# Physics 505 - Autumn 2010

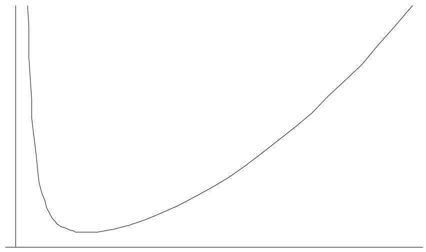
## HW II Solutions

10/13/10

1) (7 pts) Fetter & Walecka - 1.10 Here we want to analyze the $r^2$ potential much as we did the $1/r$ potential. Be certain to note the similarities and differences in the two cases, especially the fact that both produce closed, stable orbits.

<u>Solution</u>: a) (2 pts) We start by considering the effective potential corresponding to the harmonic potential

$$U_{eff}\left(\rho\right) = \frac{1}{2}k\rho^2 + \frac{L_z^2}{2m\rho^2},$$



which has the feature that it is positive everywhere, going to $+\infty$ at the origin and at infinity as in the figure. Thus for all energies above the minimum of the potential,

$$U'\left(\rho_{\min}\right) = 0 \Rightarrow k\rho_{\min} - \frac{L_z^2}{m\left(\rho_{\min}\right)^3} = 0 \Rightarrow \rho_{\min} = \sqrt[4]{\frac{L_z^2}{km}}$$

$$\Rightarrow E_{\min} = U\left(\rho_{\min}\right) = \frac{1}{2}k\sqrt{\frac{L_z^2}{km}} + \frac{L_z^2}{2m}\sqrt{\frac{km}{L_z^2}} = \sqrt{\frac{kL_z^2}{m}} = \sqrt{\frac{kl^2}{m}},$$

there are bound (classical) states with turning points given by

$$U_{eff}\left(\rho_\pm\right) = E \Rightarrow \rho_\pm^2 = \frac{E}{k} \pm \sqrt{\frac{E^2}{k^2} - \frac{L_z^2}{km}} = \frac{E}{k} \pm \sqrt{\frac{E^2}{k^2} - \frac{l^2}{km}}.$$

b) (3 pts) Next we want to verify that the corresponding orbit is an ellipse with the origin at the center. As in the text and Lecture 2 we have the orbit equation

$$\phi(\rho) = \pm\frac{L_z}{\sqrt{2m}}\int^{\rho}\frac{d\rho'}{\rho'^2}\frac{1}{\sqrt{E - \frac{L_z^2}{2m\rho'^2} - \frac{k}{2}\rho'^2}} + \phi_0$$

$$\xrightarrow[z'=\rho'^2]{} \pm\frac{1}{2}\frac{L_z}{\sqrt{2m}}\int^{z}\frac{dz}{z'}\frac{1}{\sqrt{Ez' - \frac{L_z^2}{2m} - \frac{k}{2}z'^2}} + \phi_0$$

$$= \pm\frac{1}{2}\frac{L_z}{\sqrt{2m}}\sqrt{\frac{2m}{L_z^2}}\sin^{-1}\left[\frac{Ez' - L_z^2/m}{|z'|\sqrt{E^2 - kL_z^2/m}}\right]\Bigg|_{\rho_-^2}^{\rho^2} + \phi_0$$

$$= \pm\frac{1}{2}\left\{\sin^{-1}\left[\frac{E - L_z^2/m\rho^2}{\sqrt{E^2 - kL_z^2/m}}\right] - \sin^{-1}\left[\frac{E - L_z^2/m\rho_-^2}{\sqrt{E^2 - kL_z^2/m}}\right]\right\} + \phi_0$$

$$= \pm\frac{1}{2}\left\{\sin^{-1}\left[\frac{E - L_z^2/m\rho^2}{\sqrt{E^2 - kL_z^2/m}}\right] - \sin^{-1}[-1]\right\} + \phi_0.$$

This is very similar to the $1/r$ case except for the factor of ½ and the integration variable being $\rho^2$. This latter difference (due to the different change of variables) also means that in the current analysis the angle $\phi_0$ corresponds to the location of the minor rather than major axis (*i.e.*, the minimum value of $\rho^2$ instead of $1/\rho^2$). Manipulating this expression to find the standard form for the orbit, with $\phi_0 = \pi/2$ so the major axis is along the *x*-axis, we obtain

$$\sin\left[\pm 2(\phi - \phi_0) - \frac{\pi}{2}\right] = -\cos\left[2\left(\phi - \frac{\pi}{2}\right)\right] = \frac{1}{\sqrt{E^2 - kL_z^2/m}}\left(E - \frac{L_z^2}{m\rho^2}\right)$$

$$\Rightarrow \frac{1}{\rho^2} = \frac{Em}{L_z^2}\left(1 - \sqrt{1 - \frac{kL_z^2}{E^2 m}}\cos(2\phi)\right)$$

$$= \frac{2Em}{L_z^2}\left(\frac{1}{2}\left\{1 + \sqrt{1 - \frac{kL_z^2}{E^2 m}}\right\}1 - \sqrt{1 - \frac{kL_z^2}{E^2 m}}\cos^2(\phi)\right)$$

$$= \frac{Em + \sqrt{E^2m^2 - mkL_z^2}}{L_z^2}\left(1 - 2\frac{\sqrt{E^2m^2 - mkL_z^2}}{Em + \sqrt{E^2m^2 - mkL_z^2}}\cos^2(\phi)\right).$$

We recognize this last expression as describing an ellipse symmetric about the origin. For example, we note that $\rho^2$ is maximum at $\phi = 0$ *and* $\pi$, and minimum at $\phi$

$= \pi/2$ *and* $3\pi/2$. If $a$ represents the length of the semi-major axis and $b$ the semi-minor axis, with the usual relation $b^2 = a^2\left(1-\varepsilon^2\right)$, the standard Cartesian form for the symmetric ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \rho^2\left(\frac{\cos^2\phi}{a^2}+\frac{\sin^2\phi}{b^2}\right)=1$$
$$\Rightarrow \frac{1}{\rho^2}=\frac{1}{b^2}\left(\left(1-\varepsilon^2\right)\cos^2\phi+1-\cos^2\phi\right)=\frac{1}{b^2}\left(1-\varepsilon^2\cos^2\phi\right).$$

Thus we identify

$$b^2=\frac{L_z^2}{Em+\sqrt{E^2m^2-mkL_z^2}}=\frac{l^2}{Em+\sqrt{E^2m^2-mkl^2}},$$
$$\varepsilon^2=2\frac{\sqrt{E^2m^2-mkL_z^2}}{Em+\sqrt{E^2m^2-mkL_z^2}}=2\frac{\sqrt{E^2m^2-mkl^2}}{Em+\sqrt{E^2m^2-mkl^2}},$$
$$a^2=\frac{L_z^2}{Em-\sqrt{E^2m^2-mkL_z^2}}=\frac{l^2}{Em-\sqrt{E^2m^2-mkl^2}}.$$

In the (hyperbolic) notation suggested by F&W we have (*i.e.*, we define)

$$\frac{E}{E_{\min}}=\sqrt{\frac{E^2m}{kL_z^2}}=\sqrt{\frac{E^2m}{kl^2}}\equiv\cosh\xi,$$

which we can substitute into the above expressions to obtain the required results

$$b^2 = \frac{l/\sqrt{mk}}{\sqrt{\frac{E^2 m}{kl^2}} + \sqrt{\frac{E^2 m}{kl^2} - 1}} = \frac{l}{\sqrt{mk}} \frac{1}{\cosh\xi + \sinh\xi} = \frac{l}{\sqrt{mk}} e^{-\xi},$$

$$a^2 = \frac{l/\sqrt{mk}}{\sqrt{\frac{E^2 m}{kl^2}} - \sqrt{\frac{E^2 m}{kl^2} - 1}} = \frac{l}{\sqrt{mk}} \frac{1}{\cosh\xi - \sinh\xi} = \frac{l}{\sqrt{mk}} e^{\xi},$$

$$\varepsilon^2 = 2\frac{\sqrt{\frac{E^2 m}{kl^2} - 1}}{\sqrt{\frac{E^2 m}{kl^2}} + \sqrt{\frac{E^2 m}{kl^2} - 1}} = \frac{2\sinh\xi}{\cosh\xi + \sinh\xi} = 1 - e^{-2\xi}.$$

Clearly the two limits correspond to

$$E = E_{\min} \Rightarrow \xi = 0 \Rightarrow \begin{cases} a^2 = \dfrac{l}{\sqrt{mk}} \\ b^2 = \dfrac{l}{\sqrt{mk}}, \\ \varepsilon^2 = 0 \end{cases}$$

$$E \to \infty \Rightarrow \xi \to \infty \Rightarrow \begin{cases} a^2 \to \infty \\ b^2 \to 0. \\ \varepsilon^2 = 1 \end{cases}$$

The first case is the expected circle, while the second is a *very* asymmetric ellipse.

c) (2 pts) From the general result relating the period to the area we have

$$\tau = \frac{2mA}{l} = \frac{2\pi mab}{l} = \frac{2\pi m}{l}\frac{l}{\sqrt{mk}} = 2\pi\sqrt{\frac{m}{k}}.$$

The essential feature here is that this period is *independent* of $l$ and $E$, *i.e.*, $\xi$. This means that the period is independent of the initial conditions. It is clearly fixed by the underlying harmonic oscillator frequency, $\omega_0 = \sqrt{k/m}$, and is independent of the size of the oscillations (just like a 1-D oscillator). This feature is *un*like the 1/*r* case. Like the 1/*r* case the orbit is closed, *i.e.*, the change in $\phi$ in a cycle of $\rho$, $\rho_+ \to \rho_- \to \rho_+$, is a simple fraction of $2\pi$, $\Delta\phi = \pi$. Two complete cycles in $\rho$ correspond to exactly 1 complete cycle in $\phi$. Hence the trajectory is a closed, periodic elliptical orbit that is stable under perturbations in the initial conditions and the physical parameters, *m* and *k*. The difference from the 1/*r* case is that in that case 1 cycle in $\rho$ corresponds to 1 complete cycle in $\phi$. If we define a winding number *W* as the ratio of the $\Delta\phi$ in a one (half) cycle in $\rho$, $\rho_+ \to \rho_-$, to $2\pi$, the orbit in a 1/*r* potential has *W* = ½ while the harmonic potential corresponds to *W* = ¼. Generally periodic behavior corresponds to a rational value for *W* (= *m*/*n*, *m*, *n* are integers). A rational *W* value arises only for these two central potentials.

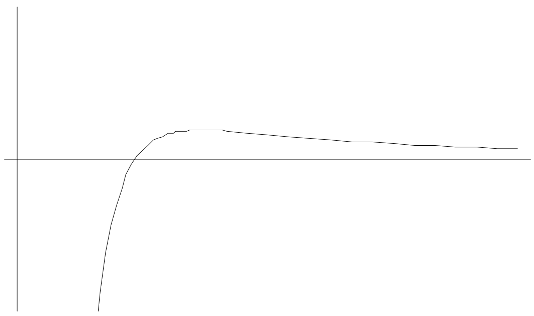
2) (4 pts) Fetter & Walecka - 1.11 The question here is what happens when the central potential is more singular than the angular momentum barrier. Consider whether our methods are reliable for motion near the origin in such a potential, *i.e.*, do any of our approximation break down?

Solution: Recall that we used angular momentum and energy conservation (and cylindrical coordinates) to obtain the first order, 1-D equation, which we apply here

$$\dot{\rho} = \pm\sqrt{\frac{2}{m}}\sqrt{E - U_{eff}(\rho)},$$

$$U_{eff}(\rho) = -\frac{\lambda}{\rho^n} + \frac{L_z^2}{2m\rho^2}.$$

With the constraint $n > 2$ the true potential is more singular at small radius values than the angular momentum barrier and hence dominates the effective potential near the origin. Far from the origin the angular momentum eventually dominates and the effective potential will approach its asymptotic value of zero from *above*. The characteristic shape of $U_{eff}$ is indicated in the following figure.



There are bound states with energies from large negative values (formally -∞) up to the height of the (positive) maximum of $U_{eff}$. These bound states have trajectories described by $E - U_{eff}\left(\rho\right) \geq 0$, $0 \leq \rho \leq \rho_+$ with only an outer classical turning point, $U_{eff}\left(\rho_+\right) = E$. While the derivative of the radius, $\dot{\rho}$, vanishes at this turning point, it diverges (to negative infinity) as the origin is approached. It is also straightforward to verify that the second derivative

$$\ddot{\rho} = \frac{L_z^2}{m^2\rho^3} - \frac{n\lambda}{\rho^{n+1}},$$

is negative at $\rho_+$ (*i.e.*, it is a turning point) and diverges to -∞ as we approach the origin. Remarkably, even with this singular behavior, both the time to reach the origin from the (outer) turning point and the corresponding change in azimuthal angle are finite. We can see this result by considering the following familiar equations

$$\Delta t = \sqrt{\frac{m}{2}}\int_0^{\rho_+} \frac{d\rho'}{\sqrt{E - U\left(\rho'\right) - \frac{L_z^2}{2m\rho'^2}}} = \sqrt{\frac{m}{2}}\int_0^{\rho_+} \frac{d\rho'}{\sqrt{E + \frac{\lambda}{\rho'^n} - \frac{L_z^2}{2m\rho'^2}}}$$

$$= \sqrt{\frac{m}{2}}\int_0^{\rho_+} \frac{d\rho'\rho'^{n/2}}{\sqrt{\lambda + E\rho'^n - \frac{L_z^2\rho'^{n-2}}{2m}}},$$

$$\Delta\phi = \frac{L_z}{\sqrt{2m}}\int_0^{\rho_+} \frac{d\rho'}{\rho'^2}\frac{1}{\sqrt{E + \frac{\lambda}{\rho'^n} - \frac{L_z^2}{2m\rho'^2}}}$$

$$= \frac{L_z}{\sqrt{2m}}\int_0^{\rho_+} \frac{d\rho'\rho'^{n/2-2}}{\sqrt{\lambda + E\rho'^n - \frac{L_z^2\rho'^{n-2}}{2m}}}.$$

Since the range of the integral is finite, $\rho_+ < \infty$, the integral can diverge only if the integrand itself diverges. The expressions above demonstrate that for $n > 2$, both of these integrands are well behaved as $\rho' \to 0$ (the quantity in the *denominator* diverges in this limit). So the remaining question is what happens as $\rho' \to \rho_+$. In this limit we know that the denominator (which is $\dot{\rho}$) vanishes and the integrand diverges, but the integral may still be finite (in fact, we know it is because very similar behavior occurs near both turning points of the 1/r potential, elliptical orbits with finite periods discussed in the Lecture). To test the behavior explicitly we expand the integrand near the turning point,

$$
\begin{aligned}
&\rho \to \rho_+ - \delta \left( \delta \ll \rho_+ \right) \\
&E - U_{eff} = E + \frac{\lambda}{\left(\rho_+ - \delta\right)^n} - \frac{L_z^2}{2m\left(\rho_+ - \delta\right)^2} \\
&\qquad \simeq E + \frac{\lambda}{\left(\rho_+\right)^n}\left(1 + n\frac{\delta}{\rho_+}\right) - \frac{L_z^2}{2m\left(\rho_+\right)^2}\left(1 + 2\frac{\delta}{\rho_+}\right) + \mathcal{O}\left(\frac{\delta^2}{\rho_+^2}\right) \\
&\qquad \simeq \frac{\delta}{\rho_+}\left(\frac{n\lambda}{\left(\rho_+\right)^n} - \frac{L_z^2}{m\left(\rho_+\right)^2}\right) + \mathcal{O}\left(\frac{\delta^2}{\rho_+^2}\right) > 0,
\end{aligned}
$$

Where the last step uses the fact that this expression vanishes at $\rho_+$. So the most singular contribution to the integral from this region near the turning point behaves like

$$\int_0 \frac{d\delta}{\sqrt{\delta}} < \infty.$$

While the integrand is divergent, the integral is perfectly well behaved.

But the question remains, can we say anything about what happens after the particle reaches the origin (in this finite time)? The above results (the divergent negative first and second time derivatives) confirm that the origin is not a second turning point and thus we do not expect the particle to come back out from the origin. On the other hand, our methods really assume velocities that are small compared to $c$ (and bounded accelerations), which is not the case here. To accurately study the behavior near the origin of such a singular potential, we really need a formalism that accounts for relativistic velocities. We might also expect the situation near the origin to look quite different in the highly accelerated frame of the particle itself. We really need to be using General Relativity (think black holes) to treat such singular situations.

3) (2 pts) Fetter & Walecka - 1.14 The strong (nuclear) interactions are short range and serve to define the size of the nucleus. The question here is – how much is the size of the cross section for the interaction between 2 nuclei reduced by their repulsive E&M interaction? HINT: Recall problem 1.13, Ex. 4) in HW I.

Solution: We can use our analysis of the rocket in Exercise 4 of HW I (F&W 1.13) if we switch from the 1/r gravitational potential to the 1/r E&M potential and note that the potential is now repulsive instead of attractive. Hence we replace $k = GMm = \gamma m$ by $k = -zZe^2 \Rightarrow \gamma = -zZe^2/m$. In Exercise 4 we noted that for a given initial energy, $E = mv_\infty^2/2$, all impact parameters out to $b_{\max} = \sqrt{R^2 + 2R\gamma/v_\infty^2}$ will suffer a collision. Making the substitutions relevant to the problem of colliding nuclei we have

$$b_{\max} = \sqrt{R^2 + \frac{2R\gamma}{v_\infty^2}} = R\sqrt{1 - \frac{2zZe^2}{mv_\infty^2 R}} = R\sqrt{1 - \frac{V_c}{E}}$$

$$\Rightarrow \sigma = 2\pi \int_0^{b_{\max}} b\,db = \pi b_{\max}^2 = \pi R^2\left(1 - \frac{V_c}{E}\right)\begin{cases} V_c = zZe^2/R \\ E = mv_\infty^2/2 \end{cases}.$$

Thus, as you might expect, the impact of the repulsive E&M interaction is to reduce the cross section and, at low enough energies, the coulomb barrier can have a substantial effect. You are encouraged to estimate the typical magnitude of $V_c$, *i.e.*, the coulomb potential energy at the surface of a nucleus.

4) (7 pts) Fetter & Walecka - 1.18 Here we want to think about the scattering problem for a general central potential V($\rho$) in the small angle (large impact parameter), (so-called) impulse approximation (the interaction effectively occurs for only a short time). This is often a useful starting point.

Solution: a) (2 pts) The game here is to calculate the transverse impulse transferred to the particle assuming that the trajectory is unperturbed in first approximation, *i.e.*, the trajectory is treated as being straight along impact parameter *b*. At any radial distance $\rho$ the force is in the radial direction ($\hat{\rho}$) with magnitude given by $dV/d\rho$. The component of this force transverse to the trajectory is given by $dV/d\rho\left(b/\rho\right)$. We then proceed by integrating this component of the force over time, which we rewrite as an integral over $\rho$ (actually over ½ the range and then multiply by 2)

$$\int F_{\text{transverse}} dt = \Delta p = \int \frac{dV}{d\rho}\frac{b}{\rho} dt = 2\left| \int_b^\infty \sqrt{\frac{m}{2}} \frac{d\rho}{\sqrt{E - V(\rho) - \frac{L_z^2}{2m\rho^2}}} \frac{dV}{d\rho}\frac{b}{\rho} \right|$$

$$= \sqrt{2m} b \left| \int_b^\infty \frac{d\rho}{\sqrt{\rho^2\left(\frac{mv_\infty^2}{2} - V(\rho)\right) - \frac{(mv_\infty b)^2}{2m}}} \frac{dV}{d\rho} \right|$$

$$= \frac{2b}{v_\infty} \left| \int_b^\infty \frac{d\rho}{\sqrt{\rho^2\left(1 - \frac{2V(\rho)}{mv_\infty^2}\right) - b^2}} \frac{dV}{d\rho} \right|.$$

Now we use the assumption that the scattering is weak by ignoring the $2V/mv_\infty^2$ term in the square root and by writing the change in the momentum that arises from this impulse as $\Delta p = \theta p = \theta m v_\infty$. Putting these pieces together we have the required *approximate* result for the scattering angle

$$\theta \approx \frac{2b}{mv_\infty^2} \left| \int_b^\infty \frac{d\rho}{\sqrt{\rho^2 - b^2}} \frac{dV}{d\rho} \right|.$$

b) (2 pts) Now we want to consider the above small angle (weak scattering) approximation applied to potentials of the form $V(\rho) = \gamma / r^n, (\gamma, n) > 0$. With this specific form we have the following relationship between the scattering angle and the impact parameter

$$\theta \approx \frac{2b}{mv_\infty^2}\left|\int_b^\infty \frac{d\rho}{\sqrt{\rho^2-b^2}}\frac{n\gamma}{\rho^{n+1}}\right| \xrightarrow[x=\rho/b]{} \frac{2n\gamma b^{-n}}{mv_\infty^2}\int_1^\infty \frac{dx}{\sqrt{x^2-1}}\frac{1}{x^{n+1}}$$

$$\xrightarrow[y=1/x\wedge 2]{} \frac{2n\gamma b^{-n}}{mv_\infty^2}\frac{1}{2}\int_0^1 \frac{y^{(n-1)/2}dy}{\sqrt{1-y}} = \frac{n\gamma b^{-n}}{mv_\infty^2}B\left(\frac{n+1}{2},\frac{1}{2}\right)$$

$$= \frac{\gamma b^{-n}}{mv_\infty^2}\frac{\sqrt{\pi}\,\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}.$$

The essential point for the current analysis is that, for small $\theta$, large $b$, we have $\theta \propto b^{-n}$, or $b \propto \theta^{-1/n}$. Thus the small angle behavior of the differential cross section is given by (see Eq. (5.23) in F&W)

$$\left.\frac{d\sigma}{d\Omega}\right|_{\theta\ll 1} \simeq \frac{b}{\theta}\left|\frac{db}{d\theta}\right| = \frac{b}{\theta}\frac{1}{n}\frac{1}{\theta^{1+1/n}} \propto \frac{1}{\theta^{2+2/n}}.$$

In particular, for $n = 1$, $d\sigma/d\Omega \propto \theta^{-4}$, as expected for the Rutherford result, and for the case $n = 2$ (F&W Exercise 1.16) we have $d\sigma/d\Omega \propto \theta^{-3}$, again as expected. Since for small angles $d\Omega \propto \theta d\theta$, neither of these cross section yield finite total (elastic) cross sections, $\int_0 \theta d\theta/\theta^3\left(\text{or } \theta^4\right) \to \infty$. For general $n$ the integral for the total cross section behaves like

$$\int_{\theta_{\min}} \frac{d\sigma}{d\Omega}d\Omega \propto \int_{\theta_{\min}} \frac{1}{\theta^{2+2/n}}\theta d\theta \propto \frac{1}{\theta_{\min}^{2/n}} \xrightarrow[\theta_{\min}\to 0]{} \infty.$$

The classical total (elastic) cross section is divergent for any value of (0 <) $n < \infty$.

c) (2 pts) For $V(\rho) = \gamma e^{-\lambda\rho}, \gamma > 0$ we have

$$\theta \approx \frac{2b}{mv_\infty^2}\left|\int_b^\infty \frac{d\rho}{\sqrt{\rho^2-b^2}}\gamma\lambda e^{-\lambda\rho}\right| = \frac{2b\gamma\lambda}{mv_\infty^2}\left|\int_1^\infty \frac{dx}{\sqrt{x^2-1}}e^{-\lambda bx}\right|$$
$$= \frac{2b\gamma\lambda}{mv_\infty^2}K_0(\lambda b)\xrightarrow[b\to\infty]{}\frac{\sqrt{2\pi\lambda b}\gamma}{mv_\infty^2}e^{-\lambda b}$$
$$\Rightarrow \theta \propto \sqrt{\lambda b}e^{-\lambda b} \Rightarrow b \propto -\frac{1}{\lambda}\ln\theta,$$

where we have recognized the integral representation of the modified Bessel function. Thus the corresponding small angle differential cross section is given by

$$\left.\frac{d\sigma}{d\Omega}\right|_{\theta\ll 1} \simeq \frac{b}{\theta}\left|\frac{db}{d\theta}\right| = \frac{b}{\theta}\frac{1}{\lambda}\frac{1}{\theta} \propto \frac{\ln(1/\theta)}{\theta^2}$$
$$\Rightarrow \sigma_T \propto \int \theta d\theta \frac{\ln(1/\theta)}{\theta^2} + \text{const} \propto \ln^2\theta \xrightarrow[\theta\to 0]{} \infty.$$

This potential is more damped at large radius than those discussed earlier leading to less singular behavior at small scattering angle. However, the total (elastic) cross section is still (logarithmically) divergent.

d) (1 pt) As noted, classical physics differs from quantum physics at small angles. We have seen that, as long as the potential is nonzero at asymptotic values of the radius (even if it is rapidly decreasing), the classical total (elastic) cross section diverges due to the small angle behavior. The classical cross section is only finite if the classical potential actually vanishes for radius values larger than some minimum value. In the QM case it is only necessary that the potential vanishes at large radius faster than $r^{-2}$. What is distinctly classical in the current analysis is the explicit one-to-one relationship between the value of *b* and the value of $\theta$, especially for large b/small angle. In QM one instead works with amplitudes (or probabilities) that the scattering angle takes the value $\theta$, and these amplitudes generally do not correspond to a local behavior in the impact parameter, rather the effective location of the particle is smeared out. The small angle QM scattering amplitude depends on a volume integral over the potential ( $f(\theta\sim 0) \propto \int r^2 dr V(r)$) rather than just the behavior of the potential (or its derivative) locally at large impact parameter. (The wavelength of the effective interaction is large and the scattering occurs coherently from the entire volume.) If this integral is finite, the scattering cross section remains finite at small angles in QM. It is, in fact, remarkable that our classical analysis of the $1/r$ potential yields the correct QM result, the Rutherford cross section.

5) (5 pts) Fetter & Walecka – 2.4 This is a problem in analyzing motion in a rotating frame.

Solution: We can use the discussion in Lecture 4 notes to help us analyze this problem. We start with Eq. (4.24)

$$m\ddot{\vec{r}}' \simeq -mg_{eff}\hat{z}' - 2m\vec{\omega}\times\dot{\vec{r}}'.$$

We will work in the same approximations as in class (the gravitational force is taken to be uniform and we ignore the change in the Coriolis force induced by the velocity change that is caused by the Coriolis force). The vertical motion (in the rotating frame) is then as in freshman physics. We can use the Ansatz

$$\begin{aligned}
&\dot{\vec{r}}'(t) \equiv \left(v_0 - g_{eff}t\right)\hat{z}' + \dot{\vec{\delta}}(t), \\
&v_0 = \sqrt{2g_{eff}h},
\end{aligned}$$

where the second line ensures that the particle reaches a height $h$. Thus the times for reaching the peak altitude and for reaching the ground again are

$$\begin{aligned}
t_{\text{peak}} &= \frac{v_0}{g_{eff}} = \sqrt{\frac{2h}{g_{eff}}}, \\
t_{\text{ground}} &= 2t_{\text{peak}} = 2\sqrt{\frac{2h}{g_{eff}}}.
\end{aligned}$$

Now we want to solve the (approximate) equation

$$\begin{aligned}
\ddot{\vec{\delta}} &\simeq -2\vec{\omega}\times\left(-g_{eff}t\hat{z}' + v_0\hat{z}' + \dot{\vec{\delta}}\right) \simeq 2\vec{\omega}\times\hat{z}'\left(g_{eff}t - v_0\right) \\
&\simeq 2\omega_E\left(\hat{z}\times\hat{z}'\right)\left(g_{eff}t - v_0\right) \\
&= 2\omega_E\left(g_{eff}t - v_0\right)\sin\theta\,\hat{y}' = 2\omega_E g_{eff}\left(t - t_{\text{peak}}\right)\sin\theta\,\hat{y}'.
\end{aligned}$$

Note that, as expected, the sign of the Coriolis force (and thus the acceleration) changes sign as the primary velocity switches from being upward to being downward (just what our spinning skater image tells us). On the other hand, the resulting contribution to the velocity is always westward, but vanishes at both the beginning and the end of the motion,

$$\begin{aligned}
\dot{\vec{\delta}}(t) &\simeq -\omega_E g_{eff} t\left(2t_{\text{peak}} - t\right)\sin\theta\,\hat{y}', \\
\dot{\vec{\delta}}(0) &= \dot{\vec{\delta}}\left(2t_{\text{peak}} = t_{\text{ground}}\right) = 0.
\end{aligned}$$

Finally the displacement is given by

$$\begin{aligned}
\vec{\delta}(t) &\simeq \omega_E\left(g_{eff}\frac{t^3}{3} - v_0 t^2\right)\sin\theta\,\hat{y}' = \frac{\omega_E g_{eff} t^2}{3}\left(t - 3t_{\text{peak}}\right)\sin\theta\,\hat{y}', \\
\vec{\delta}\left(t_{\text{ground}}\right) &\simeq \frac{\omega_E g_{eff}\left(2t_{\text{peak}}\right)^2}{3}\left(-t_{\text{peak}}\right)\sin\theta\,\hat{y}' \\
&= -\frac{4\omega_E g_{eff}}{3}\sin\theta\left(\sqrt{\frac{2h}{g_{eff}}}\right)^3\hat{y}' \\
&= -\frac{8\omega_E}{3}\sin\theta\sqrt{\frac{2h^3}{g_{eff}}}\hat{y}'.
\end{aligned}$$

As already noted, due to the minus sign, this displacement is to the *west*! At the peak the displacement is half the final displacement,

$$\begin{aligned}
\vec{\delta}\left(t_{\text{peak}}\right) &\simeq \frac{\omega_E g_{eff}\left(t_{\text{peak}}\right)^2}{3}\left(-2t_{\text{peak}}\right)\sin\theta\,\hat{y}' \\
&= -\frac{2\omega_E g_{eff}}{3}\sin\theta\left(\sqrt{\frac{2h}{g_{eff}}}\right)^3\hat{y}' \\
&= -\frac{4\omega_E}{3}\sin\theta\sqrt{\frac{2h^3}{g_{eff}}}\hat{y}'.
\end{aligned}$$

For our approximations to be relevant we require that

$$\left|\dot{\vec{\delta}}\right| = \omega_E \sin\theta\left(2v_0 t - g_{eff}t^2\right) = \omega_E g_{eff} t \sin\theta\left(2t_{\text{peak}} - t\right)$$
$$\ll \left|v_0 - g_{eff}t\right| = g_{eff}\left|t_{\text{peak}} - t\right| \Rightarrow \omega_E t_{\text{peak}} \sin\theta = \omega_E\sqrt{\frac{2h}{g_{eff}}}\sin\theta \ll 1,$$

which will generally be true because the period of the earth's rotation, $2\pi/\omega_E$ , is large compared to the time to fall, except for times just at the peak of the vertical motion when $\left(t_{\text{peak}} - t\right)$ can be small.