Physics 505 - Autumn 2010

HW IX

Due 12/1/10

Overview: Recall that solving physics problems is not (just) about solving differential equations. Use physical reasoning to help solve the following exercises and be certain to show your work. It is also important that you practice completely solving these exercises, checking for errors as you go along.

1) (6 pts) We want to look more carefully at nonlinear behavior in oscillators as discussed in Lecture 12. We have in mind a pendulum and so we write the equation of motion as

$$\ddot{\theta} + \omega_0^2 \theta \left(1 + \alpha\theta^2\right) = 0,$$

where, as noted in Lecture 12, $\alpha = -1/6$ for the pendulum (*i.e.*, from expanding $\sin\theta$). In the lecture we focused on a first order analysis, *i.e.*, keeping only terms up to first order in α (note that here α is dimensionless). Here we want to see how much the result changes if we keep terms up to second order. Assume the following form for the solution

$$\theta(t) \simeq A_1 \cos(\omega_\tau t) + \alpha A_3 \cos(3\omega_\tau t) + \alpha^2 A_5 \cos(5\omega_\tau t),$$

and find the form of the frequency ω_τ (in terms of ω_0, α and A_1) to second order in α. For the α value appropriate to the pendulum, how large can the amplitude A_1 be before the frequency ω_τ differs by more than 20% from the linear result ω_0.

2) (10 pts) Consider the motion of a pendulum of length l and (velocity dependent) viscous damping described by γ. The equation of motion is

$$\ddot{\theta} + \gamma\dot{\theta} + \omega_0^2 \sin\theta = 0,\ \omega_0^2 = \frac{g}{l}.$$

a) Define a dimensionless time unit by $\tau = \omega_0 t$ (really an phase) and rewrite the equation of motion in terms of this new "time". What single parameter describes the behavior of the pendulum?

b) Write a fourth-order Runge-Kutta script or use *Mathematica* to solve this equation of motion for arbitrary initial conditions, $\theta(0) = \theta_0, \dot{\theta}(0) = \dot{\theta}_0$.

HINT: We can always rewrite a 2nd order differential equation as two 1st order equations,

$$\ddot{y} + b\dot{y} + f(y) = 0 \Rightarrow \begin{cases} \dot{y} = z \\ \dot{z} = -bz - f(y) \end{cases},$$

and use the Runge-Kutta technique to solve for the 2-D vector $\vec{x} = (y, z)$,

$$\dot{\vec{x}} = \begin{pmatrix} \dot{y} \\ \dot{z} \end{pmatrix} = \vec{F}(t, \vec{x}) = \begin{pmatrix} z \\ -bz - f(y) \end{pmatrix}.$$

The R-K solution is then given by

$$\vec{z}(t_0 + \delta) \simeq \vec{z}_0 + \frac{1}{6}\left(\vec{k}_1 + 2\vec{k}_2 + 2\vec{k}_3 + \vec{k}_4\right)\left[+O\left(\delta^5\right)\right],$$

$$\vec{k}_1 = \delta\,\vec{F}(t_0, \vec{z}_0),$$

$$\vec{k}_2 = \delta\,\vec{F}\left(t_0 + \frac{\delta}{2}, \vec{z}_0 + \frac{\vec{k}_1}{2}\right),$$

$$\vec{k}_3 = \delta\,\vec{F}\left(t_0 + \frac{\delta}{2}, \vec{z}_0 + \frac{\vec{k}_2}{2}\right),$$

$$\vec{k}_4 = \delta\,\vec{F}\left(t_0 + \delta, \vec{z}_0 + \vec{k}_3\right).$$

c) Ignore the friction term for the moment. Start the pendulum at $\theta_0 = 0.1$ ($\dot{\theta}(0) = 0$) and plot $\theta(\tau)$ for the range $\tau = 0$ to $\tau = 4\pi$. On the same graph plot the corresponding result for the small angle approximation, $\sin\theta \to \theta$, that we have studied earlier. Repeat this comparison for the case $\theta_0 = 1.0$. What do you learn from these 2 comparisons? What angular frequency ω_τ do you obtain in the second case?

d) Still ignoring friction, use your code to graph the angular frequency ω_τ as a function of the initial amplitude θ_0 over the range $0 \le \theta_0 < \pi$. At what value of θ_0 does ω_τ deviate by 20% from its value in the linear (small angle) problem? How does this compare to the result in the previous exercise?

e) Consider the initial conditions $\theta_0 = 3.0, \dot{\theta}_0 = 0$ and, ignoring friction, graph the phase space trajectory $\left(\theta, p_\theta = \dot{\theta}\right)$ over a full cycle, indicating the direction of increasing τ on the trajectory with arrows. With the same initial conditions but now with $\gamma/\omega_0 = 0.2$ graph the phase space trajectory for the range $0 \le \tau \le 30$.

3) (6 pts) Let us think about the previous problem in a bit more detail in phase space. We want to contrast the situations with and without damping.

a) First without friction, make a plot (sketch) of the region in phase space occupied by the trajectories corresponding to the initial conditions $\dot{\theta}(0) = 0$, $0.8 \le \theta_0 \le 1.0$ (show the region as a shaded area). Now make a second plot (sketch) for the corresponding system with $\gamma/\omega_0 = 0.2$. As usual *Mathematic*a may be very useful.

b) Verify that the damped oscillator is dissipative. How is the feature realized in the plots of a)?

c) Do either of the plots in a) show evidence of an attractor? Explain (This would be a good time to reread the Appendix to Lecture 7 and start reading Baker and Gollub.) For the oscillator with dissipation, discuss the general structure of all attractors that may be present, including the issue of convergence of trajectories.