates and the canonical momenta, the canonical variables. As at the start of this lecture we wish to consider a conservative system described by a Lagrangian of *f* unconstrained generalized coordinates and velocities,

$$L\left(q_1,\ldots,q_f,\dot{q}_1,\ldots,\dot{q}_f,t\right)=T-U. \tag{7.22}$$

With the canonical momenta defined as in Eq. (7.2) ($p_k = \partial L/\partial \dot{q}_k$), we can use the Legendre transform to construct the Hamiltonian as a function of the canonical variables (q, p)

$$H\left(q_1,\ldots,q_f,p_1,\ldots,p_f,t\right)=\sum_{k=1}^{f} p_k\dot{q}_k - L, \tag{7.23}$$

i.e., we are to think of the Hamiltonian as a function of the 2*f* + 1 variables $\left(q_1,\ldots,q_f,p_1,\ldots,p_f,t\right)$. As usual (now) we can analyze this function by looking at small variations on both sides of Eq. (7.23)

$$\begin{aligned}
dH &= \sum_{k=1}^{f}\left(\frac{\partial H}{\partial q_k}dq_k + \frac{\partial H}{\partial p_k}dp_k\right) + \frac{\partial H}{\partial t}dt,\\
d\sum_{k=1}^{f} p_k\dot{q}_k - dL &\\
&= \sum_{k=1}^{f}\left(dp_k\dot{q}_k + p_k d\dot{q}_k - \frac{\partial L}{\partial q_k}dq_k - \frac{\partial L}{\partial \dot{q}_k}d\dot{q}_k\right) - \frac{\partial L}{\partial t}dt\\
&= \sum_{k=1}^{f}\left(dp_k\dot{q}_k + p_k d\dot{q}_k - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)dq_k - p_k d\dot{q}_k\right) - \frac{\partial L}{\partial t}dt\\
&= \sum_{k=1}^{f}\left(dp_k\dot{q}_k - \dot{p}_k dq_k\right) - \frac{\partial L}{\partial t}dt,
\end{aligned} \tag{7.24}$$

where we have used both the definition of the canonical momentum and Lagrange's equation. By equating the coefficients of the various terms we obtain Hamilton's canonical version of the equations of motion,

$$\begin{aligned} \dot{q}_k &= \frac{\partial H}{\partial p_k}, \\ \dot{p}_k &= -\frac{\partial H}{\partial q_k}, \\ \frac{\partial H}{\partial t} &= -\frac{\partial L}{\partial t}. \end{aligned} \tag{7.25}$$

These first order equations define a trajectory for the system in the $2f$ dimensional phase space $\left(q_1,\ldots,q_f, p_1,\ldots, p_f\right)$. It follows from Eq. (7.25) that the Hamiltonian is a conserved quantity if the Lagrangian has no explicit times dependence, *i.e.*, is invariant under translations in time,

$$\begin{aligned} \frac{dH}{dt} &= \sum_{k=1}^{f}\left(\frac{\partial H}{\partial q_k}\dot{q}_k + \frac{\partial H}{\partial p_k}\dot{p}_k\right) + \frac{\partial H}{\partial t} \\ &= \sum_{k=1}^{f}\left(-\dot{p}_k\dot{q}_k + \dot{q}_k\dot{p}_k\right) + \frac{\partial H}{\partial t} \\ &= \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \end{aligned} \tag{7.26}$$

For the familiar case that the kinetic energy is quadratic in the generalized coordinates (implying that the transformations from Cartesian coordinates are time-independent) we have

$$T = \frac{1}{2}\sum_{k,l=1}^{f} m_{kl}\dot{q}_k\dot{q}_l, \tag{7.27}$$

where the "metric" is time independent and symmetric, $m_{kl} = m_{lk}$. It follows that

$$
\begin{aligned}
p_j &= \frac{\partial T}{\partial \dot{q}_j} = \sum_{k=1}^{f} m_{jk}\dot{q}_k, \\
T &= \frac{1}{2}\sum_{k=1}^{f} p_k \dot{q}_k.
\end{aligned}
\tag{7.28}
$$

Thus the Hamiltonian is just the total mechanical energy

$$
H = \sum_{k=1}^{f} p_k \dot{q}_k - L = 2T - T + U = T + U = E. \tag{7.29}
$$

Since, by assumption, T has no explicit time dependence, if U is also free of explicit time dependence, then the total mechanical energy E is conserved.

As a simple example consider the usual Cartesian description of a single point particle,

$$
\begin{aligned}
L &= \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - U\left(x, y, z\right), \\
p_x &= \frac{\partial L}{\partial \dot{x}} = m\dot{x},\ p_y = m\dot{y},\ p_z = m\dot{z}, \\
H &= p_x\dot{x} + p_y\dot{y} + p_z\dot{z} - L \\
&= \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + U.
\end{aligned}
\tag{7.30}
$$

Hamilton's equations are then

$$
\begin{aligned}
&\dot{x} = \frac{p_x}{m}, \dot{y} = \frac{p_y}{m}, \dot{z} = \frac{p_z}{m}, \\
&\dot{p}_x = -\frac{\partial U}{\partial x}, \dot{p}_y = -\frac{\partial U}{\partial y}, \dot{p}_z = -\frac{\partial U}{\partial z},
\end{aligned}
\tag{7.31}
$$

i.e., just the usual definitions and Newton equations. The results are a bit more interesting in spherical coordinates where

$$
\begin{aligned}
H &= p_r\dot{r} + p_\theta\dot{\theta} + p_\phi\dot{\phi} - L \\
&= \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta} \right) + U\left(r,\theta,\phi\right) \\
&\Rightarrow \dot{r} = \frac{p_r}{m}, \dot{\theta} = \frac{p_\theta}{mr^2}, \dot{\phi} = \frac{p_\phi}{mr^2\sin^2\theta}, \\
&\dot{p}_r = \frac{p_\theta^2}{mr^3} - \frac{\partial U}{\partial r}, \dot{p}_\theta = \frac{p_\phi^2\cos\theta}{mr^2\sin^3\theta} - \frac{\partial U}{\partial \theta}, \dot{p}_\phi = -\frac{\partial U}{\partial \phi}.
\end{aligned}
\tag{7.32}
$$

An important feature of the Hamiltonian formalism is the similarity to the equations of fluid flow. This connection helps us to visualize the solutions of Hamilton's equations as a flow through phase space. To see the connection, consider the flow of an incompressible fluid in 2-dimensions. The continuity equation is

$$
\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\left(\rho\vec{V}\right) = 0 \tag{7.33}
$$

where ρ is the fluid density and $\vec{V} = \left(V_x, V_y\right)$ is the velocity field describing the motion of the fluid. From the fact that the density is assumed to be constant in space and time (incompressible) it follows that the velocity field is divergence free. Like the case of the magnetic field, it follows that we can define the velocity field as the curl of vector field, which points in the direction orthogonal to the 2-D motion. We can write

$$\vec{V} = \vec{\nabla} \times \vec{A},$$
$$\vec{A} = \hat{z}\psi\left(x, y\right), \tag{7.34}$$

where ψ is the stream function of the flow, a construction only possible in 2-D. In Cartesian coordinates this flow is described by

$$V_x = \frac{\partial \psi}{\partial y}, V_y = -\frac{\partial \psi}{\partial x}. \tag{7.35}$$

It is the similarity of Eq. (7.35) to Hamilton's equations that suggests the picture of flow in phase space. However, the latter have no restriction to Cartesian coordinates. In the flow case particles flowing in the fluid follow streamlines that are lines of constant ψ and that are everywhere tangent to the velocity field. In the incompressible case the stream function is time independent,

$$\begin{aligned} \frac{d\psi}{dt} &= \frac{\partial \psi}{\partial x} V_x + \frac{\partial \psi}{\partial y} V_y \\ &= \frac{\partial \psi}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \psi}{\partial x} = 0, \end{aligned} \tag{7.36}$$

a constant of the flow. We will make use of this idea of Hamilton's equations describing flows in phase space when we consider chaotic behavior.