Physics 505 - Autumn 2010

HW I Solutions

10/6/10

Overview: Recall that solving physics problems is not (just) about solving differential equations. Use physical reasoning to solve the following exercises and be certain to show your work.

1) (5 pts) Problem 1.2 in Fetter & Walecka:

Solution: This is a typical (advanced) introductory mechanics problem describing a mechanical system with several parts, each described by Newton's Laws (with no complications like friction). Let x denote the horizontal position of the spool measured with respect to the edge of the table, T denote the tension in the massless string, y denote the vertical position of the weight and θ denote the rotation angle of the spool. This is a freshman physics problem, just with more degrees of freedom (and thus more algebra). Applying Newton 3 times we have the following equations and boundary conditions

(a) $m\ddot{y} = T - mg : y(0) = 0,\ \dot{y}(0) = 0,$

(b) $I\ddot{\theta} = \frac{d}{2}T : \theta(0) = 0, \dot{\theta}(0) = 0,$

(c) $M\ddot{x} = T : x(0) = -l, \dot{x}(0) = 0.$

To proceed we need an expression for the moment of inertia of the spool, taken to be a cylinder of uniform mass density ρ and height h. In terms of the density we have

$$M = \rho V = \rho h \int_0^{d/2} r dr \oint d\theta = \rho 2\pi h \frac{d^2}{8} = \frac{\pi \rho h d^2}{4},$$

and

$$I = \rho h \int_0^{d/2} r dr\, r^2 \oint d\theta = 2\pi\rho h \frac{d^4}{64} = \frac{\pi\rho h d^4}{32} = M\frac{d^2}{8}.$$

Eliminating the tension by using (c) in (a) and (b) we have

$$\begin{aligned} m\ddot{y} &= M\ddot{x} - mg \\ \frac{Md^2}{8}\ddot{\theta} &= \frac{d}{2}M\ddot{x} \end{aligned} \Rightarrow \begin{aligned} \ddot{y} &= \frac{M}{m}\ddot{x} - g \\ d\ddot{\theta} &= 4\ddot{x} \end{aligned}.$$

To eliminate θ we apply the constraint due to the fixed length of the string

$$-(x+y) = l + \frac{d}{2}\theta.$$

The LHS is the length of the string in terms of *x* and *y* (note the sign) while the RHS exhibits the initial length plus the amount that has unwound from the spool. Taking 2 derivatives we have

$$\ddot{x} + \ddot{y} = -\frac{d}{2}\ddot{\theta} \Rightarrow \ddot{\theta} = -\frac{2}{d}(\ddot{x} + \ddot{y}),$$
$$\Rightarrow -2(\ddot{x} + \ddot{y}) = 4\ddot{x} \Rightarrow \ddot{y} = -3\ddot{x}.$$

Combining with the first equation above we have

$$g = \left(3 + \frac{M}{m}\right)\ddot{x}$$
$$\Rightarrow x(t) = \frac{g}{2\left(3 + \frac{M}{m}\right)}t^2 + \dot{x}(0)t + x(0) = \frac{g}{2\left(3 + \frac{M}{m}\right)}t^2 - l.$$

Hence the spool reaches the table's edge at time ($x(t_e) = 0$)

$$t_e = \sqrt{\frac{2l}{g}\left(3 + \frac{M}{m}\right)}.$$

The velocity of the mass is given by

$$\ddot{y} = -3\ddot{x} = -\frac{3g}{3+\frac{M}{m}} \Rightarrow \dot{y}(t) = -\frac{3g}{3+\frac{M}{m}}t + \dot{y}(0) = -\frac{3g}{3+\frac{M}{m}}t.$$

So finally the velocity of the mass as the spool reaches the edge is

$$\dot{y}(t_e) = -\frac{3g}{3+\frac{M}{m}}\sqrt{\frac{2l}{g}\left(3+\frac{M}{m}\right)} = -3\sqrt{\frac{2gl}{3+\frac{M}{m}}}.$$

2) (5 pts) Problem 1.7 in Fetter & Walecka:

Solution: In this problem we can proceed most easily using the conservation of energy.

a) Let us first consider the parameters describing the circular orbit. The magnitude of the required centrifugal force in the radial direction is $m v^2/R = m\left|\vec{\omega}\times(\vec{\omega}\times\vec{r})\right|$, which must supplied by the gravitational force $GM_E m/R^2$. Hence the kinetic energy associated with the circular orbit is

$$T_0 = \frac{mv_0^2}{2} = \frac{GM_E m}{2R}.$$

In order to get out to infinity just as the kinetic energy goes to zero we need enough kinetic energy to overcome the potential barrier, $U_0 = -GM_E m/R = -2T_0$, *i.e.*,

$$T_1 + U_0 = 0 \Rightarrow T_1 = 2T_0 = \frac{GM_E m}{R}.$$

Thus the impulse must be such as to double the kinetic energy. With the impulse tangential and in the same direction as the instantaneous velocity we have

$$v_1^2 = (v_0 + \Delta v)^2 = 2v_0^2 \Rightarrow \Delta v = \frac{1}{2}\left(-2v_0 \pm \sqrt{4v_0^2 + 4v_0^2}\right)$$
$$= -v_0 \pm \sqrt{2}v_0 = \left(\pm\sqrt{2} - 1\right)v_0.$$

We see that there are 2 solutions. The negative solution is clearly where we drive the rocket off to infinity by providing an impulse in the direction opposite to the initial motion (the dumb solution). We will focus on the other solution, which says the necessary tangential impulse is given by

$$\int F dt = m\Delta v = mv_0\left(\sqrt{2} - 1\right) = m\left(\sqrt{2} - 1\right)\sqrt{\frac{GM_E}{R}}.$$

b) We know from the general properties of trajectories in the inverse square law problem that a trajectory with zero total energy is a parabola. In this case we are interested in just one half of the parabola from the point of closest approach out to infinity. The associated angular momentum can be calculated at the initial point,

$$L = L_z = l = mv_1 R = m\sqrt{2}v_0 R = m\sqrt{2GM_E R}.$$

From Eq. (3.16b) in F&W we have

$$E = 0 \Rightarrow \varepsilon = 1, C = \frac{m^2 M_E G}{l^2} = \frac{m^2 G M_E}{2GM_E m^2 R} = \frac{1}{2R}.$$

Thus the trajectory is described by

$$\frac{1}{\rho} = \frac{1}{2R}\left(1 - \cos\phi\right).$$

c) Now we consider a rocket at rest at the same initial point and wish to provide an impulse in the radial direction that will send it to infinity. To achieve escape velocity we still need the same velocity as above after the force acts so that

$$m\Delta v = mv_1 = m\sqrt{\frac{2GM_E}{R}}.$$

Thus the ratio of the impulses in the two cases is

$$\frac{m\Delta v|_{\text{radial}}}{m\Delta v|_{\text{tangential}}} = \frac{\sqrt{2}}{\sqrt{2}\mp 1},$$

where the – sign is the "smart" tangential case and the + sign is the "dumb" tangential case.

3) (5 pts) Problem 1.12 in Fetter & Walecka: We want to characterize the precession of the elliptic orbit of Mercury (ε = 0.206, τ = 0.241 yr) in terms of the perturbed solar gravitational potential

$$V(r) = -\frac{mMG}{r}\left(1+\alpha\frac{GM}{rc^2}\right): \quad \frac{GM}{c^2} \approx 1.475 \text{ km.}$$

Solution: The corresponding effective potential is then

$$V_{eff}(r) = -\frac{mMG}{r}\left(1+\alpha\frac{GM}{rc^2}\right) + \frac{l^2}{2mr^2}.$$

The equation of the orbit (see Eq. (3.13) in F&W) is

$$\phi = \pm\frac{l}{\sqrt{2m}}\int^r \frac{dr}{r^2}\frac{1}{\sqrt{E + \frac{mMG}{r} - \frac{1}{2mr^2}\left(l^2 - 2\alpha\left(\frac{GMm}{c}\right)^2\right)}} + \phi_0.$$

As usual we can simplify this integral with the change of variable $u = 1/r$, $du = -dr/r^2$,

$$\phi = \mp\int^{u} du \frac{1}{\sqrt{\frac{2mE}{l^2}+\frac{2m^2MG}{l^2}u-u^2\left(1-2\alpha\left(\frac{GMm}{lc}\right)^2\right)}}+\phi_0$$

$$=\mp\frac{1}{\sqrt{1-2\alpha\left(\frac{\gamma m}{lc}\right)^2}}\int^{u} du \frac{1}{\sqrt{\frac{2mE'}{l^2}+\frac{2m^2\gamma'}{l^2}u-u^2}}+\phi_0,$$

where we have defined

$$E' = \frac{E}{1-2\alpha\left(\frac{\gamma m}{lc}\right)^2}, \gamma = GM, \gamma' = \frac{\gamma}{1-2\alpha\left(\frac{\gamma m}{lc}\right)^2}.$$

We recognize the integral as the arccosine so that

$$\phi-\phi_0 = \pm\frac{1}{\sqrt{1-2\alpha\left(\frac{\gamma m}{lc}\right)^2}}\cos^{-1}\left(\frac{1-\frac{ul^2}{m^2\gamma'}}{\sqrt{1+\frac{2E'l^2}{m^3\gamma'^2}}}\right)$$

$$\Rightarrow\sqrt{1+\frac{2E'l^2}{m^3\gamma'^2}}\cos\left[\sqrt{1-2\alpha\left(\frac{\gamma m}{lc}\right)^2}\,(\phi-\phi_0)\right]=1\text{-}\frac{1}{r}\frac{l^2}{m^2\gamma'}$$

$$\Rightarrow\frac{1}{r}=\frac{m^2\gamma'}{l^2}\left\{1-\sqrt{1+\frac{2E'l^2}{m^3\gamma'^2}}\cos\left[\sqrt{1-2\alpha\left(\frac{\gamma m}{lc}\right)^2}\,(\phi-\phi_0)\right]\right\}.$$

From this last result we see that the radius of the orbit reaches its maximum value (the end of the ellipse furthest from the origin) when the RHS of this equation is minimum, *i.e.*, when the argument of the cosine is an even multiple of π. Thus we can determine the precession per orbit by calculating how much ϕ - ϕ_0 advances per orbit,

$$0: \sqrt{1-2\alpha\left(\frac{\gamma m}{lc}\right)^2}\left(\phi-\phi_0\right)=0$$

$$2\pi: \sqrt{1-2\alpha\left(\frac{\gamma m}{lc}\right)^2}\left(\phi-\phi_0\right)=2\pi$$

$$\Rightarrow\left(\phi-\phi_0\right)=\frac{2\pi}{\sqrt{1-2\alpha\left(\frac{\gamma m}{lc}\right)^2}}\approx 2\pi\left(1+\alpha\left(\frac{\gamma m}{lc}\right)^2\right): \alpha\left(\frac{\gamma m}{lc}\right)^2 \ll 1$$

$$\Rightarrow \delta\left(\phi-\phi_0\right)_{\text{per orbit}}=\left(\phi-\phi_0\right)_{1\text{ orbit}}-2\pi=2\pi\alpha\left(\frac{\gamma m}{lc}\right)^2.$$

We want to compare this result to the fact that the perihelion of Mercury is observed to advance 43 seconds of arc per 100 years. We have for the advance in 100 years

$$2\pi\left(\text{radians}\right)\alpha\left(\frac{\gamma m}{lc}\right)^2_{\text{per orbit}}\left(\frac{1\text{ orbit}}{0.241\text{ yr}}\right)\left(100\text{ yr}\right)\left(\frac{360\cdot 60\cdot 60\text{ arc sec}}{2\pi\text{ radians}}\right)$$

$$=\alpha\left(\frac{\gamma m}{lc}\right)^2\cdot 5.4\times 10^8\text{ arc sec}=43\text{ arc sec}$$

$$\Rightarrow\alpha\left(\frac{\gamma m}{lc}\right)^2=8.0\times 10^{-8}.$$

To obtain the value of the parameter α we need a value for the combination of constants in the parentheses. Since the term we are trying to evaluate is already first order in the small perturbation (labeled by a single power of α), we can evaluate the rest of the factors to zeroth order in the perturbation. Hence we can use the unperturbed expression for elliptic orbits in a $1/r$ potential, *i.e.*, ignoring the perturbation. This is the standard approach for simplifying an analysis in terms of a *small* perturbation, *i.e.*, we keep only the leading terms and ignore the rest. We have for the eccentricity and period,

$$\varepsilon = \sqrt{1+\frac{2El^2}{m^3\gamma^2}}, \tau = 2\pi\sqrt{\frac{a^3}{\gamma}} = 2\pi\sqrt{\frac{m^3\gamma^3}{8|E|^3\gamma}} = \pi\gamma\sqrt{\frac{m^3}{2|E|^3}}$$

$$\Rightarrow \frac{l^2c^2}{m^2\gamma^2} = \frac{\varepsilon^2-1}{2E}mc^2 = \frac{1-\varepsilon^2}{2}mc^2\left(\frac{\sqrt{2}\tau}{\pi\gamma}\right)^{2/3}\frac{1}{m} = \left(1-\varepsilon^2\right)\left(\frac{c^3\tau}{2\pi\gamma}\right)^{2/3}$$

$$= \left(1-\varepsilon^2\right)\left(\frac{c\tau}{2\pi\left(\gamma/c^2\right)}\right)^{2/3} = \left(1-\left(0.206\right)^2\right)\left(\frac{0.241\cdot 9.5\times 10^{12}\text{ km}}{2\pi\left(1.475\text{ km}\right)}\right)^{2/3}$$

$$\simeq 3.7\times 10^7.$$

Thus we have

$$\alpha \simeq \left(3.7\times 10^7\right)\times\left(8.0\times 10^{-8}\right) \simeq 3.0.$$

4) (5 pts) Problem 1.13 in Fetter & Walecka: We want to consider a rocket approaching a planet and consider the conditions under which it does or does not collide with the planet (under the influence of gravity).

Solution: Given the initial conditions we can calculate the kinetic energy and the angular momentum (m_R is the rocket mass),

$$T_0 = \frac{m_R v_\infty^2}{2} \Rightarrow E = T_0\ \left[U_\infty = 0\right],$$
$$L_z = l = m_R v_\infty b,$$

From the discussion in F&W and in class we know that the trajectory of the rocket will be a hyperbola described by

$$\varepsilon = \sqrt{1+\frac{2El^2}{mk^2}} = \sqrt{1+\frac{m_R v_\infty^2 \left(m_R v_\infty b\right)^2}{m_R \left(GMm_R\right)^2}} = \sqrt{1+\frac{v_\infty^4 b^2}{\left(GM\right)^2}} = \sqrt{1+\frac{v_\infty^4 b^2}{\gamma^2}},$$

$$C = \frac{mk}{l^2} = \frac{GMm_R^2}{\left(m_R v_\infty b\right)^2} = \frac{GM}{\left(v_\infty b\right)^2} = \frac{\gamma}{\left(v_\infty b\right)^2},$$

$$\frac{1}{\rho} = \frac{1}{r} = C\left(1-\varepsilon\cos\left(\phi\right)\right).$$

Note that, as implied by the problem description, the results do not depend on the mass of the rocket (m_R), as long as we ignore the perturbation in the planet's position due to the rocket, *i.e.*, treat this as a rocket in a *fixed* gravitational well. This approximation is OK as long as $m_R \ll M$. If this is not true, we should replace γ by $\bar{\gamma}$ in these expressions, *i.e.*, $M \to M + m_R$ in γ.

Clearly the closest approach of the rocket to the planet occurs for $\phi = \pi$ yielding

$$\rho_- = r_{\min} = \frac{1}{\left(1+\varepsilon\right)C} = \frac{v_\infty^2 b^2}{\left(1+\sqrt{1+\frac{v_\infty^4 b^2}{\gamma^2}}\right)\gamma} = b\frac{\sqrt{\varepsilon^2-1}}{1+\varepsilon} = b\sqrt{\frac{\varepsilon-1}{\varepsilon+1}}$$

$$= b\frac{v_\infty^2 b}{\left(\gamma+\sqrt{\gamma^2+\left(v_\infty^2 b\right)^2}\right)} \le b.$$

So the condition to strike the planet is

$$\rho_- = r_{\min} = b\sqrt{\frac{\varepsilon-1}{\varepsilon+1}} < R_0$$

$$\Rightarrow \frac{v_\infty^2 b^2}{\left(\gamma+\sqrt{\gamma^2+\left(v_\infty^2 b\right)^2}\right)} < R_0.$$

If we think of this result as a constraint on the initial conditions, b and v_∞, there is a range of values that can lead to a collision. For $b \le R_0$, any (finite) initial velocity leads to a collision. For $b > R_0$ we must require that $v_\infty^2 < 2R_0\gamma \big/ \left(b^2 - R_0^2\right)$ in order to hit the planet. Alternatively, for a given v_∞, rockets with impact parameters in the range $b^2 < R_0^2 + 2R_0\gamma \big/ v_\infty^2 = R_0^2\left(1 + \left(m_R\gamma/R_0\right)/E\right)$ will collide with the planet. For $r_{\min} = R_0^+$ (*i.e.*, just larger than R_0) the rocket just misses the planet and we can determine the angle of deflection from the equation for the trajectory. Actually as suggested above there is a family of trajectories (*i.e.*, choices of b and v_∞) that just miss the planet. For a given value of b larger than R_0, the velocity squared should be just larger than $2R_0\gamma \big/ \left(b^2 - R_0^2\right)$ in order to just miss the planet.

The equation for the angle of deflection as $\rho \to \infty$ is

$$\left(1 - \varepsilon\cos\phi\right)\Big|_{\rho\to\infty} = 0 \Rightarrow \cos\phi_\infty = \frac{1}{\varepsilon}.$$

Hence the angle of deflection for a trajectory defined by the above initial conditions is given by

$$\theta = \pi - 2\cos^{-1}\left(\frac{1}{\varepsilon}\right) = \pi - 2\cos^{-1}\left(\frac{1}{\sqrt{1 + \frac{v_\infty^4 b^2}{\gamma^2}}}\right)$$

$$\Rightarrow \cot\left(\frac{\theta}{2}\right) = \frac{\cos\left(\frac{\pi}{2} - \cos^{-1}\left(\frac{1}{\varepsilon}\right)\right)}{\sin\left(\frac{\pi}{2} - \cos^{-1}\left(\frac{1}{\varepsilon}\right)\right)} = \frac{\sin\left(\cos^{-1}\left(\frac{1}{\varepsilon}\right)\right)}{\cos\left(\cos^{-1}\left(\frac{1}{\varepsilon}\right)\right)} = \frac{\sqrt{1 - \frac{1}{\varepsilon^2}}}{\frac{1}{\varepsilon}}$$

$$= \sqrt{\varepsilon^2 - 1} = \frac{bv_\infty^2}{\gamma},$$

$$\theta = 2\cot^{-1}\left(\frac{bv_\infty^2}{\gamma}\right) = 2\cot^{-1}\left(\frac{bv_\infty^2}{GM}\right) = 2\cot^{-1}\left(\frac{2bR_0}{b^2 - R_0^2}\right),$$

valid for any $b > R_0$. For the case of b just larger than R_0 and very large energy the rocket just misses the planet and experiences a very small angle of deflection,

$\theta_{b\to R_0^+} \approx \left(b^2 - R_0^2\right)\big/ R_0^2 \approx 2\left(b - R_0\right)\big/ R_0$. For very large impact parameter and low energy, the rocket just misses the planet but experiences an angle of deflection of π.

5) (5 pts) No summary/review of undergraduate classical mechanics is complete without an inclined plane problem. So consider an inclined plane of mass M that rests on a rough floor with coefficient of static friction μ (the horizontal frictional force between the block with the inclined plane and the floor is less than or equal to μ times the normal force between the inclined plane and the floor). A mass m_1 is suspended by a (massless) string, which passes over a smooth (frictionless) peg at the upper end of the incline and attaches to a mass m_2. The second mass slides on the inclined plane without friction and the incline makes an angle θ with the horizontal. The local acceleration due to gravity is g.

a) Solve for the accelerations of m_1 and m_2 and the tension in the string if μ is very large.

b) Find the smallest coefficient of friction μ for which the inclined plane remains at rest.

Solution: a) In the case that μ is large enough to ensure that the inclined plane remains at rest (*i.e.*, in the very large μ limit), we consider free-body diagrams for the two masses labeling the tension in the string as T. For mass 1 there are only vertical forces while for mass 2 the only interesting motion and forces are parallel to the inclined plane defined by angle θ,

$$\begin{aligned} m_1 a &= m_1 g - T, \\ m_2 a &= T - m_2 g \sin\theta, \end{aligned}$$

where we are assuming (as usual) that the string does not break, $a_1 = a_2 = a$. In these equations $a > 0$ corresponds to m_1 moving down vertically and m_2 moving up the inclined plane (*i.e.*, $m_1 > m_2 \sin\theta$). These equations can be solved to find

$$\begin{aligned} a &= \frac{\left(m_1 - m_2 \sin\theta\right) g}{m_1 + m_2}, \\ T &= \frac{m_1 m_2 \left(1 + \sin\theta\right) g}{m_1 + m_2}. \end{aligned}$$

b) If we now allow the possibility that the inclined plane can slide, we must include a free body diagram for the inclined plane also, including the normal force exchanged with mass 2 and the floor and the forces transferred from the string to the peg. The normal force (normal to the inclined plane) due to mass 2 on the inclined plane is

$$N_2 = m_2 g \cos\theta,$$

which resolves into vertical (down) and horizontal (towards the tall end of the inclined plane) forces

$$N_{2,V} = m_2 g \cos^2\theta,$$
$$N_{2,H} = m_2 g \cos\theta \sin\theta.$$

The force of the string on the peg resolves as (the vertical component is down while here the horizontal component is towards the low end of the inclined plane, *i.e.*, opposite to $N_{2,H}$)

$$P_V = T\left(1+\sin\theta\right) = \frac{m_1 m_2 \left(1+\sin\theta\right)^2 g}{m_1+m_2},$$
$$P_H = T\cos\theta = \frac{m_1 m_2 \left(1+\sin\theta\right)\cos\theta g}{m_1+m_2}.$$

Thus the normal force (up) from the floor to the inclined plane must be

$$N = Mg + m_2 g \cos^2\theta + \frac{m_1 m_2 \left(1+\sin\theta\right)^2 g}{m_1+m_2}.$$

On the other hand the frictional force f, must balance a horizontal force given by

$$f = \frac{m_1 m_2 \left(1+\sin\theta\right)\cos\theta g}{m_1+m_2} - m_2 g \cos\theta \sin\theta.$$

Here a positive value means the frictional force supplied by the floor on the block points towards the tall end of the inclined plane. A negative value for this quantity implies that the required frictional force points in the opposite direction. In either case the magnitude of the frictional force is bounded by μN, $|f| \le \mu N$, so that

$$\mu \ge \mu_{\min} = \frac{|f|}{N} = \frac{m_2 \cos\theta \left| m_1 - m_2 \sin\theta \right|}{M\left(m_1 + m_2\right) + m_1 m_2 \left(1 + \sin\theta\right)^2 + \left(m_1 + m_2\right) m_2 \cos^2\theta}.$$