Physics 505 - Autumn 2010

HW IV Solutions

10/27/10

Overview: Recall that solving physics problems is not (just) about solving differential equations. Use physical reasoning to help solve the following exercises and be certain to show your work. It is also important that you practice completely solving these exercises, checking for errors as you go along.

1) Fetter & Walecka – 3.6 (6 pts) Same idea as the last problem in HW III. Be certain to work out the details.

Solution: (a) (1 pt) Let the length of string above the hole be ρ, *i.e.*, we use cylindrical coordinates to describe the motion on the table, and the length of string below the hole be *z*. The constraint for the motion is thus $\rho + z = l$. If we take the gravitational potential to be zero at the level of the table, the Lagrangian is

$$L = \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) + mgz.$$

This looks like a pretty standard expression except for the sign of the last term. Since we are not interested in the constraint force (*i.e.*, the tension in the string), we can just eliminate the variable ρ in terms of *z* to find

$$L = \frac{m}{2}\left(2\dot{z}^2 + \left(l - z\right)^2\dot{\phi}^2\right) + mgz.$$

(b) (2 pts) First we want to look for "time independent" motion. Lagrange tells us that the equations of motion look like

$$2m\ddot{z} + m(l-z)\dot{\phi}^2 - mg = 0,$$
$$\frac{d}{dt}\left(m(l-z)^2\dot{\phi}\right) = 0 \Rightarrow m(l-z)^2\dot{\phi} = L_z = \text{constant},$$
$$\Rightarrow \ddot{z} = \frac{1}{2}\left(g - \frac{L_z^2}{m^2(l-z)^3}\right).$$

The hanging (lower) mass will remain stationary if $\ddot{z} = 0$ (and $\dot{z} = 0$ initially). Thus for a stationary solution at $z = z_0$ we require

$$g = \frac{L_z^2}{m^2(l-z_0)^3} \Rightarrow \dot{\phi}_0 = \sqrt{\frac{g}{l-z_0}}.$$

For a given value of z_0 (< l) there is a corresponding angular velocity $\dot{\phi}_0$ of the mass on the table such that the hanging mass does not move (the centrifugal force on one is just the gravitational force on the other). The corresponding value for the initial angular momentum is

$$L_{z,0} = m\sqrt{g}\,(l-z_0)^{\frac{3}{2}}.$$

(c) (2 pts) Now we want to consider perturbations to the "fixed" motion of part (b) caused by pulling the hanging mass down slightly. The corresponding perturbing force is vertical as seen by the lower mass and purely radial as seen by the upper mass. Hence the angular momentum, $L_{z,0}$, describing the motion of the upper mass remains the same (is conserved) and has the value exhibited in the previous equation. (This result is true even if the vertical perturbation is not small.)

We can also consider what happens to the total energy,

$$E = \frac{m}{2}\left(2\dot{z}^2 + (l-z)^2\dot{\phi}^2\right) - mgz = \frac{m}{2}\left(2\dot{z}^2 + \frac{L_z^2}{m^2(l-z)^2}\right) - mgz,$$

under such a perturbation. Since the forces are all initially in equilibrium, the work we do on the system is second order in the small perturbation. If we release the perturbed system with the hanging mass at rest after a small perturbation, $\eta(0) = \eta_0 \ll l - z_0, \dot{\eta}(0) = 0$, the total energy changes only at second order in the perturbation

$$E = \frac{m}{2}\left(\frac{g\left(l-z_0\right)^3}{\left(l-z_0-\eta_0\right)^2}\right) - mg\left(z_0+\eta_0\right)$$
$$= \frac{m}{2}g\left(l-z_0\right)\left(1+2\frac{\eta_0}{l-z_0}\right) - mg\left(z_0+\eta_0\right) + \mathcal{O}\left[\frac{m}{2}g\left(l-z_0\right)\left(\frac{\eta_0}{l-z_0}\right)^2\right]$$
$$= \frac{m}{2}g\left[l-3z_0\right] + \mathcal{O}\left[\frac{m}{2}g\left(l-z_0\right)\left(\frac{\eta_0}{l-z_0}\right)^2\right]$$
$$= E_0 + \mathcal{O}\left[\frac{m}{2}g\left(l-z_0\right)\left(\frac{\eta_0}{l-z_0}\right)^2\right].$$

(d) (2 pts) To solve for the subsequent motion of the perturbed system we can proceed as in the previous exercise (in HW III) by including a small perturbation about the equilibrium point. We can then solve for its subsequent evolution (assuming that it is small), $z_0 \to z_0 + \eta, \eta \ll z_0$ and linearizing the equation of motion. We expand our previous equation of motion for *z* about the equilibrium configuration, using the specific value of the angular momentum. We find

$$2\ddot{z} + \left(l-z\right)\left(\frac{L_{z,0}}{m\left(l-z\right)^2}\right)^2 - g = 0$$
$$\Rightarrow 2\ddot{\eta} + \frac{L_{z,0}^2}{m^2\left(l-z_0-\eta\right)^3} - g \simeq 2\ddot{\eta} + \frac{L_{z,0}^2}{m^2\left(l-z_0\right)^3} - g + \eta\frac{3L_{z,0}^2}{m^2\left(l-z_0\right)^4} = 0$$
$$\Rightarrow \ddot{\eta} + \eta\frac{3L_{z,0}^2}{2m^2\left(l-z_0\right)^4} = \ddot{\eta} + \frac{3g}{2\left(l-z_0\right)}\eta = 0.$$

Thus the motion corresponds to oscillations (a second linear differential equation with the correct sign for a restoring force) and we can simply read off the frequency

$$\omega = \sqrt{\frac{3g}{2\left(l-z_0\right)}} = \sqrt{\frac{3}{2}}\dot{\phi}_0.$$

If the hanging mass is displaced by a (small) distance η_0 and released at rest, the oscillations will be described by the following functions of time (the solutions of the equations above)

$$\eta(t) = \eta_0 \cos\omega t = \eta_0 \cos\left(\sqrt{\frac{3g}{2(l - z_0)}} t\right)$$
$$z(t) = z_0 + \eta_0 \cos\left(\sqrt{\frac{3g}{2(l - z_0)}} t\right).$$

A similar expression will describe the radius of the orbit followed by the mass on the table. Note that, since $\omega / \dot{\phi}_0 = \sqrt{3/2}$ is not a rational number, the trajectory of the mass on the table will not be a closed orbit (the frequency of the radial motion and the frequency of the angular motion are not related by a ratio of integers).

2) Fetter & Walecka – 3.17 (6 pts) This problem illustrates the usefulness of the method of Lagrange multipliers, *i.e.*, we can easily evaluate when the constraint force goes to zero.

Solution: (a) (3 pts) The motion of the point mass on the smooth sphere is just motion in uniform gravity but with the constraint that the trajectory be a circle (say with radius R) until it falls off. We use Lagrange multipliers in this problem so that we obtain the magnitude of the constraint force (the "reaction of the sphere") automatically and the condition for falling off the sphere is when this force vanishes. We employ spherical coordinates on the sphere with the *z* direction up (opposite gravity). With an appropriate choice of the zero of the potential energy, the Lagrangian and the constraint function are

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - mgr\cos\theta,$$
$$f(r,\theta) = r - R = 0.$$

Thus we want to solve the following equations (from Lagrange)

$$r: m\frac{d}{dt}\left(\dot{r}\right) - mr\dot{\theta}^2 + mg\cos\theta = m\left(\ddot{r} - r\dot{\theta}^2 + g\cos\theta\right) = -\lambda,$$
$$\theta: m\frac{d}{dt}\left(r^2\dot{\theta}\right) - mgr\sin\theta = m\left(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} - gr\sin\theta\right) = 0.$$

From the constraint we have $r = R, \dot{r} = \ddot{r} = 0$ so that the equations of motion become

$$\dot{\theta}^2 - \frac{g}{R}\cos\theta = \frac{\lambda}{Rm},$$
$$\ddot{\theta} - \frac{g}{R}\sin\theta = 0.$$

The second equation we can integrate with the integrating factor $\dot{\theta}$ (as we did in the lecture – note the result here is just conservation of energy and got be obtained that way)

$$\dot{\theta}\ddot{\theta} = \frac{g}{R}\dot{\theta}\sin\theta \Rightarrow \frac{d}{dt}\left(\frac{\dot{\theta}^2}{2}\right) = -\frac{d}{dt}\left(\frac{g\cos\theta}{R}\right)$$
$$\Rightarrow \dot{\theta}^2 = -\frac{2g}{R}\cos\theta + \text{constant} = \frac{2g}{R}\left(1 - \cos\theta\right),$$

where the last step matches the initial conditions $\theta\left(0\right) = \dot{\theta}\left(0\right) = 0$, *i.e.*, at rest at the top of the sphere. Without further work we can solve for the constraint force through the Lagrange multiplier,

$$\lambda = mR\left(\dot{\theta}^2 - \frac{g}{R}\cos\theta\right) = mR\left(\frac{2g}{R}\left(1 - \cos\theta\right) - \frac{g}{R}\cos\theta\right)$$
$$= mg\left(2 - 3\cos\theta\right).$$

The particle falls off when this force vanishes

$$\theta_{off} = \cos^{-1}\left(\frac{2}{3}\right) \simeq 48.19°,$$
$$h_{off} = R\left(1+\cos\theta_{off}\right) = \frac{5}{3}R.$$

(b) (3 pts) Now we want to consider how the situation changes when the point mass is replaced by a small roughened sphere that rolls on the original sphere without slipping. Clearly the change is that now there is kinetic energy associated with the rotation of the rolling sphere. If the rolling sphere has radius *a*, its moment of inertia is $I_a = 2ma^2/5$. We can also define the angle of its rotation to be θ_a (measured, for example, with respect to the vertical), while its location on the original sphere is still defined by the variable θ. The new Lagrangian and the 2 constraint equations look like (*r* is the location of the CM of the rolling sphere)

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{m}{2}\left(\frac{2a^2}{5}\right)\dot{\theta}_a^2 - mgr\cos\theta,$$
$$f(r,\theta) = r - R - a = 0,$$
$$f_a(\theta,\theta_a) = R\theta - a(\theta_a - \theta) = 0.$$

The second constraint says the sphere rolls without slipping. Note the important point that if, in fact, it was slipping down the original sphere with the same point always in contact, it would still rotate by an angle θ (recall the hoop on a sphere discussed in the lecture). In matching the arc lengths in the second constraint this kinematic fact must be subtracted out (see Eq. 19.38 in F&W). So now Lagrange says the equations of motion are

$$m\left(\ddot{r} - r\dot{\theta}^2 + g\cos\theta\right) = -\lambda,$$
$$m\left(r^2\ddot{\theta} - gr\sin\theta\right) = -\lambda_a(R+a)$$
$$\frac{2m}{5}a^2\ddot{\theta}_a = a\lambda_a.$$

When we include the constraints (especially $\ddot{\theta}(R+a) = \ddot{\theta}_a a$), we have

$$\dot{\theta}^2 - \frac{g}{R+a}\cos\theta = \frac{\lambda}{m(R+a)},$$

$$\ddot{\theta} - \frac{g}{R+a}\sin\theta = -\frac{\lambda_a}{m(R+a)}$$

$$\ddot{\theta}_a = \frac{5\lambda_a}{2ma} \Rightarrow \ddot{\theta} = \frac{5\lambda_a}{2m(R+a)}$$

$$\Rightarrow \ddot{\theta} - \frac{g}{R+a}\sin\theta = -\frac{2}{5}\ddot{\theta} \Rightarrow \ddot{\theta} = \frac{5}{7}\frac{g}{R+a}\sin\theta.$$

Employing the same integrating factor as above we find

$$\dot{\theta}\ddot{\theta} = \frac{5}{7}\frac{g}{R+a}\dot{\theta}\sin\theta \Rightarrow \frac{d}{dt}\left(\frac{\dot{\theta}^2}{2}\right) = -\frac{d}{dt}\left(\frac{5}{7}\frac{g\cos\theta}{R+a}\right)$$

$$\Rightarrow \dot{\theta}^2 = -\frac{10g}{7(R+a)}\cos\theta + \text{constant} = \frac{10g}{7(R+a)}(1-\cos\theta).$$

Hence the normal constraint force at the surface of the sphere behaves like

$$\lambda = m(R+a)\left(\dot{\theta}^2 - \frac{g}{R+a}\cos\theta\right)$$

$$= m(R+a)\left(\frac{10g}{7(R+a)}(1-\cos\theta) - \frac{g}{R+a}\cos\theta\right)$$

$$= \frac{mg}{7}(10 - 17\cos\theta).$$

Hence in this new situation separation occurs for

$$\theta_{off,a} = \cos^{-1}\left(\frac{10}{17}\right) \simeq 53.97°,$$

$$h_{off,a} = (R+a)\left(1+\cos\theta_{off,a}\right) = \frac{27}{17}(R+a).$$

Since some of the gain in kinetic energy (from the loss in potential energy) goes into the rotational energy of the rolling sphere, it takes longer (a larger angle) before separation velocity is reached.

For fun lets summarize the 4 cases we have looked at with the following table

Object	I	θ_{off}
Point Mass	0	48.19°
Sphere	$2ma^2/5$	53.97°
Cylinder	$ma^2/2$	55.15°
Hoop	ma^2	60°

As the moment of inertia increases, absorbing more of the energy in rotation, the angle at separation also increases.

3) Fetter & Walecka – 3.18 (7 pts) This is basically a freshman physics problem, but here more carefully analyzed. It is similar to the previous problem, but with a bit more complexity. Be certain to answer all of the parts of the question.

Solution: Since the ladder will experience rotation, we need to evaluate its moment of inertia. The relevant mass density per unit length is $\rho = M/L$, which yields

$$I_L = 2\int_0^{L/2} r^2 \rho dr = \frac{2}{3}\rho r^3\Big|_0^{L/2} = \frac{ML^2}{12}.$$

If we define (x, y) to be the horizontal and vertical coordinates of the CM of the ladder (measured with respect to an origin where the floor meets the wall and with zero gravitational potential at $y = 0$) and θ to be the angle of rotation about the CM in the clockwise sense ($\theta(0) = \theta_0$), we can express the Lagrangian as

$$L = \frac{M}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}\frac{ML^2}{12}\dot{\theta}^2 - Mgy.$$

Note that thinking about the motion of the CM, plus motion around it is the preferred way to proceed. We can also express the constraints that the ladder continues to touch the wall and the floor as

$$x - \frac{L}{2}\cos\theta = 0,$$
$$y - \frac{L}{2}\sin\theta = 0.$$

As usual we use Lagrange to write down the equations of motion

$$M\,\ddot{x} = \lambda_w \quad [\hat{x} \text{ force due to wall}],$$
$$M\,\ddot{y} + Mg = \lambda_f \quad [\hat{y} \text{ force due to floor}],$$
$$\frac{ML^2}{12}\ddot{\theta} = \frac{L}{2}\lambda_w \sin\theta - \frac{L}{2}\lambda_f \cos\theta.$$

From the second derivative of the constraint equations we also have

$$M\,\ddot{x} = -\frac{ML}{2}\left(\sin\theta\ddot{\theta} + \cos\theta\dot{\theta}^2\right) = \lambda_w,$$
$$M\,\ddot{y} + Mg = \frac{ML}{2}\left(\cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2\right) + Mg = \lambda_f.$$

Combining these equations we can eliminate *x* and *y* in terms of θ,

$$\begin{aligned}\frac{ML^2}{12}\ddot{\theta} &= \frac{L}{2}\sin\theta\left[-\frac{ML}{2}\left(\sin\theta\ddot{\theta} + \cos\theta\dot{\theta}^2\right)\right] \\ &\quad - \frac{L}{2}\cos\theta\left[\frac{ML}{2}\left(\cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2\right) + Mg\right] \\ &= \frac{ML^2}{4}\left[-\ddot{\theta}\right] - \frac{MLg}{2}\cos\theta \\ \Rightarrow \ddot{\theta} &+ \frac{3}{2}\frac{g}{L}\cos\theta = 0.\end{aligned}$$

This last equation can be integrated again using the integrating factor $\dot{\theta}$ (as we did in the earlier problems, or by using the conservation of energy) to find

$$\dot{\theta}^2 = 3\frac{g}{L}\left(\sin\theta_0 - \sin\theta\right),$$
$$\dot{\theta} = -\sqrt{\frac{3g}{L}\left(\sin\theta_0 - \sin\theta\right)}.$$

As usual we could obtain this result directly from energy conservation. In terms of the θ coordinate only (*i.e.*, use the constraints to eliminate *x* and *y* directly), we have

$$E_0 = \frac{MgL}{2}\sin\theta_0 = \frac{M}{2}\left(\frac{L}{2}\dot{\theta}\right)^2\left(\sin^2\theta + \cos^2\theta\right) + \frac{ML^2}{24}\dot{\theta}^2 + \frac{MgL}{2}\sin\theta$$
$$= \frac{ML^2}{6}\left(\dot{\theta}^2 + \frac{3g}{L}\sin\theta\right).$$

At least formally we can solve for the angle as a function time by separating variables (as in our orbit problems) and integrating,

$$t = \int_0^t dt' = -\sqrt{\frac{L}{3g}}\int_{\theta_0}^{\theta}\frac{d\theta'}{\sqrt{\sin\theta_0 - \sin\theta'}},$$

which produces an elliptic integral (as in the analysis of a finite angle pendulum). Note, this is a good place to use *Mathematic* to proceed numerically.

Here we are interested in the force from the wall that has the form

$$\lambda_w = -\frac{ML}{2}\left(\sin\theta\ddot{\theta} + \cos\theta\dot{\theta}^2\right)$$
$$= \frac{ML}{2}\left(\sin\theta\frac{3}{2}\frac{g}{L}\cos\theta + \cos\theta 3\frac{g}{L}\left(\sin\theta - \sin\theta_0\right)\right)$$
$$= \frac{3Mg\cos\theta}{4}\left(3\sin\theta - 2\sin\theta_0\right).$$

This force vanishes (and the ladder loses contact with wall) when

$$\theta_{off} = \sin^{-1}\left(\frac{2}{3}\sin\theta_0\right) < \theta_0.$$

At that angle the elevation of the upper end of the ladder is

$$y_{top,off} = L\sin\theta_{off} = \frac{2}{3}L\sin\theta_0,$$

as desired. Note that the initial elevation of the end of the ladder is $L\sin\theta_0$. At the same time the force from the floor is

$$\begin{aligned}
\lambda_f &= \frac{ML}{2}\left(\cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2\right) + Mg \\
&= \frac{ML}{2}\left(-\cos\theta\frac{3}{2}\frac{g}{L}\cos\theta + \sin\theta 3\frac{g}{L}\left(\sin\theta - \sin\theta_0\right)\right) + Mg \\
&= \frac{Mg}{4}\left(-3\cos^2\theta + 6\sin^2\theta - 6\sin\theta\sin\theta_0 + 4\right) \\
&= \frac{Mg}{4}\left(1 + 9\sin^2\theta - 6\sin\theta\sin\theta_0\right) = \frac{Mg}{4} + \lambda_w\tan\theta.
\end{aligned}$$

At the time the top pulls away from the wall ($\lambda_w = 0$) we have $\lambda_f = Mg/4 > 0$. The bottom end of the ladder is still in contact with the floor.

After losing contact with the wall ($\theta < \theta_{off}$), the equations of motion look like

$$\begin{aligned}
&M\ddot{x} = 0, \\
&M\ddot{y} + Mg = \lambda_f = \frac{ML}{2}\left(\cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2\right) + Mg, \\
&\frac{ML^2}{12}\ddot{\theta} = -\frac{L}{2}\lambda_f\cos\theta = -\frac{ML^2}{4}\cos\theta\left(\cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2\right) - \frac{MLg}{2}\cos\theta \\
&\Rightarrow \frac{L}{12}\ddot{\theta}\left(1 + 3\cos^2\theta\right) - \frac{L}{4}\sin\theta\cos\theta\dot{\theta}^2 + \frac{g}{2}\cos\theta = 0.
\end{aligned}$$

One more time we can use the integrating factor $\dot{\theta}$,

$$\frac{d}{dt}\left[\frac{L}{24}\dot{\theta}^2+\frac{L}{8}\dot{\theta}^2\cos^2\theta+\frac{g}{2}\sin\theta\right]=0$$
$$\Rightarrow \frac{ML^2}{24}\dot{\theta}^2+\frac{ML^2}{8}\dot{\theta}^2\cos^2\theta+\frac{gML}{2}\sin\theta=\text{constant}=E.$$

As we see the corresponding conserved quantity is the energy of the ladder corresponding to just its rotation and the motion in the *y* direction. After the ladder leaves the wall the energy associated with the *x* motion is $M\dot{x}^2/2\big|_{\theta=\theta_{off}}=MgL\sin^3\theta_0/18=E_0\sin^2\theta_0/9$, which thereafter remains constant. The constant *E* itself (as defined above) has the value

$$\begin{aligned}E&=\frac{ML^2}{24}\dot{\theta}^2+\frac{ML^2}{8}\dot{\theta}^2\cos^2\theta+\frac{gML}{2}\sin\theta\bigg|_{\theta=\theta_{off}}\\&=-\frac{ML^2}{24}\left(1+3\cos^2\theta_{off}\right)\times 3\frac{g}{L}\left(\sin\theta_{off}-\sin\theta_0\right)+\frac{gML}{2}\sin\theta_{off}\\&=\frac{gML}{24}\left(4-\frac{4}{3}\sin^2\theta_0\right)\sin\theta_0+\frac{gML}{3}\sin\theta_0\\&=\frac{gML}{2}\sin\theta_0-\frac{gML}{18}\sin^3\theta_0=E_0\left(1-\frac{\sin^2\theta_0}{9}\right).\end{aligned}$$

As expected, when we add the *x* motion related energy we get back to the initial (total) energy $E_0=MgL\sin\theta_0/2$. Knowing *E* we can again setup an integral for $\theta(t)$, much as we did when we studied orbits and in the previous exercises,

$$\dot{\theta}^2=\frac{12g}{L}\frac{\left[\frac{2E}{MgL}-\sin\theta\right]}{1+3\cos^2\theta}\Rightarrow\frac{d\theta}{dt}=\pm\sqrt{\frac{\frac{12g}{L}\left[\frac{2E}{MgL}-\sin\theta\right]}{1+3\cos^2\theta}}$$
$$\Rightarrow t=\int_{t_{off}}^{t}dt'=-\sqrt{\frac{L}{12g}}\int_{\theta_{off}}^{\theta}\sqrt{\frac{1+3\cos^2\theta'}{\left[\sin\theta_0\left(1-\sin^2\theta_0/9\right)-\sin\theta'\right]}}d\theta'.$$

Finally we can consider the behavior of the force from the floor. We have

$$\lambda_f = \frac{ML}{2}\left(\cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2\right) + Mg$$

$$= \frac{ML}{2}\cos\theta\left(-\frac{6}{ML}\cos\theta\lambda_f\right) - \frac{ML}{2}\sin\theta\frac{12g}{L}\frac{\left[\frac{2E}{MgL} - \sin\theta\right]}{1+3\cos^2\theta} + Mg$$

$$\Rightarrow \lambda_f\left(1+3\cos^2\theta\right) = Mg - 6Mg\sin\theta\frac{\left[\frac{2E}{MgL} - \sin\theta\right]}{1+3\cos^2\theta}$$

$$\Rightarrow \lambda_f = \frac{Mg}{\left(1+3\cos^2\theta\right)^2}\left[1+3\cos^2\theta + 6\sin^2\theta - 12\frac{E}{MgL}\sin\theta\right]$$

$$= \frac{Mg}{\left(4-3\sin^2\theta\right)^2}\left[4+3\sin^2\theta - 6\sin\theta\sin\theta_0\left(1-\frac{1}{9}\sin^2\theta_0\right)\right]$$

$$= \frac{Mg}{\left(4-3\sin^2\theta\right)^2}\left[4+3\left(\sin\theta - \sin\theta_0\left(1-\frac{1}{9}\sin^2\theta_0\right)\right)^2 \right.$$

$$\left. -3\sin^2\theta_0\left(1-\frac{1}{9}\sin^2\theta_0\right)^2\right].$$

Since the last term in the numerator is always smaller than 3 in magnitude, and the first two terms always sum to greater than 4, the force due to the floor never vanishes (in the angular range $0 \le \theta \le \theta_{off}$). Note that the force goes to $\lambda_f = Mg/4$ at both ends of the range, $\theta = \theta_{off}$ and $\theta = 0$, and has a minimum value near $\sin\theta_{\min} \simeq \sin\theta_{off}/2 \simeq \sin\theta_0/3$ with a value between about $0.19\,Mg$ and $0.25\,Mg$.

4) (6 pts) Lets consider a different sort of problem where we start with the Lagrangian,

$$L = \frac{1}{2}m\left(\dot{x}^2 - \omega^2 x^2\right)e^{\gamma t},$$

describing the motion of a particle of mass *m* in one dimension (*x*). The constants m, γ and ω are real and positive.

a) (1 pt) Find the equation of motion for the particle.

Solution: From Lagrange's equation we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$
$$\Rightarrow \frac{d}{dt}\left(m\dot{x}e^{\gamma t}\right) + m\omega^2 x e^{\gamma t} = 0$$
$$\Rightarrow m\ddot{x} + m\gamma\dot{x} + m\omega^2 x = 0.$$

b) (1 pt) You have seen this system before. What is it, and what kinds of forces act on the particle?

Solution: We recognize this equation as describing a damped harmonic oscillator. There is a linear restoring force, $-m\omega^2 x$, and a (viscous) damping force, $-m\gamma\dot{x}$ (proportional to the speed).

c) (2 pts) Find the canonical momentum, and from this construct the Hamiltonian function.

Solution: We simply use our general formulae to find

$$p = \frac{\partial L}{\partial \dot{x}} = me^{\gamma t}\dot{x},$$
$$H = p\dot{x} - L$$
$$= me^{\gamma t}\dot{x}^2 - \frac{1}{2}me^{\gamma t}\dot{x}^2 + \frac{1}{2}m\omega^2 e^{\gamma t}x^2$$
$$= \frac{p^2}{2m}e^{-\gamma t} + \frac{1}{2}m\omega^2 e^{\gamma t}x^2.$$

d) (1 pt) Is the Hamiltonian a constant of the motion? Is the energy conserved? Explain.

Solution: In this case there is explicit time dependence and the Hamiltonian is not a constant of the motion. The energy is not conserved as the damping force is continually doing negative work on the system, dissipating energy until $\dot{x} = 0$.

e) (1 pt) For the initial conditions $x(0) = 0$ and $\dot{x}(0) = v_0$, what is $x(t)$ asymptotically as $t \to \infty$?

Solution: We try an Ansatz of the general form $x \sim e^{i\Omega t}$. Substitution in the equation of motion yields

$$-\Omega^2 + i\gamma\Omega + \omega^2 = 0$$
$$\Rightarrow \Omega = \frac{i}{2}\left(\gamma \pm \sqrt{\gamma^2 - 4\omega^2}\right).$$

The general solution is then

$$x(t) = A\exp\left[-\frac{1}{2}\left(\gamma + \sqrt{\gamma^2 - 4\omega^2}\right)t\right] + B\exp\left[-\frac{1}{2}\left(\gamma - \sqrt{\gamma^2 - 4\omega^2}\right)t\right].$$

The initial conditions yield

$$B = -A,\ A = -\frac{v_0}{\sqrt{\gamma^2 - 4\omega^2}}.$$

Thus we have damped, damped oscillation or critically damped behavior

$$x = \frac{2v_0 e^{-\gamma t/2}}{\sqrt{\gamma^2 - 4\omega^2}}\sinh\left(\frac{\sqrt{\gamma^2 - 4\omega^2}}{2}t\right) \quad [\gamma > 2\omega],$$
$$x = \frac{2v_0 e^{-\gamma t/2}}{\sqrt{4\omega^2 - \gamma^2}}\sin\left(\frac{\sqrt{4\omega^2 - \gamma^2}}{2}t\right) \quad [\gamma < 2\omega],$$
$$x = v_0 t e^{-\gamma t/2} \quad [\gamma = 2\omega].$$

In any case we have $x(\infty) = 0$, as expected for damped motion.