Physics 505 - Autumn 2010

HW IX Solutions

12/1/10

Overview: Recall that solving physics problems is not (just) about solving differential equations. Use physical reasoning to help solve the following exercises and be certain to show your work. It is also important that you practice completely solving these exercises, checking for errors as you go along.

1) (6 pts) We want to look more carefully at nonlinear behavior in oscillators as discussed in Lecture 12. We have in mind a pendulum and so we write the equation of motion as

$$\ddot{\theta} + \omega_0^2 \theta \left(1 + \alpha \theta^2\right) = 0,$$

where, as noted in Lecture 12, $\alpha = -1/6$ for the pendulum (*i.e.*, from expanding $\sin\theta$). In the lecture we focused on a first order analysis, *i.e.*, keeping only terms up to first order in α (note that here α is dimensionless). Here we want to see how much the result changes if we keep terms up to second order. Assume the following form for the solution

$$\theta(t) \simeq A_1 \cos\left(\omega_\tau t\right) + \alpha A_3 \cos\left(3\omega_\tau t\right) + \alpha^2 A_5 \cos\left(5\omega_\tau t\right),$$

and find the form of the frequency ω_τ (in terms of ω_0, α and A_1) to second order in α. For the α value appropriate to the pendulum, how large can the amplitude A_1 be before the frequency ω_τ differs by more than 20% from the linear result ω_0.

<u>Solution</u>: We proceed as in class to expand the solution and the equation to order α^2. However, note that we are expressing everything in terms of the (perturbed) amplitude A_1 and not in terms of the actual initial condition θ_0 (or *A* in the lecture). There will be differences at order α^2. The current approach simplifies the arithmetic. Substituting we have

$$\begin{aligned}
&-\omega_\tau^2\left[A_1\cos(\omega_\tau t)+9\alpha A_3\cos(3\omega_\tau t)+25\alpha^2A_5\cos(5\omega_\tau t)\right]\\
&+\omega_0^2\left[A_1\cos(\omega_\tau t)+\alpha A_3\cos(3\omega_\tau t)+\alpha^2A_5\cos(5\omega_\tau t)\right]\\
&+\alpha\omega_0^2\left[A_1\cos(\omega_\tau t)+\alpha A_3\cos(3\omega_\tau t)+\alpha^2A_5\cos(5\omega_\tau t)\right]^3\\
&\simeq A_1\cos(\omega_\tau t)\left\{-\omega_\tau^2+\omega_0^2\right\}+\alpha A_3\cos(3\omega_\tau t)\left\{-9\omega_\tau^2+\omega_0^2\right\}\\
&\quad+\alpha^2A_5\cos(5\omega_\tau t)\left\{-25\omega_\tau^2+\omega_0^2\right\}\\
&\quad+\alpha\omega_0^2\left\{\left(A_1\cos(\omega_\tau t)\right)^3+3\alpha A_3\left(A_1\cos(\omega_\tau t)\right)^2\cos(3\omega_\tau t)\right\}\\
&\simeq A_1\cos(\omega_\tau t)\left\{-\omega_\tau^2+\omega_0^2+\frac{3}{4}\alpha\omega_0^2A_1^2\left(1+\alpha\frac{A_3}{A_1}\right)\right\}\\
&\quad+\alpha\cos(3\omega_\tau t)\left\{-9\omega_\tau^2A_3+\omega_0^2A_3+\frac{1}{4}\omega_0^2A_1^3+\frac{3}{2}\alpha\omega_0^2A_1^2A_3\right\}\\
&\quad+\alpha^2\cos(5\omega_\tau t)\left\{-25\omega_\tau^2A_5+\omega_0^2A_5+\frac{3}{4}\omega_0^2A_1^2A_3\right\}\simeq 0.
\end{aligned}$$

To obtain the last expressions we have used three trigonometric identities, $\cos^3\omega t=\left(3\cos\omega t+\cos 3\omega t\right)/4$, $\cos^2\omega t=\left(1+\cos 2\omega t\right)/2$ and $\cos^2\omega t\cos 3\omega t=\left(\cos 5\omega t+2\cos 3\omega t+\cos\omega t\right)/4$. Setting each of the coefficients of the independent time dependences to zero we have first ($A_1\neq 0$ by assumption)

$$\omega_\tau^2=\omega_0^2\left(1+\frac{3}{4}\alpha A_1^2+\frac{3}{4}\alpha^2A_1A_3\right),$$

which says that to obtain ω_τ to order α^2, we need A_3 only to order α^0. This we can get from the $\cos 3\omega_\tau t$ term,

$$\begin{aligned}
&9\omega_\tau^2A_3=\omega_0^2\left(A_3+\frac{1}{4}A_1^3+\frac{3}{2}\alpha A_1^2A_3\right)=9\omega_0^2A_3\left(1+\frac{3}{4}\alpha A_1^2+\frac{3}{4}\alpha^2A_1A_3\right)\\
&\Rightarrow A_3\left[8+\frac{21}{4}\alpha A_1^2+\frac{3}{4}\alpha^2A_1A_3\right]=\frac{1}{4}A_1^3\Rightarrow A_3\simeq\frac{1}{32}A_1^3.
\end{aligned}$$

Substituting back in the initial result we have

$$\omega_\tau^2 \simeq \omega_0^2\left(1+\frac{3}{4}\alpha A_1^2+\frac{3}{4}\alpha^2 A_1\frac{1}{32}A_1^3\right)=\omega_0^2\left(1+\frac{3}{4}\alpha A_1^2+\frac{3}{128}\left(\alpha A_1^2\right)^2\right).$$

To see a deviation of 20 % we want

$$\begin{aligned}
&\left(1+\frac{3}{4}\alpha A_1^2+\frac{3}{128}\left(\alpha A_1^2\right)^2\right)=\left(\frac{\omega_\tau}{\omega_0}\right)^2=(1.2)^2=\frac{36}{25}\\
&\Rightarrow\left(\alpha A_1^2\right)^2+32\alpha A_1^2-\frac{1408}{75}=0\\
&\Rightarrow\alpha A_1^2=\frac{-32\pm\sqrt{1024+1536/75}}{2}=-16\left[1\mp\sqrt{1+\frac{11}{150}}\right]\\
&\qquad\quad=\begin{matrix}0.576288\\-32.5763\end{matrix}.
\end{aligned}$$

We are interested in the positive but smaller amplitude number and, with $\alpha=-1/6$ for the case of a pendulum, we have

$$A_1(120\%)\simeq\sqrt{6\times 0.576288}=1.86$$

when the nonlinear effects have increased the frequency by 20%, at least as found by analytic approximation. However, note that we are really expanding here in the quantity $\alpha A_1^2\simeq 0.576$, which is not all that small! At the same time the coefficients in the expansion are rapidly decreasing,

$$\left(1+\frac{3}{4}\alpha A_1^2+\frac{3}{128}\left(\alpha A_1^2\right)^2\right)_{\alpha A_1^2=0.576}=1+0.432+0.008.$$

The bulk of the correction is from the first order term. Note (not required) that, if we proceed as in the lecture in terms of the actual initial angle θ_0, we can solve for A_1 in the form $A_1\simeq\theta_0\left(1-\alpha/32+23\alpha^2/1024\right)$.

2) (10 pts) Consider the motion of a pendulum of length l and (velocity dependent) viscous damping described by γ. The equation of motion is

$$\ddot{\theta} + \gamma\dot{\theta} + \omega_0^2 \sin\theta = 0, \omega_0^2 = \frac{g}{l}.$$

General Comment: Since the pendulum is the basic system studied in Baker and Gollub, this exercise furthers our preparation.

a) (1 pt) Define a dimensionless time unit by $\tau = \omega_0 t$ (really an phase) and rewrite the equation of motion in terms of this new "time". What single parameter describes the behavior of the pendulum?

Solution: With the definitions $\tau = \omega_0 t$ and $\alpha = \gamma/\omega_0$ (and with $\dot{\theta} = d\theta/d\tau$ now) we have the equation of motion in the form

$$\ddot{\theta} + \alpha\dot{\theta} + \sin\theta = 0.$$

Now the behavior of this system depends on just the single parameter α, simplifying the analysis.

b) (0 pts) NOT REQUIRED Write a fourth-order Runge-Kutta script or use *Mathematica* to solve this equation of motion for arbitrary initial conditions, $\theta(0) = \theta_0, \dot{\theta}(0) = \dot{\theta}_0$.

HINT: We can always rewrite a 2nd order differential equation as two 1st order equations,

$$\ddot{y} + b\dot{y} + f(y) = 0 \Rightarrow \begin{cases} \dot{y} = z \\ \dot{z} = -bz - f(y) \end{cases},$$

and use the Runge-Kutta technique to solve for the 2-D vector $\vec{x} = (y, z)$,

$$\dot{\vec{x}} = \begin{pmatrix} \dot{y} \\ \dot{z} \end{pmatrix} = \vec{F}(t, \vec{x}) = \begin{pmatrix} z \\ -bz - f(y) \end{pmatrix}.$$

The R-K solution is then given by

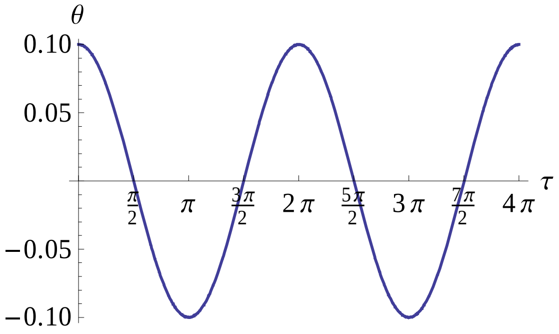
$$\begin{aligned}
&\vec{z}\left(t_0+\delta\right) \simeq \vec{z}_0 + \frac{1}{6}\left(\vec{k}_1 + 2\vec{k}_2 + 2\vec{k}_3 + \vec{k}_4\right)\left[+O\left(\delta^5\right)\right], \\
&\vec{k}_1 = \delta\,\vec{F}\left(t_0, \vec{z}_0\right), \\
&\vec{k}_2 = \delta\,\vec{F}\left(t_0 + \frac{\delta}{2}, \vec{z}_0 + \frac{\vec{k}_1}{2}\right), \\
&\vec{k}_3 = \delta\,\vec{F}\left(t_0 + \frac{\delta}{2}, \vec{z}_0 + \frac{\vec{k}_2}{2}\right), \\
&\vec{k}_4 = \delta\,\vec{F}\left(t_0 + \delta, \vec{z}_0 + \vec{k}_3\right).
\end{aligned}$$

Solution: See the code in the appended *Mathematica* notebook.

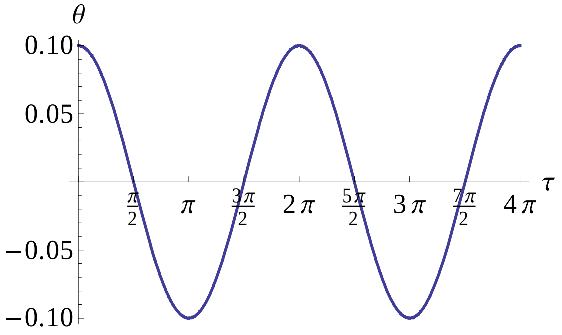
c) Ignore the friction term for the moment. Start the pendulum at $\theta_0 = 0.1$ ($\dot{\theta}(0) = 0$) and plot $\theta(\tau)$ for the range $\tau = 0$ to $\tau = 4\pi$. On the same graph plot the corresponding result for the small angle approximation, $\sin\theta \to \theta$, that we have studied earlier. Repeat this comparison for the case $\theta_0 = 1.0$. What do you learn from these 2 comparisons? What angular frequency ω_τ do you obtain in the second case?

Solution: (5 pts) Here we simplify our analysis by setting the viscous damping to zero, $\gamma = \alpha = 0$, and numerically compare the behavior of the pendulum including its nonlinear behavior (see the details in the attached *Mathematica* notebook). With amplitude 0.1 the nonlinear and linear plots are

Nonlinear

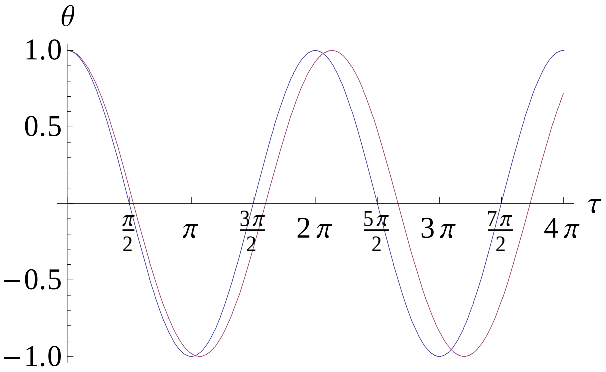


Linear



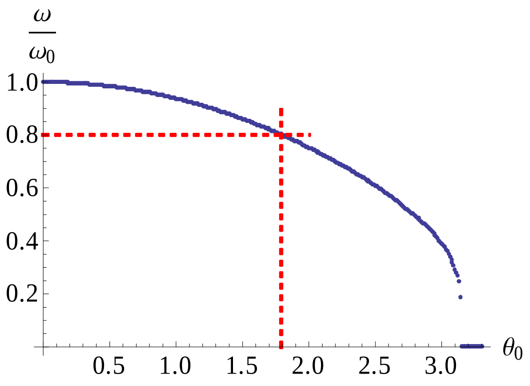
They are essentially identical!

Now consider amplitude 1.0. Now we can tell the difference, but they are still very similar. As expected the frequency in the nonlinear case is slightly smaller (the first nonlinear terms is negative – recall the previous problem). By using the fit routine in *Mathematica* we find the nonlinear frequency to be $\omega_\tau/\omega_0 = 0.938$.



d) (2 pts) Still ignoring friction, use your code to graph the angular frequency ω_τ as a function of the initial amplitude θ_0 over the range $0 \le \theta_0 < \pi$. At what value of θ_0 does ω_τ deviate by 20% from its value in the linear (small angle) problem? How does this compare to the result in the previous exercise?

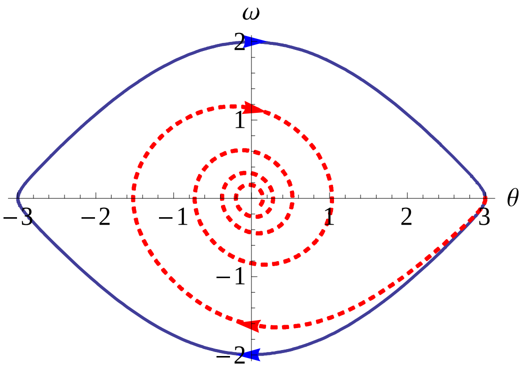
Solution: Again this plot is straightforward to generate using Mathematica, *e.g.*, using the FindFit function. The resulting plot is shown here with the 80% point highlighted at initial amplitude 1.79. Note the quite rapid turn off at the end for initial π, where the restoring force is lost. In the previous (second order) problem we found the necessary amplitude to be $A_1 = 1.71$, which is pretty close (better than 10%) to the full result here, but it would take a large number of terms in the (αA_1^2) expansion to reproduce the present numerical results (*i.e.*, the convergence is slow).



e) (2 pts) Consider the initial conditions $\theta_0 = 3.0, \dot{\theta}_0 = 0$ and, ignoring friction, graph the phase space trajectory $\left(\theta, p_\theta = \dot{\theta}\right)$ over a full cycle, indicating the direction of increasing τ on the trajectory with arrows. With the same initial conditions but now with $\alpha = \gamma/\omega_0 = 0.2$ graph the phase space trajectory for the range $0 \le \tau \le 30$.

Solution: Now we want to consider how things look in (θ, ω) phase space with and without viscous damping. Using *Mathematica* we find the following plot.

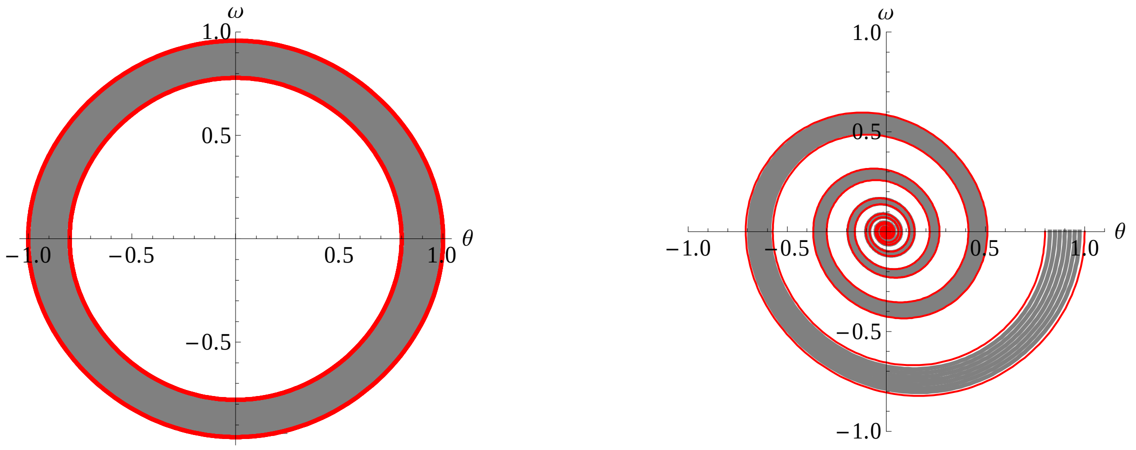
The undamped trajectory (blue, solid) reaches nearly to the hyperbolic points at $(\pm\pi,0)$ (the location of the seperatrix, the lines separating the different bases of attraction). In the damped case (red, dashed) the trajectory spirals in to the stable attractor at the origin.



3) (6 pts) Let us think about the previous problem in a bit more detail in phase space. We want to contrast the situations with and without damping.

a) (2 pts) First without friction, make a plot (sketch) of the region in phase space occupied by the trajectories corresponding to the initial conditions $\dot{\theta}(0)=0$, $0.8 \le \theta_0 \le 1.0$ (show the region as a shaded area). Now make a second plot (sketch) for the corresponding system with $\alpha = \gamma/\omega_0 = 0.2$. As usual *Mathematica* may be very useful.

<u>Solution</u>: The *Mathematica* code is available on our class web page.
The shading is performed by using trajectories at intermediate initial conditions.



b) (2 pts) Verify that the damped oscillator is dissipative. How is the feature realized in the plots of a)?

<u>Solution</u>: As we have discussed back in Lecture 7 we can think of the flow in phase space as “fluid flow-like”. Conservative, non-dissipative systems

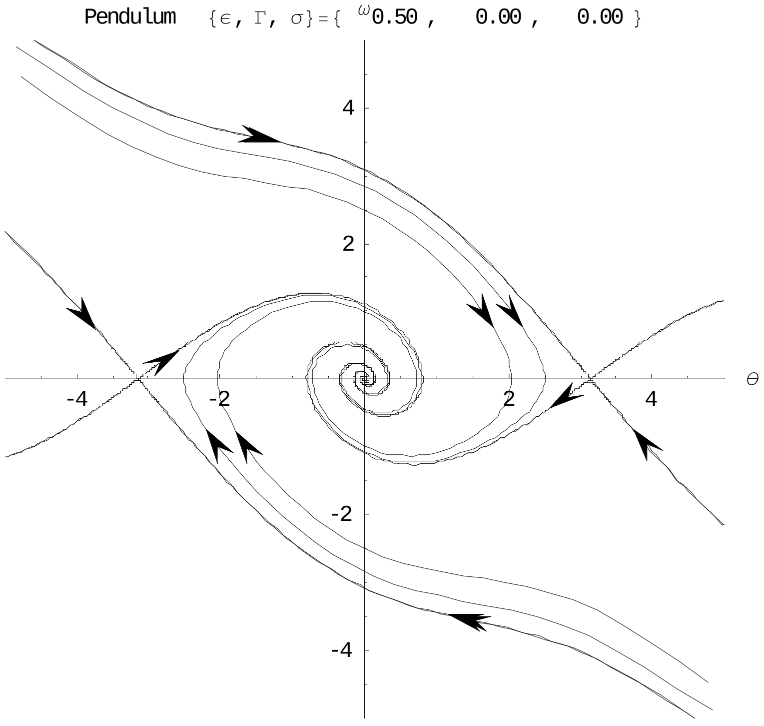
correspond to incompressible fluid flow, with zero divergence. In the present case we define the flow field in terms of the components of the "force",

$$\vec{F} = \begin{pmatrix} \omega \\ -\alpha\omega - \sin\theta \end{pmatrix},$$

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial}{\partial\theta} F_1 + \frac{\partial}{\partial\omega} F_2 = -\alpha.$$

As expected systems without damping, $\alpha = \gamma/\omega_0 = 0$, are conservative, while systems with damping, $\alpha \neq 0$, are dissipative. In the 2-D phase portrait the former system the shaded phase area between the boundary trajectories (more generally a phase volume) is conserved as we move along the trajectories (see the discussion in Lecture 7). In the non-conservative case the phase area contracts as we move along the trajectories, *i.e.*, the trajectories approach each other as we approach the attractor (sink) at the origin. (This contraction is why it is difficult to get Mathematica to do the shading for us.)

c) (2 pt) Do either of the plots in a) show evidence of an attractor? Explain (This would be a good time to reread the Appendix to Lecture 7 and start reading Baker and Gollub.) For the oscillator with dissipation, discuss the general structure of all attractors that may be present, including the issue of convergence of trajectories.



Solution: As we have and will discuss in some detail (see Baker and Gollub), the damped pendulum has an attractor at the origin to which all trajectories starting within the appropriate basin of attraction converge. There are corresponding basins of attraction centered around the periodic reflections of the attractor at $(\pm 2\pi n, 0)$. The boundaries between these basins are illustrated in the next two figures and are called separatrix. The attractors at the points $\left(\pm(2n+1)\pi, 0\right)$ are unstable in the sense that small perturbations of the trajectories headed into these points will cause them to veer away. These unstable attractors exhibit hyperbolic or saddle point structure. This same behavior is indicated in this last figure. We see trajectories spiraling into the 3 attractors $-\pi$, 0 and π. The regions of attraction are separated by the yellow lines (if you look in color).