Physics 505 - Autumn 2010

HW VIII Solutions

11/24/10→11/29/10 (Snow)

Overview: Recall that solving physics problems is not (just) about solving differential equations. Use physical reasoning to help solve the following exercises and be certain to show your work. It is also important that you practice completely solving these exercises, checking for errors as you go along.

1) Fetter & Walecka – 6.6 (7 pts) Here we apply Hamiltonian techniques to the familiar problem of the harmonic oscillator.

<u>Solution</u>: Here we want to start by considering a harmonic oscillator defined by

$$L = \frac{m}{2}\dot{q}^2 - \frac{k}{2}q^2 \Rightarrow p = \frac{\partial L}{\partial \dot{q}} = m\dot{q}$$

$$\Rightarrow H = p\dot{q} - L = \frac{m}{2}\dot{q}^2 + \frac{k}{2}q^2$$

$$= \frac{p^2}{2m} + \frac{k}{2}q^2 = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right) : \left[\omega^2 = \frac{k}{m}\right].$$

Next we consider that we have transformed to a new set of canonical variables defined by transformations

$$\left.\begin{aligned} Q &= C\left(p + im\omega q\right) \\ P &= C\left(p - im\omega q\right) \end{aligned}\right\} \Leftrightarrow \begin{cases} q = \dfrac{Q-P}{2iCm\omega} \\ p = \dfrac{Q+P}{2C} \end{cases}.$$

We want to find the generating function for this transformation, $S\left(q,P,t\right)$, which must satisfy

$$
\begin{aligned}
p &= \frac{\partial S}{\partial q}, \\
Q &= \frac{\partial S}{\partial P} = C\left(p + im\omega q\right) = \frac{P+Q}{2} + iCm\omega q \\
&= \frac{1}{2}\frac{\partial S}{\partial P} + \frac{P}{2} + iCm\omega q \\
\Rightarrow \frac{\partial S}{\partial P} &= P + 2iCm\omega q \\
\Rightarrow S\left(q,P,t\right) &= \frac{P^2}{2} + 2iCm\omega qP + f\left(q\right).
\end{aligned}
$$

Going back to the first equation and using the transformation we have

$$
\begin{aligned}
p &= \frac{Q+P}{2C} = \frac{1}{2C}\left(P + C\left(p + im\omega q\right)\right) \\
\Rightarrow\ p &= \frac{P}{C} + im\omega q = \frac{\partial S}{\partial q} = 2iCm\omega P + \frac{\partial f\left(q\right)}{\partial q}.
\end{aligned}
$$

Since $\left(P,q\right)$ are independent variables, their coefficients in this equation must match and we have

$$
\begin{aligned}
&1 = 2iC^2 m\omega \Rightarrow C = \frac{1}{\sqrt{2im\omega}} = \frac{e^{-i\pi/4}}{\sqrt{2m\omega}}, \\
&f\left(q\right) = im\omega\frac{q^2}{2}\left(+\text{constant}\right).
\end{aligned}
$$

As always we can ignore the constant and write

$$
S\left(q,P\right) = \frac{P^2}{2} + 2iCm\omega qP + im\omega\frac{q^2}{2}.
$$

With the factor *C* defined above we can rewrite the transformations as

$$\left.\begin{aligned} Q &= \frac{1}{\sqrt{2im\omega}}\left(p + im\omega q\right) \\ P &= \frac{1}{\sqrt{2im\omega}}\left(p - im\omega q\right) \end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned} q &= \frac{Q-P}{\sqrt{2im\omega}} \\ p &= \sqrt{\frac{im\omega}{2}}\left(Q+P\right) \end{aligned}\right. ,$$

and the Hamiltonian as

$$\begin{aligned} H\left(q,p\right) &= \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right) = \frac{im\omega\left(Q+P\right)^2}{4m} + \frac{m^2\omega^2\left(Q-P\right)^2}{4im^2\omega} \\ &= \frac{i\omega}{4}\left[\left(Q+P\right)^2 - \left(Q-P\right)^2\right] = i\omega QP = \tilde{H}\left(Q,P\right). \end{aligned}$$

Thus in the new coordinates the equations of motion are decoupled, which is the point of this change of coordinates, and we find simply that

$$\left.\begin{aligned} \dot{Q} &= \frac{\partial \tilde{H}}{\partial P} = i\omega Q \\ \dot{P} &= -\frac{\partial \tilde{H}}{\partial Q} = -i\omega P \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} Q &= \frac{A}{\sqrt{2im\omega}} e^{i\omega t} \\ P &= \frac{A^*}{\sqrt{2im\omega}} e^{-i\omega t} \end{aligned}\right. .$$

The choice of the coefficients was made to match the form of the transformation, which says that $\sqrt{2im\omega}\,Q = \left(\sqrt{2im\omega}P\right)^*$, *i.e.*, assuming that initial coordinates $\left(q,p\right)$ are real. This feature will also arise when we match the initial conditions, which are presumably real numbers. Now we can transform back to the original coordinates

$$\begin{aligned} q &= \frac{Q-P}{\sqrt{2im\omega}} = \frac{1}{2im\omega}\left(Ae^{i\omega t} - A^* e^{-i\omega t}\right) = \frac{1}{m\omega}\mathrm{Im}\left(Ae^{i\omega t}\right) \\ &= \frac{|A|}{m\omega}\sin\left(\omega t + \varphi\right) : \left[A = |A| e^{i\varphi}\right], \\ p &= \sqrt{\frac{im\omega}{2}}\left(Q+P\right) = \frac{1}{2}\left(Ae^{i\omega t} + A^* e^{-i\omega t}\right) = \mathrm{Re}\left(Ae^{i\omega t}\right) \\ &= |A|\cos\left(\omega t + \varphi\right). \end{aligned}$$

The initial conditions on (q, p) will determine the complex parameter A, $(|A|, \varphi)$. As noted in F&W the complex coordinates (Q, P) act like the creation and annihilation operators of QM. In particular, the form of the transformed Hamiltonian is $\tilde{H}(Q, P) = i\omega QP \sim a^\dagger a$, which just counts the number of excitations.

2) Fetter & Walecka – 6.9 (4 pts) Here we get to practice with Hamilton-Jacobi equation.

Solution: We start with a system described by generalized coordinates (q_1, q_2) and kinetic and potential energies given by

$$T = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right)\left(q_1^2 + q_2^2\right), V = \frac{1}{\left(q_1^2 + q_2^2\right)}.$$

As usual we proceed to find the Lagrangian, the canonical momenta and the Hamiltonian,

$$\begin{aligned}
L &= \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right)\left(q_1^2 + q_2^2\right) - \frac{1}{\left(q_1^2 + q_2^2\right)} \\
\Rightarrow p_1 &= \frac{\partial L}{\partial \dot{q}_1} = \dot{q}_1\left(q_1^2 + q_2^2\right),\ p_2 = \frac{\partial L}{\partial \dot{q}_2} = \dot{q}_2\left(q_1^2 + q_2^2\right) \\
\Rightarrow H &= \left(\dot{q}_1^2 + \dot{q}_2^2\right)\left(q_1^2 + q_2^2\right) - L = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right)\left(q_1^2 + q_2^2\right) + \frac{1}{\left(q_1^2 + q_2^2\right)} \\
&= \frac{1}{2\left(q_1^2 + q_2^2\right)}\left(p_1^2 + p_2^2 + 2\right) = H\left(q_1, q_2, p_1, p_2\right).
\end{aligned}$$

The Hamiltonian-Jacobi equations tells us that

$$H\left(q_1,q_2,\frac{\partial S}{\partial q_1},\frac{\partial S}{\partial q_1}\right)+\frac{\partial S}{\partial t}=0$$

$$\Rightarrow \frac{1}{2\left(q_1^2+q_2^2\right)}\left(\left(\frac{\partial S}{\partial q_1}\right)^2+\left(\frac{\partial S}{\partial q_2}\right)^2+2\right)+\frac{\partial S}{\partial t}=0.$$

Since H is time independent we write $S\left(q_1,q_2,\alpha_1,\alpha_2,t\right)=W\left(q_1,q_2,E,\alpha_2\right)-Et$ and thus

$$\left(\frac{\partial W}{\partial q_1}\right)^2+\left(\frac{\partial W}{\partial q_2}\right)^2+2=2\left(q_1^2+q_2^2\right)E.$$

So we try simplifying by separating variables again, $W\left(q_1,q_2\right)=W_1\left(q_1\right)+W_2\left(q_2\right)$, and write for Hamilton's principal function

$$\left(\frac{dW_1}{dq_1}\right)^2+\left(\frac{dW_2}{dq_2}\right)^2+2=2\left(q_1^2+q_2^2\right)E$$

$$\Rightarrow \left.\begin{matrix}\left(\frac{dW_1}{dq_1}\right)^2+1-2Eq_1^2=\alpha_2\\ \left(\frac{dW_2}{dq_2}\right)^2+1-2Eq_2^2=-\alpha_2\end{matrix}\right\}\Rightarrow\left\{\begin{matrix}\frac{dW_1}{dq_1}=\pm\sqrt{2Eq_1^2+\alpha_2-1}\\ \frac{dW_2}{dq_2}=\pm\sqrt{2Eq_2^2-\alpha_2-1}\end{matrix}\right.$$

$$\Rightarrow\begin{cases}W_1=\pm\int^{q_1}dq_1\sqrt{2Eq_1^2+\alpha_2-1}\\ W_2=\pm\int^{q_2}dq_2\sqrt{2Eq_2^2-\alpha_2-1}\end{cases}$$

$$\Rightarrow S\left(q_1,q_2,E,\alpha_2,t\right)=\pm\int^{q_1}dq_1\sqrt{2Eq_1^2+\alpha_2-1}\pm\int^{q_2}dq_2\sqrt{2Eq_2^2-\alpha_2-1}-Et.$$

The other 2 constants of the motion are defined by

$$\beta_1 = \text{constant} = \frac{\partial S}{\partial E} = \pm\int^{q_1} dq_1 \frac{q_1^2}{\sqrt{2Eq_1^2 + \alpha_2 - 1}} \pm \int^{q_2} dq_2 \frac{q_2^2}{\sqrt{2Eq_2^2 - \alpha_2 - 1}} - t$$

$$\Rightarrow \beta_1 + t = \pm\int^{q_1} dq_1 \frac{q_1^2}{\sqrt{2Eq_1^2 + \alpha_2 - 1}} \pm \int^{q_2} dq_2 \frac{q_2^2}{\sqrt{2Eq_2^2 - \alpha_2 - 1}},$$

$$\beta_2 = \text{constant} = \frac{\partial S}{\partial \alpha_2} = \pm\int^{q_1} dq_1 \frac{1}{\sqrt{2Eq_1^2 + \alpha_2 - 1}} \mp \int^{q_2} dq_2 \frac{1}{\sqrt{2Eq_2^2 - \alpha_2 - 1}}.$$

This integrals can be performed (if messily) and the resulting expressions inverted to find (q_1, q_2) in terms of the constants $(E, \alpha_2, \beta_1, \beta_2)$ and *t*. The constants can then be evaluated in terms of the initial conditions, $(q_1(0), \dot{q}_1(0), q_2(0), \dot{q}_2(0))$.

3) (6 pts) We want to try using the numerical methods outlined in the attached notes to solve first order differential equations. In particular, we recall the ladder problem from exercise 3 in HW IV (F&W 3.18). In the latter part of that problem we obtained a fairly complicated expression for the time derivative of the angle of the ladder versus time after the ladder lost contact with the wall (at $\theta = \theta_{off}$), where the initial angle is $\theta = \theta_0$,

$$\frac{d\theta}{dt} = -\sqrt{\frac{\frac{12g}{L}\left[\frac{2E}{MgL} - \sin\theta\right]}{1 + 3\cos^2\theta}},$$

$$E = \frac{gML}{2}\sin\theta_0 - \frac{gML}{18}\sin^3\theta_0.$$

Now we want to solve this equation using numerical methods.

<u>Solution:</u> In this write-up we will use *Mathematica* to solve this problem. See the attached *Mma* notebook (which had a typo initially).

a) (5 pts) Consider a ladder of length *L* = 3 meters and take the acceleration of gravity to be *g* = 9.8 m/s^2. Assume that the initial angle of the ladder with respect to the floor is $\theta_0 = 60°$ (before the ladder starts to fall). Using numerical methods find the orientation angle θ as a function of time. If you use NDSolve in

Mathematica, I find it best to solve the original second order differential equation. If you use the Runge-Kutta method outlined in the notes, you should focus on the first order equation we found in our original analysis using the conservation of energy. Using your numerical solution for the angle (and our previous analytic analysis of the constraint forces) determine the time and the angle, $t = t_{off}, \theta = \theta_{off}$, when the ladder loses contact with the wall, *i.e.*, when the constraint force due to the wall vanishes.

b) (5 pts) The subsequent motion is described slightly differently (as there is no longer a force from the wall and the upper end of the ladder no longer touches the wall). Numerically determine the subsequent motion, *i.e.*, find $\theta(t)$ using $\theta = \theta_{off}$ (and possibly $d\theta/dt$ at t_{off}) as the relevant initial condition, until the time when the ladder hits the floor, $\theta = 0$. To fully specify the subsequent motion you need to also specific the motion of the CM of the ladder. Also find the time variation of the constraint force of the floor, (recall)

$$\lambda_f = \frac{Mg}{\left(4 - 3\sin^2\theta\right)^2}\left[4 + 3\sin^2\theta - 12\frac{E}{MgL}\sin\theta\right],$$

and verify that it never goes to zero.

To perform the numerical analysis you can use either *Mathematica* or the (4^{th} order) Runge-Kutta method as outlined in the attached notes. Your results can be presented either as a printed copy of your *Mathematica* notebook or as a table of $\theta(t)$ on an evenly spaced grid of time values for $\theta_0 \geq \theta \geq 0$ (some trial & error may be required to find an appropriate time step size and number of steps). Make plots (sketches) of both $\theta(t)$ and $\lambda_f(t)$ versus *t*. You are strongly encouraged to use *Mathematica* to perform this analysis, which will also happily do the plots for you. Finally you are encouraged to make an animation of the motion of the ladder like the one that appears on our web page.