

Lecture 8 – Small Oscillations [Chapter 4 in F & W]

We want to discuss the small oscillations that a mechanical system exhibits when perturbed around an equilibrium point as has already appeared in the HW exercises (and the appendix to the previous lecture). As usual we are imagining that the systems can be described by potentials, $U(q_1, \dots, q_f)$ and Lagrangians,

$L(q_1, \dots, q_f, \dot{q}_1, \dots, \dot{q}_f)$. We consider systems that exhibit equilibrium points in phase space where both $\dot{q}_k = 0$ and $\dot{p}_k = 0$ (all k). If we expand the behavior in terms of small perturbations about this point, we expect to have a linear description of the system, in which case we can always solve for the behavior of the system in terms of the normal modes (the eigenfunctions) with generic time dependence $\exp(i\omega_j t)$. For expansions about a stable equilibrium (a true minimum of the potential) we find real values for the frequencies ω_j . (For future reference we note that the corresponding flow in phase space – recall the appendix to the last lecture - is elliptical, including circular.) For expansions about an unstable equilibrium (a maximum or saddle point of the potential) the frequency picks up an imaginary part and the system tends to run away from the initial point (thereby lowering its energy). (This corresponds to hyperbolic flow in phase space.)

We consider systems described by a positive kinetic energy defined in terms of a possibly coordinate dependent metric (which includes any mass factors left over after the definition of q),

$$T = \frac{1}{2} \sum_{k,l} m_{kl}(q_1, \dots, q_f) \dot{q}_k \dot{q}_l. \quad (8.1)$$

We are implicitly assuming that all conserved canonical momenta have been identified and replaced by their constant values in the effective potential, which we will continue to call $U(q_1, \dots, q_f)$. The remaining coordinates are taken to be f in number. Since T is positive, the matrix m_{kl} (sometimes represented as g_{kl}) is both symmetric and positive definite. Hence it also has an inverse, m^{-1}_{kl} . As usual we have,

$$L = T - U,$$

$$\begin{aligned} p_k &= \frac{\partial L}{\partial \dot{q}_k} = \frac{1}{2} \sum_l m_{kl} (q_1, \dots, q_f) \dot{q}_l + \frac{1}{2} \sum_l m_{lk} (q_1, \dots, q_f) \dot{q}_l \\ &= \sum_l m_{kl} (q_1, \dots, q_f) \dot{q}_l, \end{aligned} \quad (8.2)$$

$$\sum_{k,l} p_k \dot{q}_l = \sum_{kl} m_{kl} (q_1, \dots, q_f) \dot{q}_k \dot{q}_l = 2T \Rightarrow H = T + U = E.$$

Employing the inverse of the metric we can write

$$\begin{aligned} \dot{q}_l &= \sum_k m^{-1}_{lk} p_k \\ \Rightarrow T &= \frac{1}{2} \sum_{kl} m_{lk} m^{-1}_{lm} m^{-1}_{kn} p_m p_n = \frac{1}{2} \sum_{kl} m^{-1}_{lk} p_l p_k \equiv \frac{1}{2} \tilde{p} m^{-1} p, \end{aligned} \quad (8.3)$$

where we have introduced the usual matrix notation in the last step (the transpose \tilde{p} is a row vector instead of a column vector). The equilibria are defined by Hamilton's equations set to zero, which yield

$$\left. \begin{aligned} \dot{q}_l &= \frac{\partial H}{\partial p_l} = \frac{\partial T}{\partial p_l} = m^{-1}_{lk} p_k = 0 \\ \dot{p}_l &= -\frac{\partial H}{\partial q_l} = -\tilde{p} \frac{\partial m^{-1}}{\partial q_l} p - \frac{\partial U}{\partial q_l} = 0 \end{aligned} \right\} \Rightarrow \begin{cases} p_k = 0 \\ \frac{\partial U}{\partial q_k} = 0 \end{cases}. \quad (8.4)$$

We focus first on a solution, $\bar{q} = \bar{q}_1, \dots, \bar{q}_f$, to these equations corresponding to a stable (elliptic) equilibrium (with later mention of the hyperbolic case). In terms of a small perturbation about the equilibrium point, $\delta q_k = q_k - \bar{q}_k$, we have

$$\begin{aligned}
U(q) &\simeq U(\bar{q}) + \frac{1}{2} \sum_{k,l} \left. \frac{\partial^2 U}{\partial q_k \partial q_l} \right|_{q=\bar{q}} \delta q_k \delta q_l \\
&\equiv U(\bar{q}) + \frac{1}{2} \sum_{k,l} v_{kl} \delta q_k \delta q_l,
\end{aligned} \tag{8.5}$$

$$T(p) \simeq \frac{1}{2} \sum_{kl} m_{kl}(\bar{q}_1, \dots, \bar{q}_f) \delta \dot{q}_k \delta \dot{q}_l,$$

where we have kept only terms (up) to quadratic order in the small perturbations. This will lead to linear equations of motion, which we know we can solve. By assumption of being at a stable equilibrium we expect the matrix v to be real, positive semi-indefinite and symmetric.

Now we move entirely to matrix notation (summations assumed) and subtract the (constant) value of the potential at the equilibrium point, which yields

$$\eta \equiv \begin{pmatrix} \delta q_1 \\ \vdots \\ \delta q_f \end{pmatrix} \Rightarrow \begin{cases} T = \frac{1}{2} \dot{\eta} m \dot{\eta} \\ U = \frac{1}{2} \tilde{\eta} v \eta \end{cases}. \tag{8.6}$$

In this notation (the linear) Lagrange's equations of motion are (the metric is time independent by assumption)

$$m \ddot{\eta} = -v \eta. \tag{8.7}$$

As usual such linear equations with constant coefficients suggests looking for exponential solutions of the form

$$\eta = \text{Re} \sum_k c_k e^{i\omega_k t}, \tag{8.8}$$

i.e., the magic of exponentials is that any number of derivatives still yields the same exponential. Substituting in the generic form $\eta \sim \exp(i\omega t)$ (F&W tend to work with

just the real form but we will work with the complex form and then take the real part at the end) produces the eigenvalue equation

$$\omega^2 m\eta = v\eta. \quad (8.9)$$

As real, symmetric matrices both v and m can be diagonalized, but we want to diagonalize both at once (which is not possible if the matrices do not commute). We change Eq. (8.9) from a vector equation to a scalar equation by multiplying on the left by $\eta^\dagger = \tilde{\eta}^*$ (*i.e.*, multiply into the remaining free index) and use the facts that v and m are real and symmetric to obtain 2 equations

$$\begin{aligned} \eta_j^\dagger v\eta_k &= \lambda_k \eta_j^\dagger m\eta_k, \\ \eta_j^\dagger v\eta_k &= \lambda_j^* \eta_j^\dagger m\eta_k, \end{aligned} \quad (8.10)$$

In these expressions η_k is not the k^{th} component of a vector but rather the vector representing the k^{th} eigenvector of m with eigenvalue $\lambda_k = \omega_k^2$. By taking the difference of these two equations we find

$$(\lambda_k - \lambda_j^*) \eta_j^\dagger m\eta_k = 0. \quad (8.11)$$

Thus we can conclude from the diagonal terms, $j = k$, with the constraint that m is positive definite ($\eta_k^\dagger m\eta_k > 0$), that the eigenvalues are real,

$$\lambda_k = \lambda_k^*. \quad (8.12)$$

From the off-diagonal terms, $j \neq k$, assuming non-degenerate eigenvalues $\lambda_j \neq \lambda_k$, we learn that the eigenvectors are orthogonal (in terms of the metric),

$$\eta_j^\dagger m\eta_k = 0. \quad (8.13)$$

In the case of degeneracy, $\lambda_j = \lambda_k$, we can always choose orthogonal eigenvectors (by hand). We will assume this choice has been made in the following discussion.

Since this is a linear problem, there is an overall normalization ambiguity. We fix the normalization with the constraint

$$\eta_j^\dagger m \eta_j = 1. \quad (8.14)$$

We are also free to choose the eigenvectors to be real (the only ambiguity is an overall phase). From Eqs. (8.10) and (8.14) it follows that

$$\lambda_k = \frac{\eta_k^\dagger v \eta_k}{\eta_k^\dagger m \eta_k} = \eta_k^\dagger v \eta_k. \quad (8.15)$$

Thus, as we expect, the eigenvalues λ_k are real and positive for an elliptic equilibrium point, ($v = U'' > 0$), corresponding to real values of the frequencies ω_k . At hyperbolic points at least some of the eigenvalues will be negative ($v < 0$) yielding complex frequencies.

The eigenvalues are found by solving the f -order equation

$$\det[v - \lambda m] = 0, \quad (8.16)$$

which must hold in order to find non-trivial eigenfunctions (*i.e.*, otherwise the inverse of the matrix $v - \lambda m$ exists and we can solve Eq. (8.9) to find $\eta = 0$). Assuming that we have found the corresponding eigenvectors, we can (at least formally) define the normal mode transformation or modal matrix, which is a $f \times f$ matrix in which each column is one of the eigenvectors ($\eta_{j,k}$ is the k^{th} component of eigenvector j),

$$N = \mathcal{A} = \begin{pmatrix} \eta_{1,1} & \eta_{2,1} & \cdots & \eta_{f,1} \\ \vdots & & & \vdots \\ \eta_{1,f} & \eta_{2,f} & \cdots & \eta_{f,f} \end{pmatrix}. \quad (8.17)$$

From the orthonormal properties of the eigenvectors, Eqs. (8.13) and (8.14) we have (note, if everything is real, we have, as in F&W, $N^\dagger = N^T = \mathcal{A}^\dagger = \mathcal{A}^T$)

$$\begin{aligned} \left(N^\dagger m N \right)_{jk} &= \sum_{lm} \eta_{j,l}^* m_{lm} \eta_{k,m} = \eta_j^\dagger m \eta_k = \delta_{jk} \\ N^\dagger m N &= \mathbf{1}. \end{aligned} \quad (8.18)$$

Thus the transformation N serves to diagonalize m , or, said the other way, with respect to the metric m , N is a unitary matrix, $N^{-1} = N^\dagger$. Actually, with our choice of real eigenvectors, N is (real and) orthogonal, $N^{-1} = N^\dagger = \tilde{N} = N^T$, all defined with respect to the metric m . In this language Eq. (8.18) is a standard similarity transformation, $N^{-1}mN$ (or $\tilde{N}m\tilde{N}^{-1}$).

Thus by construction we have

$$\begin{aligned} (vN)_{jk} &= \sum_l v_{jl} N_{lk} = \sum_l v_{jl} \eta_{k,l} = \lambda_k \sum_l m_{jl} \eta_{k,l} \\ \Rightarrow vN &= mN\Lambda, \Lambda_{jk} = \delta_{jk} \lambda_k, \\ \Rightarrow N^\dagger vN &= N^\dagger mN\Lambda = \Lambda. \end{aligned} \quad (8.19)$$

So, as desired the transformation provided by the normal mode (modal) matrix diagonalizes both v and m . If we consider the linear transformation provided by N in terms of the general states, $\eta = N\zeta$, $\zeta = N^{-1}\eta$, we have

$$\begin{aligned} U &= \frac{1}{2} \eta^\dagger v \eta = \frac{1}{2} \zeta^\dagger N^\dagger v N \zeta = \frac{1}{2} \zeta^\dagger \Lambda \zeta \\ T &= \frac{1}{2} \dot{\eta}^\dagger m \dot{\eta} = \frac{1}{2} \dot{\zeta}^\dagger N^\dagger m N \dot{\zeta} = \frac{1}{2} \dot{\zeta}^\dagger \dot{\zeta} \\ E &= \text{Tr}[T + U] = \frac{1}{2} \sum_k \left(\dot{\zeta}_k^* \dot{\zeta}_k + \omega_k^2 \zeta_k^* \zeta_k \right). \end{aligned} \quad (8.20)$$

Thus we have a system composed of f independent harmonic oscillators with normal modes $\zeta_k(t) = c_k \exp(i\omega_k t)$, where we will use just the real parts (as in Eq. (8.8)). The question of whether the overall motion is periodic boils down to whether the individual frequencies have ratios that are rational or not. Every pair with a rational

ratio, $\omega_j/\omega_k = m/n$, will yield a 2-D subspace of the full configuration space in which the projected trajectory is periodic.

A classical example of such a system is the case of two (identical) coupled pendulums, as illustrated in Fig. 23.1 in F&W, of length l and mass m . In terms of the usual angles that define the deviation from the local down direction (*i.e.*, the direction of gravity) we have

$$\begin{aligned} T &= \frac{ml^2}{2}(\dot{\theta}_1^2 + \dot{\theta}_2^2), \\ U &= mgl(2 - \cos\theta_1 - \cos\theta_2) + kl^2(1 - \cos(\theta_1 - \theta_2)), \end{aligned} \quad (8.21)$$

where k is the spring constant of the spring coupling the two pendulums. The corresponding Lagrangian in the small angle limit is (the variables in F&W are $\eta_k = l\theta_k$)

$$L = \frac{ml^2}{2}(\dot{\theta}_1^2 + \dot{\theta}_2^2) - \frac{mgl}{2}(\theta_1^2 + \theta_2^2) - \frac{kl^2}{2}(\theta_1 - \theta_2)^2. \quad (8.22)$$

We can simplify the analysis (or at least the notation) by switching to dimensionless quantities

$$\begin{aligned} L' &= \frac{L}{mgl}, \\ t' &= t\sqrt{\frac{g}{l}} \equiv \omega_p t, \\ \alpha &= \frac{kl}{gm} \equiv \frac{\omega_s^2}{\omega_p^2}, \end{aligned} \quad (8.23)$$

where we have identified the two natural frequencies in the problem: the usual pendulum oscillator frequency, $\omega_p = \sqrt{g/l}$, and the usual spring oscillator frequency, $\omega_s = \sqrt{k/m}$. In terms of these quantities we want to study

$$L' = \frac{1}{2} \left(\left(\frac{d\theta_1}{dt'} \right)^2 + \left(\frac{d\theta_2}{dt'} \right)^2 \right) - \frac{1}{2} (\theta_1^2 + \theta_2^2) - \frac{\alpha}{2} (\theta_1 - \theta_2)^2. \quad (8.24)$$

Switching to matrix, vector notation we have

$$\begin{aligned} \eta &= \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}, \\ T' &= \frac{\dot{\eta}^\dagger m \dot{\eta}}{2} : m = \mathbf{1}, \\ U' &= \frac{\eta^\dagger v \eta}{2} : v = \begin{pmatrix} 1 + \alpha & -\alpha \\ -\alpha & 1 + \alpha \end{pmatrix}, \end{aligned} \quad (8.25)$$

where the dots in the second line mean $\partial/\partial t'$. The eigenvalue problem is

$$\begin{aligned} \det \begin{bmatrix} 1 + \alpha - \lambda & -\alpha \\ -\alpha & 1 + \alpha - \lambda \end{bmatrix} &= 0 \\ \Rightarrow (1 + \alpha - \lambda)^2 - \alpha^2 &= 1 + 2\alpha - 2(1 + \alpha)\lambda + \lambda^2 = 0 \\ \Rightarrow \lambda_{\pm} &= 1 + \alpha \pm \sqrt{(1 + \alpha)^2 - 1 - 2\alpha} = \begin{cases} 1 + 2\alpha \\ 1 \end{cases}. \end{aligned} \quad (8.26)$$

The 2 eigenfrequencies are (recall the factor of $\omega_p = \sqrt{g/l}$ in our definition of t')

$$\begin{aligned} \omega_1 &= \sqrt{\lambda_-} \omega_p = \sqrt{\frac{g}{l}}, \\ \omega_2 &= \sqrt{\lambda_+} \omega_p = \sqrt{1 + 2\alpha} \omega_p = \sqrt{\omega_p^2 + 2\omega_s^2} = \sqrt{\frac{g}{l} + 2\frac{k}{m}}. \end{aligned} \quad (8.27)$$

The corresponding eigenvectors are

$$\begin{aligned}
v\eta_k &= \lambda_k \eta_k \\
\Rightarrow \eta_1 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \eta_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix},
\end{aligned} \tag{8.28}$$

where we identify the first eigenfunction with the case where the 2 pendulums move in parallel, with no stretching of the spring, and the second with the two pendulums moving oppositely with maximum spring stretching/compression. The normal modes are then

$$\begin{aligned}
N &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, N^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\
\Rightarrow N^{-1}\eta &= \frac{1}{\sqrt{2}} \begin{pmatrix} \theta_1 + \theta_2 \\ \theta_1 - \theta_2 \end{pmatrix} = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \\
\Rightarrow \eta &= N\zeta = \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta_1 + \zeta_2 \\ \zeta_1 - \zeta_2 \end{pmatrix} = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}.
\end{aligned} \tag{8.29}$$

In terms of the normal modes the (original) Lagrangian decouples

$$L = \frac{1}{2}(\dot{\zeta}_1^2 + \dot{\zeta}_2^2) - \frac{1}{2}(\omega_1^2 \zeta_1^2 + \omega_2^2 \zeta_2^2). \tag{8.30}$$

In terms of the scaled coordinates with $\bar{\omega}_1 = \omega_1/\omega_p = 1$ and $\bar{\omega}_2 \equiv \bar{\omega} = \omega_2/\omega_p = \sqrt{1+2\alpha}$ we have

$$\begin{aligned}
\zeta_1 &= a_1 \sin t' + b_1 \cos t' = c_1 \cos(t' + \phi_1), \\
\zeta_2 &= a_2 \sin \bar{\omega}t' + b_2 \cos \bar{\omega}t' = c_2 \cos(\bar{\omega}t' + \phi_2) \\
\Rightarrow \theta_1 &= \frac{1}{\sqrt{2}}(a_1 \sin t' + b_1 \cos t' + a_2 \sin \bar{\omega}t' + b_2 \cos \bar{\omega}t'), \\
\theta_2 &= \frac{1}{\sqrt{2}}(a_1 \sin t' + b_1 \cos t' - a_2 \sin \bar{\omega}t' - b_2 \cos \bar{\omega}t').
\end{aligned} \tag{8.31}$$

If we assume initial conditions of the form $\theta_1(0) = \theta_2(0) = d\theta_2/dt' \Big|_{t'=0} = 0$ and $d\theta_1/dt' \Big|_{t'=0} = c$ (a kick applied to pendulum 1 at $t = 0$), we have

$$\begin{aligned}\theta_1(t) &= \frac{c}{2} \left(\sin t' + \frac{1}{\bar{\omega}} \sin \bar{\omega} t' \right), \\ \theta_2(t) &= \frac{c}{2} \left(\sin t' - \frac{1}{\bar{\omega}} \sin \bar{\omega} t' \right).\end{aligned}\tag{8.32}$$

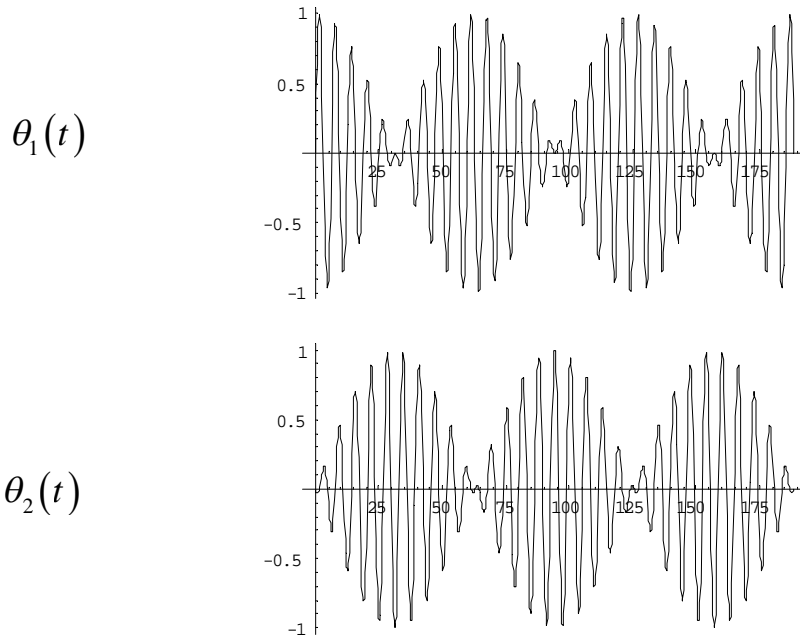
Now consider this solution in the limit of small frequency difference ($\alpha \ll 1$) for the two normal modes ($\omega_s \ll \omega_p$) where

$$\begin{aligned}\sin t' \pm \frac{1}{\bar{\omega}} \sin \bar{\omega} t' &\simeq \sin t' \pm \sin \bar{\omega} t' = \\ &= \sin \left(\frac{1+\bar{\omega}}{2} t' \right) \pm \sin \left(\frac{1-\bar{\omega}}{2} t' \right) \\ &= \begin{cases} 2 \sin \frac{1+\bar{\omega}}{2} t' \cos \frac{1-\bar{\omega}}{2} t' = 2 \sin \frac{\omega_1 + \omega_2}{2} t \cos \frac{\omega_1 - \omega_2}{2} t \\ \quad \simeq 2 \sin t' \cos \frac{\alpha}{2} t' \\ 2 \cos \frac{1+\bar{\omega}}{2} t' \sin \frac{1-\bar{\omega}}{2} t' = 2 \cos \frac{\omega_1 + \omega_2}{2} t \sin \frac{\omega_1 - \omega_2}{2} t \\ \quad \simeq -2 \cos t' \sin \frac{\alpha}{2} t' \end{cases}.\end{aligned}\tag{8.33}$$

Thus, in this limit, the solutions look like

$$\begin{aligned}\theta_1(t) &\simeq c \sin t' \cos \frac{\alpha}{2} t', \\ \theta_2(t) &\simeq -c \cos t' \sin \frac{\alpha}{2} t',\end{aligned}\tag{8.34}$$

corresponding to a rapid oscillation at frequency ω_p (amplitude) modulated by a slower frequency oscillation at $\alpha\omega_p/2 = \omega_s^2/2\omega_p$. This is the phenomenon of “beats”, *i.e.*, the amplitude of the motion of the two pendulums go up and down at the (smaller) difference frequency. This behavior is indicated in the final two figures ($\alpha = 0.1$).



Note that this is the same mathematics that describes the phenomenon of neutrino oscillation that has created so much excitement in the last 10 years.