Physics 505 - Autumn 2010

HW III Solutions

10/20/10

Overview: Recall that solving physics problems is not (just) about solving differential equations. Use physical reasoning to solve the following exercises and be certain to show your work. Note: the last 3 exercises in this HW set are quite similar to Classical Mechanics exercises in past Qualifying Exams!

1) (6 pts) Fetter & Walecka - 2.5 We want to exercise our understanding of motion in a rotating frame by considering how the rotation of the earth changes the apparent trajectory of cannon shells. Note that we simplify the analysis by expanding in powers of the rotation frequency ω and keeping only the first power. Strictly the dimensionless small parameter is the angular frequency ω times the characteristic time interval of the problem.

<u>Solution</u>: (a) (3 pts) The cannon at colatitude θ pointed east with muzzle velocity V_0 and elevation α provides the following initial conditions in the rotating frame affixed to the surface of the earth (as defined in the lecture notes, with $\omega = \omega_E$),

$$\vec{r}'(0) = 0, \dot{\vec{r}}'(0) = V_0\left(\cos\alpha\,\hat{y}' + \sin\alpha\,\hat{z}'\right).$$

We proceed (as in the Lecture notes) by expressing the subsequent trajectory in terms of the non-rotating result plus a small perturbation due to the rotation. We write

$$\dot{\vec{r}}'(t) \equiv \dot{\vec{r}}_0'(t) + \dot{\vec{\delta}}(t) = -g_{eff}t\hat{z}' + V_0\left(\cos\alpha\,\hat{y}' + \sin\alpha\,\hat{z}'\right) + \dot{\vec{\delta}}(t).$$

Hence the unperturbed trajectory (if the earth were not rotating) in the earth fixed system is given by

$$\vec{r}_0'(t) = -g_{eff}\frac{t^2}{2}\hat{z}' + V_0\left(\cos\alpha\,\hat{y}' + \sin\alpha\,\hat{z}'\right)t,$$

returning to the ground after a time $\tau_0 = 2V_0 \sin\alpha / g_{eff}$. We can express Newton's equation for the perturbation in the rotating frame (keeping only the leading terms in $|\omega|$, or really the dimensionless quantity $|\omega| V_0 / g_{eff}$) as

$$\begin{aligned}
\ddot{\vec{\delta}} &\simeq -2\vec{\omega} \times \left[-g_{eff} t \hat{z}' + V_0 \left(\cos\alpha \hat{y}' + \sin\alpha \hat{z}' \right) + \dot{\vec{\delta}}(t) \right] \\
&\simeq 2\vec{\omega} \times \left[g_{eff} t \hat{z}' - V_0 \left(\cos\alpha \hat{y}' + \sin\alpha \hat{z}' \right) \right] \\
&\simeq 2\omega \left[g_{eff} t \, \hat{z} \times \hat{z}' - V_0 \left(\cos\alpha \hat{z} \times \hat{y}' + \sin\alpha \hat{z} \times \hat{z}' \right) \right] \\
&= 2 g_{eff} \omega t \sin\theta \hat{y}' - 2\omega V_0 \left(-\cos\alpha \cos\theta \hat{x}' - \cos\alpha \sin\theta \hat{z}' + \sin\alpha \sin\theta \hat{y}' \right) \\
&= 2\omega \sin\theta \left(g_{eff} t - V_0 \sin\alpha \right) \hat{y}' + 2\omega V_0 \cos\alpha \left(\cos\theta \hat{x}' + \sin\theta \hat{z}' \right).
\end{aligned}$$

Hence the lateral motion (transverse to the initial velocity) is in the $\cos\theta \hat{x}'$ or southerly direction in the northern hemisphere and the northerly direction in the southern hemisphere (towards the equator in both hemispheres). The magnitude of the lateral motion is given by

$$\begin{aligned}
&\delta_{x'}(t) = \omega V_0 \cos\alpha \cos\theta t^2 \\
&\Rightarrow \delta_{x'}(\tau_0) = \omega V_0 \cos\alpha \cos\theta \left(\frac{2V_0 \sin\alpha}{g_{eff}} \right)^2 = \frac{4V_0^3 \sin^2\alpha}{g_{eff}^2} \omega \cos\alpha \cos\theta.
\end{aligned}$$

Note that the rotation does perturb the $\hat{z}'$ motion also, but this leads to a higher order, ω^2, correction to the lateral motion.

(b) (3 pts) Now we want to consider how the rotation impacts the range of the cannon. The unperturbed range is

$$R = r'_{0,y'}(\tau_0) = V_0 \cos\alpha \frac{2V_0 \sin\alpha}{g_{eff}} = 2V_0^2 \frac{\sin\alpha \cos\alpha}{g_{eff}}.$$

The perturbations to the $\hat{x}'$ and $\hat{y}'$ motion are

$$\delta_{x'}(t) = \omega V_0 \cos\theta \cos\alpha t^2,$$

$$\delta_{x'}(\tau_0) = \omega V_0 \cos\theta \cos\alpha \left(\frac{2V_0 \sin\alpha}{g_{eff}} \right)^2 = 4\omega \frac{V_0^3 \sin^2 \alpha \cos\alpha \cos\theta}{g_{eff}^2}.$$

and

$$\delta_{y'}(t) = 2\omega \sin\theta \left(g_{eff} \frac{t^3}{6} - V_0 \sin\alpha \frac{t^2}{2} \right)$$

$$\Rightarrow \delta_{y'}(\tau_0) = \omega \sin\theta \left(\frac{(2V_0 \sin\alpha)^3}{3g_{eff}^2} - V_0 \sin\alpha \frac{(2V_0 \sin\alpha)^2}{g_{eff}^2} \right)$$

$$= -\frac{4}{3}\omega \frac{V_0^3}{g_{eff}^2} \sin^3 \alpha \sin\theta.$$

Finally in order to get the complete first order (in ω) perturbation we need to consider the change in the time to return to earth, which will also yield a perturbation in $r'_{0,y}$. The full z' motion is

$$r'_{z'}(t) = -g_{eff} \frac{t^2}{2} + V_0 \sin\alpha t + \omega V_0 \cos\alpha \sin\theta t^2$$

$$\Rightarrow \tau = \frac{2V_0 \sin\alpha}{g_{eff} - 2\omega V_0 \cos\alpha \sin\theta} \simeq \tau_0 \left(1 + \frac{2\omega V_0 \cos\alpha \sin\theta}{g_{eff}} \right).$$

Hence the perturbed range due to the change in the z' motion is

$$r'_{0,y'}(\tau) \simeq 2V_0^2 \frac{\sin\alpha \cos\alpha}{g_{eff}} \left(1 + \frac{2\omega V_0 \cos\alpha \sin\theta}{g_{eff}} \right)$$

$$\simeq R \left(1 + \frac{2\omega V_0 \cos\alpha \sin\theta}{g_{eff}} \right).$$

Including both the z' motion and the motion in the other directions from above, the complete change in the range (to first order in ω) is

$$
\begin{aligned}
\Delta R &= \sqrt{\left(r'_{0,y'}\left(\tau\right)+\delta_{y'}\right)^2+\delta_{x'}^2}-R=\sqrt{\left(R+r'_{0,y'}\left(\tau\right)+\delta_{y'}-R\right)^2+\delta_{x'}^2}-R \\
&= \sqrt{R^2+2R\left(r'_{0,y'}\left(\tau\right)+\delta_{y'}-R\right)+\left(r'_{0,y'}\left(\tau\right)+\delta_{y'}-R\right)^2+\delta_{x'}^2}-R \\
&\simeq R\sqrt{1+2\left(r'_{0,y'}\left(\tau\right)+\delta_{y'}-R\right)\big/R}-R \\
&\simeq r'_{0,y'}\left(\tau\right)+\delta_{y'}-R=R\left(\frac{2\omega V_0\cos\alpha\sin\theta}{g_{eff}}\right)-\frac{4}{3}\omega\frac{V_0^3}{g_{eff}^2}\sin^3\alpha\sin\theta \\
&\simeq 4\omega\frac{V_0^3\sin\alpha\sin\theta}{g_{eff}^2}\left(\cos^2\alpha-\frac{\sin^2\alpha}{3}\right) \\
&\simeq \omega\sin\theta\sqrt{\frac{2R^3}{g_{eff}}}\left(\sqrt{\cot\alpha}-\frac{\sqrt[3]{\tan\alpha}}{3}\right),
\end{aligned}
$$

where the last step uses the fact that $V_0=\sqrt{Rg_{eff}\big/2\cos\alpha\sin\alpha}$. Note that the 3rd and 4th terms in the square root on the second line are order ω^2.

2) (6 pts) Fetter & Walecka – 2.7 In this exercise we study how the rotation of the earth perturbs the effective gravitational potential near the earth and hence its shape.

Solution: (a) (1 pt) We evaluate the indicated gradient to find the desired force (using the ε symbol),

$$\begin{aligned}
-m\vec{\nabla}\Phi_c &= -m\vec{\nabla}\left(-\frac{1}{2}(\vec{\omega}\times\vec{r})\cdot(\vec{\omega}\times\vec{r})\right) \\
\Rightarrow -m\vec{\nabla}_j\Phi_c &= \frac{m}{2}\vec{\nabla}_j\left(\varepsilon_{klm}\omega_l r_m\right)\left(\varepsilon_{knp}\omega_n r_p\right) \\
&= \frac{m}{2}\varepsilon_{klm}\omega_l\varepsilon_{knp}\omega_n\left(\delta_{jm}r_p+\delta_{jp}r_m\right) \\
&= \frac{m}{2}\left[\left(\varepsilon_{klj}\omega_l\varepsilon_{knp}\omega_n r_p\right)+\left(\varepsilon_{klm}\omega_l\varepsilon_{knj}\omega_n r_m\right)\right] \\
&= -\frac{m}{2}\left[\left(\varepsilon_{jlk}\omega_l\varepsilon_{knp}\omega_n r_p\right)+\left(\varepsilon_{jnk}\omega_n\varepsilon_{klm}\omega_l r_m\right)\right] \\
&= -m\left[\vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right)\right]_j.
\end{aligned}$$

(b) (1 pt) As we discussed (briefly) in class the forces on a (still) fluid must be normal to the surface of the fluid. If the force has a tangential component, the fluid will move in response, but, by assumption, the fluid is not moving. Hence, for a force defined as the gradient of a potential, the surface of the fluid will be normal to the gradient, *i.e.*, the surface will (necessarily) lie along an equi-potential surface (in 3-D). Hence in the frame rotating with the surface of the earth where the forces (on still water) arise from gravity and the centrifugal force of part (a), the surface of the water will be an equi-potential surface,

$$\Phi_g + \Phi_c = \text{ constant.}$$

(c) (2 pts) Using the provided forms of the two potentials we have

$$-M_E\frac{G}{r}\left[1-J_2\left(\frac{R_E}{r}\right)^2 P_2\left(\cos\theta\right)\right]-\frac{1}{2}\omega^2 r^2\sin^2\theta = \text{constant.}$$

In particular, we want to guarantee the same value at $\theta = 0$ (the north pole) and $\theta = \pi/2$ (the equator) and we want the values of the radius at the surface at the 2 points that yield this result. We define

$$\Delta R \equiv R_E - R_P \ll R_E, R_P$$

and use the smallness of this perturbation to expand the expressions for the potentials to first order in this quantity. In particular we can write

$$\begin{aligned}
\Phi\left(\theta=0\right)\Big|_{R_P} &= -M_E\frac{G}{R_P}\left[1-J_2\left(\frac{R_E}{R_P}\right)^2\right] \\
&= -M_E\frac{G}{R_E-\Delta R}\left[1-J_2\left(\frac{R_E}{R_E-\Delta R}\right)^2\right] \\
&\simeq -M_E\frac{G}{R_E}\left[1-J_2\left(1+2\frac{\Delta R}{R_E}\right)\right]\left(1+\frac{\Delta R}{R_E}\right) \\
&\simeq -M_E\frac{G}{R_E}\left[1-J_2\right]+\frac{2J_2GM_E}{R_E}\frac{\Delta R}{R_E}-\frac{\left(1-J_2\right)GM_E}{R_E}\frac{\Delta R}{R_E} \\
&\simeq -M_E\frac{G}{R_E}\left[1-J_2\right]-\frac{GM_E}{R_E}\frac{\Delta R}{R_E},
\end{aligned}$$

and

$$\begin{aligned}
\Phi\left(\theta=\frac{\pi}{2}\right)\Big|_{R_E} &= -M_E\frac{G}{R_E}\left[1+\frac{1}{2}J_2\left(\frac{R_E}{R_E}\right)^2\right]-\frac{1}{2}\omega^2{R_E}^2 \\
&= -M_E\frac{G}{R_E}\left[1+\frac{1}{2}J_2\right]-\frac{1}{2}\omega^2{R_E}^2.
\end{aligned}$$

The constraint that these expressions are equal yields the following relation

$$\begin{aligned}
&-M_E\frac{G}{R_E}\left[1-J_2\right]-\left[\frac{GM_E}{R_E}\right]\frac{\Delta R}{R_E}\simeq -M_E\frac{G}{R_E}\left[1+\frac{1}{2}J_2\right]-\frac{1}{2}\omega^2{R_E}^2 \\
&\Rightarrow\left[\frac{GM_E}{R_E}\right]\frac{\Delta R}{R_E}\simeq\frac{3J_2GM_E}{2R_E}+\frac{1}{2}\omega^2{R_E}^2 \\
&\Rightarrow\frac{\Delta R}{R_E}\simeq\frac{3J_2}{2}+\frac{1}{2}\frac{\omega^2{R_E}^3}{GM_E}.
\end{aligned}$$

where we have kept only first order terms in the “physics” perturbations due to the quadrupole moment and the centrifugal force, both of which are assumed to be small. Using the provided value for J_2 and the known properties of the earth and its rotation we have

$$\frac{\Delta R}{R_E} \simeq \frac{3}{2}\left(1.083\times10^{-3}\right)+1.718\times10^{-3} \simeq 3.34\times10^{-3}$$

in good agreement with the measured values.

(d) (2 pts) As we discussed in part (b) the local gradient of the effective potential is normal to the surface and is the force we experience in the rotating frame we live in. We also saw in part (c) that to 3 parts in a thousand the earth is round and the local normal force is in the radial direction to that level of approximation. We proceed to take the radial gradient and find

$$\begin{aligned} g_{eff}\left(r,\theta\right) &\simeq \frac{\partial}{\partial r}\left(-M_E\frac{G}{r}\left[1-J_2\left(\frac{R_E}{r}\right)^2 P_2\left(\cos\theta\right)\right]-\frac{1}{2}\omega^2 r^2\sin^2\theta\right) \\ &\simeq \frac{M_E G}{r^2}-\frac{3J_2M_EGR_E^2}{2r^4}\left[3\cos^2\theta-1\right]-\omega^2 r\sin^2\theta \\ &\simeq \frac{M_E G}{r^2}+\frac{3J_2M_EGR_E^2}{2r^4}-\omega^2 r+\cos^2\theta\left[\omega^2 r-\frac{9J_2M_EGR_E^2}{2r^4}\right]. \end{aligned}$$

All the terms except the first one are small (either because the radius of the earth is large or because the angular velocity is small or because the quadrupole perturbation is small) and we can expand in the small quantities. We proceed by giving a label to the acceleration at the equator,

$$g_E = \frac{M_E G}{R_E^2}+\frac{3J_2M_EG}{2R_E^2}-\omega^2 R_E,$$

and expand in small quantities around the equator (*i.e.*, with respect to $\cos\theta$). Due to the quadrupole deformation we have $R_E - r = \Delta R\cos^2\theta$ on the surface. Thus expanding to first order in the small bits gives the desired result of Clairaut,

$$\begin{aligned} g_{eff}(\theta) &\simeq \frac{M_E G}{\left(R_E - \Delta R\cos^2\theta\right)^2} + \frac{3J_2 M_E G}{2R_E^2} - \omega^2 R_E + \cos^2\theta\left[\omega^2 R_E - \frac{9J_2 M_E G}{2R_E^2}\right] \\ &\simeq g_E + \frac{M_E G}{R_E^2}\cos^2\theta\left[\frac{2\Delta R}{R_E} + \frac{\omega^2 R_E^3}{M_E G} - \frac{9J_2}{2}\right] \\ &\simeq g_E + \frac{M_E G}{R_E^2}\cos^2\theta\left[\frac{5\omega^2 R_E^3}{2M_E G} - \frac{\Delta R}{R_E}\right] \\ &\simeq g_E\left[1 + \cos^2\theta\left[\frac{5\omega^2 R_E^3}{2M_E G} - \frac{\Delta R}{R_E}\right]\right]. \end{aligned}$$

In the next to last step we used the relation from part (c)

$$-\frac{9J_2}{2} = -3\frac{\Delta R}{R} + \frac{3\omega^2 R_P^3}{2M_E G},$$

and in the last step we pulled the leading form of g_E out of the second term, which is okay since that term is already first order in the small quantities. Finally we can look at numbers to find

$$\frac{g_P}{g_E} \simeq 1 + \frac{5\omega^2 R_E^3}{2M_E G} - \frac{\Delta R}{R_E} = \frac{983.20\,\mathrm{cm/s^2}}{978.03\,\mathrm{cm/s^2}} \simeq 1.005286.$$

Evaluating the second term is this expression numerically we find

$$\frac{\Delta R}{R_E} \simeq 0.005286 - \frac{5}{2}\left(1.718\times10^{-3}\right) \simeq 3.31\times10^{-3}.$$

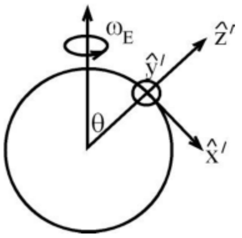
Again we find good agreement with observations.

3) (4 pts) To further strengthen our understanding of rotating reference frames we should analyze the case of a pendulum in such a frame. In particular, we can use the results of Chapter 2, Section 12 to study the Foucault Pendulum in building A. Take the pendulum bob to have mass *m* and the massless support wire to have length *l*. Determine the oscillation frequency of the pendulum and the rotation rate (angular velocity) of the plane of this oscillation as observed in the rotating frame fixed to the

surface of the earth at polar angle (colatitude) θ. How long does it take the plane of the pendulum in A wing to rotate through 180°? [Assume that the displacement angle of the pendulum is small enough that a linear analysis is applicable.]

Solution: Let the displacement angle of the pendulum be ψ and the angle describing the plane of oscillation in the (x',y') plane (the rotating coordinate system fixed on the surface of the earth, *i.e.*, that we live in) be ϕ. We expect this system to be described by 2 angular frequencies. One frequency for the oscillation of the pendulum and another (slower) frequency for the precession of the oscillation plane. From Eq. (4.19) in the lecture notes Newton's second law looks like

$$m\ddot{\vec{r}}' = m\vec{g}_{eff} + \vec{F}_{\text{support}} - 2m\vec{\omega}\times\dot{\vec{r}}',$$
$$\vec{\omega} = \omega_E\left(-\sin\theta\hat{x}', 0, \cos\theta\hat{z}'\right),$$



where the second line gives the angular velocity of the earth's rotation in the rotating coordinates. Thus the components of the Coriolis force look like (in (x,y,z) component notation)

$$\vec{\omega}\times\dot{\vec{r}}' = \left(-\omega_E\cos\theta\,\dot{y}', \omega_E\cos\theta\,\dot{x}' + \omega_E\sin\theta\,\dot{z}', -\omega_E\sin\theta\,\dot{y}'\right).$$

If we call the tension in the support cable T and take gravity to be in the $-z'$ direction, we have forces (in (x,y,z) component notation)

$$\begin{aligned}
m\vec{g}_{eff} + \vec{F}_{\text{support}} &= \left(-T\sin\psi\cos\phi, -T\sin\psi\sin\phi, T\cos\psi - mg\right) \\
&\simeq \left(-T\psi\cos\phi, -T\psi\sin\phi, T\left(1-\frac{\psi^2}{2}\right) - mg\right) \\
&\simeq \left(-T\frac{\sqrt{x'^2+y'^2}}{l}\frac{x'}{\sqrt{x'^2+y'^2}}, -T\frac{\sqrt{x'^2+y'^2}}{l}\frac{y'}{\sqrt{x'^2+y'^2}}, T\left(\frac{l-z'}{l}\right) - mg\right) \\
&\simeq \left(-T\frac{x'}{l}, -T\frac{y'}{l}, T\left(\frac{l-z'}{l}\right) - mg\right).
\end{aligned}$$

Thus Newton's equations are

$$
\begin{aligned}
m\ddot{x}' &= -T\frac{x'}{l} + 2m\omega_E \cos\theta\,\dot{y}', \\
m\ddot{y}' &= -T\frac{y'}{l} - 2m\omega_E\left(\cos\theta\,\dot{x}' + \sin\theta\,\dot{z}'\right), \\
m\ddot{z}' &= T - mg - T\frac{z'}{l} + 2m\omega_E \sin\theta\,\dot{y}'.
\end{aligned}
$$

Now there are two small parameters, ψ and the earth's rotation frequency ω_E (times time). The motion in the z' direction is order ψ^2 and we ignore it. The remaining terms give us the two expected frequencies, the usual pendulum frequency $\omega_p = \sqrt{g/l}$ and the much slower precession frequency $\omega_\perp = \omega_E \cos\theta \ll \omega_p$. Note that, due to the $\cos\theta$ factor, at the equator, $\theta = 90°$, there is no precession and that the sense (*i.e.*, sign) of the precession is opposite in the two hemispheres.

With these definitions and approximations we have two coupled oscillators

$$
\begin{aligned}
\ddot{x}' &= -\omega_p^2 x' + 2\omega_\perp \dot{y}', \\
\ddot{y}' &= -\omega_p^2 y' - 2\omega_\perp \dot{x}'.
\end{aligned}
$$

This 2D configuration is easily studied in terms of the (effectively 2D) complex variable $\varsigma = x' + iy'$, which we assume (our Ansatz) to have periodic behavior $\varsigma = Ae^{i\varpi t}$. The (single complex) differential equation then becomes an eigenvalue equation,

$$
\begin{aligned}
&\ddot{\varsigma} = -\omega_p^2\varsigma - 2i\omega_\perp\dot{\varsigma} \\
&\Rightarrow -\varpi^2 = -\omega_p^2 + 2\varpi\omega_\perp \\
&\Rightarrow \varpi = -\omega_\perp \pm \sqrt{\omega_p^2 + \omega_\perp^2} \simeq -\omega_\perp \pm \omega_p.
\end{aligned}
$$

The (approximate) eigenfunctions are

$$
\varsigma \simeq e^{-i\omega_\perp t}\left[A_1 e^{i\omega_p t} + A_2 e^{-i\omega_p t}\right],
$$

where the first factor describes the precession and the second factor the (more rapid) usual pendulum oscillation. For a pendulum of length ~ 15 m (as in the A wing) we have

$$\omega_E \simeq \frac{2\pi}{24\text{ hr}} \simeq 7.2\times 10^{-5}\frac{1}{s},$$
$$\omega_p \simeq \sqrt{\frac{9.8\text{ m}}{15\text{ m s}^2}} \simeq 0.81\frac{1}{s} \gg \omega_E.$$

In Seattle (latitude ~ $47°$, colatitude $\theta \sim 43°$) we find

$$\begin{aligned}\omega_\perp &= \omega_E\cos\left(43°\right) \simeq 5.3\times 10^{-5}\frac{\text{rad}}{\text{s}}\\ &\simeq 0.19\frac{\text{rad}}{\text{hr}} \simeq 11\frac{°}{\text{hr}}.\end{aligned}$$

So the time for the pendulum to precess through 180 degrees is about 16.4 hours.

4) (3 pts) As a first practice problem with minimization concepts and the calculus of variations consider a cord (or "chain") of indefinite length that passes freely over pulleys at heights y_1 and y_2 above the plane surface of the earth, with a horizontal distance $x_2 - x_1$ between them. If the cord has uniform linear mass density, then find the differential equation that describes the cord's shape, and solve it for that shape. How does the analysis and shape change if the cord is of fixed length, $l > \sqrt{\left(x_2-x_1\right)^2 + \left(y_2-y_1\right)^2}$? Your solution will contain arbitrary constants to fit to the given endpoints. You need not evaluate these constants unless you want to.

Solution: Let the linear mass density be ρ_0. We note that we are only interested in the gravitational potential energy of the cord between the pulleys. The energy of the two ends hanging down from the pulleys will not change. We thus want to find the minimum value of (where we drop the dimensionful factor $g\rho_0$)

$$E_{POT} = \int_{x_1}^{x_2} g y(x) \rho_0 ds = g\rho_0 \int_{x_1}^{x_2} y(x)\sqrt{1+y'^2}dx$$

$$\Rightarrow \Phi(y, y') = y\sqrt{1+y'^2}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{yy'}{\sqrt{1+y'^2}}\right) - \sqrt{1+y'^2} = 0.$$

This expression is similar in form to the brachistochrone problem discussed in Lecture 5. In particular, there is no explicit x dependence and we can use Eq. (5.12) to find immediately a first order differential equation. This yields (here we can ignore the factor of $g\rho_0$)

$$\Phi - y'\frac{\partial\Phi}{\partial y'} = y\sqrt{1+y'^2} - \frac{yy'^2}{\sqrt{1+y'^2}} = \frac{y}{\sqrt{1+y'^2}} = c,$$

with *c* a constant. We can easily convert this result to a first-order equation and integrate. We have (staying with the positive branch)

$$\frac{y}{\sqrt{1+y'^2}} = c \Rightarrow y' = \sqrt{\frac{y^2}{c^2} - 1}$$

$$\Rightarrow \int_c \frac{dy}{\sqrt{y^2 - c^2}} = \int_{x_0} \frac{dx}{c} \Rightarrow \cosh^{-1}\left(\frac{y}{c}\right) = \frac{x - x_0}{c} \Rightarrow y = c\cosh\left(\frac{x - x_0}{c}\right).$$

The constants *c* and x_0 are chosen to match the boundary points $y(x_1) = y_1$ and $y(x_2) = y_2$.

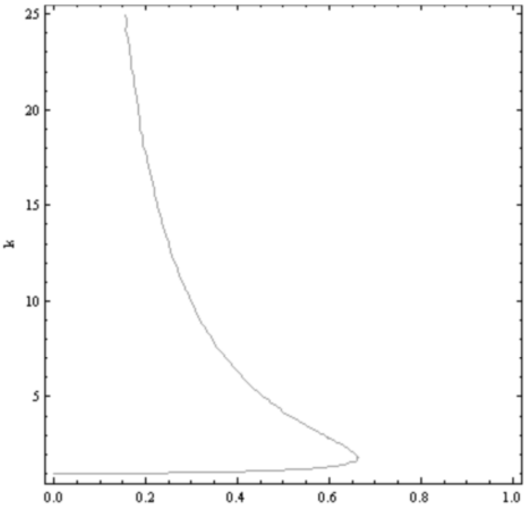
Alternatively, we can essentially guess the answer by using the change of variable $y' = \sinh\eta$ to turn the original equation into

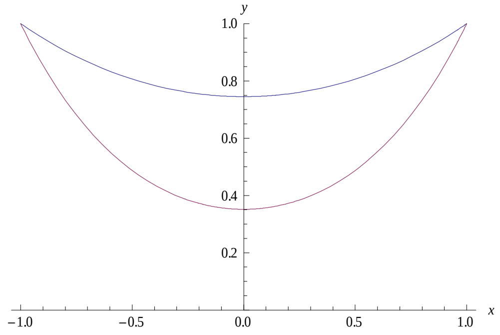
$$\sqrt{1+y'^2} = \cosh\eta \Rightarrow y = c\cosh\eta$$
$$\Rightarrow \frac{dx}{d\eta} = \frac{1}{y'}\frac{dy}{d\eta} = c \Rightarrow x = c\eta + x_0$$
$$\Rightarrow y(x) = c\cosh\left(\frac{x-x_0}{c}\right).$$

Although the exercise does not ask us to consider in detail the problem of matching the boundary conditions, or determining whether we really have a stable minimum of the potential energy, there were lots of questions about this. So let's look at a symmetric case in more detail, but NOTE to TA, this following discussion is not required for credit. Let's define the endpoints via $y_1 = y_2 = \overline{y},\, x_2 = -x_1 = \overline{x}$ and thus $x_0 = 0$ above. Also define the dimensionless parameters $\alpha = \overline{x}/\overline{y}$, $k = \overline{y}/c = y_{\max}/y_{\min}$. Then the solutions of the extremal problem must satisfy

$$k = \cosh k\alpha.$$

The solutions to this (hyperbolic) equation in the (α,k) plane fall on the curve in the figure to the right. Note that for α larger than about 0.66 there are no solutions, and for smaller α values there are 2 allowed *k* values. Only the smaller k value corresponds to the minimum potential energy (of the catenary shapes). For a given values of $\overline{x}, \overline{y}$, and thus α, we pick the value of the constant *c* to match the corresponding *k* value. For α = 0.6 the two possible catenary curves look as in the next figure (with $\overline{y} = 1$). The lower (and longer) shape, with the larger *k* value, actually has the larger energy, due to the length. The gravitational energy of such a curve is given by





$$E_{\text{pot}} = \left(\frac{\overline{y}}{k}\right)^2\left(\alpha k + \frac{\sinh\alpha k}{2}\right).$$

To understand the physical system we must also compare this potential energy to that of "Goldschmidt" solution where the chain is on the floor everywhere except at the point where it loops over the 2 pulleys. In the current notation that corresponds to an potential energy of $E_G = \bar{y}^2$. With some effort, perhaps using *Mathematica*, you should be able to convince yourself that as long as α is less than about 0.528, the small k value catenary curve (upper curve) has the lowest energy. For larger values of α, i.e., $\bar{x} > 0.528\bar{y}$ the chain falls to the floor.

Now we return to the graded exercise and consider what changes when we add the constraint of a fixed length. The constraint is represented by

$$l = \int_{x_1}^{x_2} dx\sqrt{1+y'^2}.$$

Including the arbitrary multiplying factor, we want to apply Euler-Lagrange to the new function

$$\Phi = \frac{(g\rho_0 y + \lambda)}{g\rho_0}\sqrt{1+y'^2}.$$

The subsequent steps are just as above with the replacement $y \to y - y_0$,with the definition $y_0 = -\lambda/g\rho_0$, which introduces an additive term that must be used to fit the length after fitting the end points,

$$y(x) = c\cosh\left(\frac{x-x_0}{c}\right) + y_0.$$

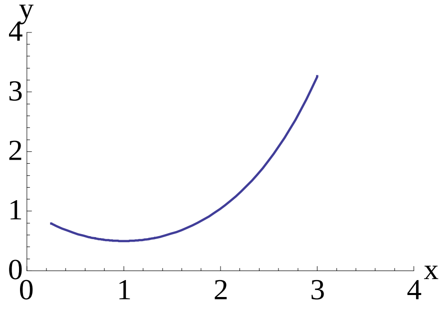


Figure 1: c = 1, x_0 = 1, y_0 = -0.5

There is clearly a minimum at the point $(x_0, c+y_0)$ with y increasing in both directions in x. The messy part of this problem, which we are not asked to address in the HW, is to convert our knowledge about the location of the endpoints and the length into values for the parameters x_0, y_0 and c. Although the form looks appealing (x_0, y_0 look like they set the origin and c sets the length scale), the relationships tend to be hyperbolic (nonlinear and implicit). In particular, the endpoints tell us that

$$
\begin{aligned}
&y(x_1) = y_1 = c\cosh\frac{x_1 - x_0}{c} + y_0 \Rightarrow y_0 = y_1 - c\cosh\frac{x_1 - x_0}{c},\\
&y(x_2) = y_2 = c\cosh\frac{x_2 - x_0}{c} - c\cosh\frac{x_1 - x_0}{c} + y_1,\\
&\Delta y = y_2 - y_1 = c\cosh\frac{x_2 - x_0}{c} - c\cosh\frac{x_1 - x_0}{c}.
\end{aligned}
$$

This last result gives an implicit value for x_0 in terms of $\Delta y = y_2 - y_1$, $\Delta x = x_2 - x_1$ and *c*. The length scale *c* is determined by the constraint

$$
\begin{aligned}
l &= \int_{x_1}^{x_2} dx\sqrt{1 + y'^2} = \int_{x_1}^{x_2} dx\cosh\left(\frac{x - x_0}{c}\right)\\
&= c\left(\sinh\left(\frac{x_2 - x_0}{c}\right) - \sinh\left(\frac{x_1 - x_0}{c}\right)\right).
\end{aligned}
$$

These equations can be simplified somewhat by choosing our coordinates so that one end of the cord is at the origin, $x_1 = y_1 = 0$, $y_2 = \Delta y,\ x_2 = \Delta x$. Then we have the equations

$$
\begin{aligned}
&\frac{y_0}{c} = -\cosh\frac{x_0}{c}\\
&\frac{y_2}{c} = \cosh\frac{x_2 - x_0}{c} - \cosh\frac{x_0}{c},\\
&\frac{l}{c} = \sinh\left(\frac{x_2 - x_0}{c}\right) + \sinh\left(\frac{x_0}{c}\right),
\end{aligned}
$$

which can be solved, at least implicitly (using inverse hyperbolic functions) for the 3 parameters. Another choice (seen on the web) is to put the origin at the lowest point of the chain. In that case we have

$$
y(x) = c\cosh\frac{x}{c} - c.
$$

The length of such a curve with this choice of coordinates is $l = c\left(\sinh\left(|x_1|/c\right) + \sinh\left(x_2/c\right)\right)$ (where $x_1 < 0$), which gives an implicit solution for *c*.

5) Fetter & Walecka – 3.1 (6 pts) This is an exercise that illustrates Lagrangian methods for a system with a (trivial) constraint (confined to motion on a loop) and rotating coordinates. It also introduces us to using small (linearized) perturbations to study stability. Note that small perturbations about a stable equilibrium yield oscillatory (harmonic oscillator-like) behavior.

Solution: In setting this problem up it is natural to use spherical coordinates (in the rotating frame). Since we are not interested in the constraint force that keeps the mass on the wire, we can eliminate the constraint by eliminating the radial variable as one of the generalized coordinates, *i.e.*, $r = a$ is a constant ($\dot{r} = \ddot{r} = 0$). The azimuthal angle is also constrained by $\dot{\phi} = \Omega$. The remaining dynamical coordinate is just θ, which we are instructed to measure from the "downward" direction. We can choose the gravitational potential energy to be zero at the "equator", $\theta = \pi/2$.

(a) (1 pt) The resulting kinetic energy (see, for example, Eq. (6.14) in the Lecture 6 notes), potential energy and Lagrangian are

$$\begin{aligned} T &= \frac{m}{2}\left[\left(a\dot{\theta}\right)^2 + \left(a\sin\theta\,\Omega\right)^2\right] \\ V &= -mga\cos\theta, \\ L &= \frac{m}{2}\left[\left(a\dot{\theta}\right)^2 + \left(a\sin\theta\,\Omega\right)^2\right] + mga\cos\theta. \end{aligned}$$

(b) (2 pts) The (corresponding) Lagrange equation is

$$\begin{aligned} &\frac{d}{dt}\left(ma^2\dot{\theta}\right) - \left[ma^2\Omega^2\sin\theta\cos\theta - mga\sin\theta\right] = 0 \\ &\Rightarrow \ddot{\theta} = \Omega^2\sin\theta\cos\theta - \frac{g}{a}\sin\theta \\ &\qquad \equiv \Omega^2\sin\theta\cos\theta - \omega_\theta^2\sin\theta\left[\omega_\theta = \sqrt{\frac{g}{a}}\right]. \end{aligned}$$

For an equilibrium orbit at fixed angle θ_0 we require $\dot{\theta} = \ddot{\theta} = 0$ and the right-hand-side of this last equation must vanish. Thus the possible equilibrium angles are

$$\Omega^2 \sin\theta_0 \cos\theta_0 - \omega_\theta^2 \sin\theta_0 = 0$$
$$\Rightarrow \begin{cases} \sin\theta_0 = 0, \theta_0 = 0, \pi \\ \cos\theta_0 = \dfrac{\omega_\theta^2}{\Omega^2} = \dfrac{g}{a\Omega^2} \left[\omega_\theta \le \Omega\right] \end{cases}.$$

Note that, for $\pi > \theta_0 > \pi/2$ and thus $\cos\theta_0 < 0$, $\ddot{\theta}$ is always nonzero and in fact negative. There are no equilibrium orbits above the "equator" except at the north pole. The 2 trivial solutions at the (two) poles correspond to no motion at all.

In the language of forces in the rotating frame the equilibrium angle corresponds to a vanishing of the force tangential to the loop (in which direction there can be no constraint force). The two contributions to this force (in the $\hat{\theta}$ direction) arise from gravity ($mg\left(-\sin\theta\hat{\theta} + \cos\theta\hat{r}\right)$) downward and the upward tangential component of the centrifugal force ($m\vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right) = m\Omega^2 a\sin\theta\left(\cos\theta\hat{\theta} + \sin\theta\hat{r}\right)$). (Note that the centrifugal force is only upward for $\theta < \pi/2$.)

(c) (2 pts) This part of the exercise is our first experience with an important technique for studying stability – looking at small perturbations (oscillations) around an equilibrium trajectory. Substituting $\theta = \theta_0 + \eta(t)$ in the Lagrange equation above and keeping terms to lowest nontrivial order in the perturbation yields

$$\begin{aligned} \ddot{\eta} &= \Omega^2 \sin(\theta_0 + \eta)\cos(\theta_0 + \eta) - \omega_\theta^2 \sin(\theta_0 + \eta) \\ &\simeq \Omega^2 \left(\sin\theta_0 + \eta\cos\theta_0\right)\left(\cos\theta_0 - \eta\sin\theta_0\right) - \omega_\theta^2\left(\sin\theta_0 + \eta\cos\theta_0\right) \\ &\simeq \Omega^2\left[\sin\theta_0\cos\theta_0 + \eta\left(\cos^2\theta_0 - \sin^2\theta_0\right)\right] - \omega_\theta^2\left(\sin\theta_0 + \eta\cos\theta_0\right) \\ &\simeq \eta\left[\Omega^2\left(\cos^2\theta_0 - \sin^2\theta_0\right) - \omega_\theta^2\cos\theta_0\right]. \end{aligned}$$

In the last step we used the equilibrium condition from above, $\Omega^2\sin\theta_0\cos\theta_0 - \omega_\theta^2\sin\theta_0 = 0$. For the conditions noted in part (b) for (nontrivial) equilibrium, *i.e.*, $\omega_\theta \le \Omega$ and $\cos\theta_0 = \omega_\theta^2/\Omega^2$, the factor on the right-hand-side of the last equation is negative

$$\left[\Omega^2\left(\cos^2\theta_0-\sin^2\theta_0\right)-\omega_\theta^2\cos\theta_0\right]=\left[\Omega^2\left(2\frac{\omega_\theta^4}{\Omega^4}-1\right)-\omega_\theta^2\frac{\omega_\theta^2}{\Omega^2}\right]$$
$$=\Omega^2\left[\frac{\omega_\theta^4}{\Omega^4}-1\right]\le 0.$$

Thus we have a "restoring" force and stable oscillations at frequency (squared) ω^2,

$$\ddot{\eta}+\omega^2\eta=0,$$
$$\omega^2=\Omega^2\left[1-\frac{\omega_\theta^4}{\Omega^4}\right]=\Omega^2\left[1-\cos^2\theta_0\right]=\Omega^2\sin^2\theta_0.$$

On the other hand the two potential "orbits" at the poles, $\theta_0=0,\pi$, correspond to

$$\ddot{\eta}\simeq\eta\left[\Omega^2\mp\omega_\theta^2\right],$$

and, with $\omega_\theta\le\Omega$, the right-hand-side corresponds to an *unstable* force ($[\;]\ge 0$), pushing the mass away from the pole once it is perturbed (these two configurations are unstable equilibria).

(d) (1pt) If the wire hoop is spinning slowly, $a\Omega^2<g$, we have $\omega_\theta>\Omega$ and there is no nontrivial physical solution for θ_0, *i.e.*, we require $\cos\theta_0>1$ in order that $\ddot{\theta}=0$ in part (b). There are no nontrivial equilibrium solutions for rotations this slow. The centrifugal force is too small to keep the point mass from falling to the bottom of the hoop ($\theta=0$). However, motion about the bottom of the hoop, the point $\theta=0$, does now have a restoring force, *i.e.*, the expression above is now negative. The perturbed system will exhibit oscillations with frequency $\omega^2=\omega_\theta^2-\Omega^2=g/a-\Omega^2$. This is just what we expect as a perturbation of the non-rotating limit, $\Omega\to 0$. The frequency is $\omega=\sqrt{g/a}$, *i.e.*, the system is the familiar pendulum.