

Physics 505 - Autumn 2010

HW VII Solutions

11/17/10

Overview: Recall that solving physics problems is not (just) about solving differential equations. Use physical reasoning to help solve the following exercises and be certain to show your work. It is also important that you practice completely solving these exercises, checking for errors as you go along.

1) Fetter & Walecka – 5.3 (7 pts) We want to practice analyzing rigid body motion.

Solution: This system can be easily reproduced with a coin on a smooth table (except for the complete absence of rolling friction). We set up the rectilinear inertial frame, \hat{e}_k^0 , and other variables as suggested in figure P5.3 in F&W. There are several ways to proceed, depending on how much of the Euler angle analysis we want to assume. We note that the body-associated frame in this problem is just the \hat{g} frame from the Euler analysis in Lecture 10. Thus the formulae in that analysis are applicable here if we set $\gamma = 0$ (with $\beta \rightarrow \theta, \alpha \rightarrow \phi$). Probably the simplest way to proceed is to directly use Newton's equations in rectangular coordinates. We label the reaction force of the table in the inertial frame, \vec{f}^0 . We have (from Newton) the rectilinear motion of the CM of the coin given by

$$m\ddot{x}_1^0 = f_1^0, m\ddot{x}_2^0 = f_2^0, m\ddot{x}_3^0 = f_3^0 - Mg,$$

and, from the definitions and trigonometry, we have (note that $\phi = 0$ is when the coin touches the 2^0 axis, *i.e.*, $\pi/2$ from the usual definition of the azimuthal angle)

$$\begin{aligned}x_1^0 &= (b - a \sin \theta) \sin \phi, \\x_2^0 &= (b - a \sin \theta) \cos \phi, \\x_3^0 &= a \cos \theta, \\ \theta &= \text{constant}, \dot{\phi} = \text{constant}.\end{aligned}$$

We also know that for a flat disk the components of the moment of inertia are $I_1 = I_2 = Ma^2/4$ and $I_3 = Ma^2/2$. The non-slipping constraint means $\dot{\gamma} = (b/a)\dot{\phi}$. By simply taking derivatives we find

$$\begin{aligned}\dot{x}_1^0 &= (b - a \sin \theta) \cos \phi \dot{\phi}, & \ddot{x}_1^0 &= -(b - a \sin \theta) \sin \phi \dot{\phi}^2 \\ \dot{x}_2^0 &= -(b - a \sin \theta) \sin \phi \dot{\phi}, & \ddot{x}_2^0 &= -(b - a \sin \theta) \cos \phi \dot{\phi}^2 \\ \dot{x}_3^0 &= \ddot{x}_3^0 = 0,\end{aligned}$$

and

$$\begin{aligned}f_1^0 &= -M(b - a \sin \theta) \sin \phi \dot{\phi}^2, \\ f_2^0 &= -M(b - a \sin \theta) \cos \phi \dot{\phi}^2, \\ f_3^0 &= Mg.\end{aligned}$$

The last component is the expected normal force. The first two components are just the horizontal force in the (inward) radial direction,

$f_\rho^0 = f_1^0 \sin \phi + f_2^0 \cos \phi = -M(b - a \sin \theta) \dot{\phi}^2$, *i.e.*, the centripetal force required to make the coin roll along a circle. The orthogonal component, f_ϕ^0 , which would accelerate the rotation, vanishes.

An alternative, and more powerful, technique is to write the general Lagrangian for this system, with 6 coordinates (3 for the location of the CM and 3 angles for the orientation of the coin) and 5 constraints arising from rolling on a circle (2), touching the horizontal surface, rolling without slipping, and being oriented at an angle θ . Using cylindrical coordinates (ρ, ϕ, z) for the CM and orientation angles α, β, γ the Lagrangian is

$$L = \frac{M}{2} \left[\dot{z}^2 + \dot{\rho}^2 + \rho^2 \dot{\phi}^2 + a^2 \frac{\dot{\alpha}^2}{4} + a^2 \frac{\dot{\beta}^2}{4} + a^2 \frac{\dot{\gamma}^2}{2} \right] - Mgz,$$

with constraint functions

$$\begin{aligned}
\varphi_1 &= \rho - b + a \sin \theta = 0, \\
\varphi_2 &= z - a \cos \theta = 0, \\
\varphi_3 &= \gamma a - b\phi = 0, \\
\varphi_4 &= \theta - \beta = 0 \\
\varphi_5 &= \phi - \alpha.
\end{aligned}$$

The equations of motion from Euler-Lagrange with Lagrange multipliers are (after substituting in the constraints)

$$\begin{aligned}
z: M\ddot{z} + Mg + \lambda_2 &= 0 \Rightarrow f_3^0 = -\lambda_2 = Mg [\ddot{z} = 0, z = a \cos \theta], \\
\rho: M\ddot{\rho} - M\rho\dot{\phi}^2 + \lambda_1 &= 0 \Rightarrow f_\rho^0 = -\lambda_1 = -M(b - a \sin \theta)\dot{\phi}^2 [\ddot{\rho} = 0, \rho = b - a \sin \theta], \\
\phi: M\ddot{\phi}\rho^2 - b\lambda_3 &= 0 \Rightarrow f_\phi^0 = \lambda_3 = 0 [\ddot{\phi} = 0], \\
\alpha: \frac{M}{4}a^2\ddot{\alpha} - \lambda_5 &= 0 \Rightarrow \lambda_5 = 0 [\ddot{\alpha} = 0, \alpha = \phi], \\
\beta: \frac{M}{4}a^2\ddot{\beta} - \lambda_4 &= 0 \Rightarrow \lambda_4 = 0 [\ddot{\beta} = 0, \beta = \theta], \\
\gamma: \frac{M}{2}a^2\ddot{\gamma} + a\lambda_3 &= 0 \Rightarrow \lambda_3 = 0 [\ddot{\gamma} = 0, a\gamma = b\phi].
\end{aligned}$$

Hence the equations and forces are just as obtained above.

(a) (2 pts) Using these forces our goal is to express them in the CM system so we can use $d\vec{L}_{CM}/dt = \vec{r} \times \vec{f}$. As usual we want to work out the relation between the unit vectors in the inertial frame and those defined in the body-associated, but not body-fixed (principle axis) frame in figure P5.3. We have

$$\begin{aligned}
\hat{e}_1^0 &= \cos \phi \hat{e}_1 + \sin \theta \sin \phi \hat{e}_2 + \cos \theta \sin \phi \hat{e}_3, \\
\hat{e}_2^0 &= -\sin \phi \hat{e}_1 + \sin \theta \cos \phi \hat{e}_2 + \cos \theta \cos \phi \hat{e}_3, \\
\hat{e}_3^0 &= -\cos \theta \hat{e}_2 + \sin \theta \hat{e}_3.
\end{aligned}$$

This leads to

$$\begin{aligned}
\vec{f} &= f_{0,1}\hat{e}_1^0 + f_{0,2}\hat{e}_2^0 + f_{0,3}\hat{e}_3^0 \\
&= \left(-M(b - a\sin\theta)\sin\phi\dot{\phi}^2\right)(\cos\phi\hat{e}_1 + \sin\theta\sin\phi\hat{e}_2 + \cos\theta\sin\phi\hat{e}_3) \\
&+ \left(-M(b - a\sin\theta)\cos\phi\dot{\phi}^2\right)(-\sin\phi\hat{e}_1 + \sin\theta\cos\phi\hat{e}_2 + \cos\theta\cos\phi\hat{e}_3) \\
&+ (Mg)(-\cos\theta\hat{e}_2 + \sin\theta\hat{e}_3) \\
&= M\left(-M(b - a\sin\theta)\sin\phi\dot{\phi}^2\cos\phi + M(b - a\sin\theta)\sin\phi\dot{\phi}^2\cos\phi\right)\hat{e}_1 \\
&+ M\left(-(b - a\sin\theta)\dot{\phi}^2\sin\theta - g\cos\theta\right)\hat{e}_2 \\
&+ M\left(-(b - a\sin\theta)\dot{\phi}^2\cos\theta + g\sin\theta\right)\hat{e}_3 \\
&= M\left[\left(-(b - a\sin\theta)\dot{\phi}^2\sin\theta - g\cos\theta\right)\hat{e}_2 + \left(-(b - a\sin\theta)\dot{\phi}^2\cos\theta + g\sin\theta\right)\hat{e}_3\right].
\end{aligned}$$

Noting that $\vec{r} = a\hat{e}_2$, the torque is quite simple (this is the reason for this choice of axes) and we have

$$\begin{aligned}
\frac{d\vec{L}_{CM}}{dt} &= af_{0,3}(\hat{e}_2 \times \hat{e}_3) \\
&= Ma\left[g\sin\theta - \dot{\phi}^2\cos\theta(b - a\sin\theta)\right]\hat{e}_1,
\end{aligned}$$

Which is the desired result

(b) (2 pts) Now consider the general motion of the coin. In the body-associated frame it rotates with angular velocity $\dot{\gamma}$ about the axis \hat{e}_3 . As viewed in the inertial frame the body associated frame rotates about the $\hat{e}_3^0 = -\cos\theta\hat{e}_2 + \sin\theta\hat{e}_3$ axis with angular velocity $-\dot{\phi}$ and about the \hat{e}_1 axis with angular velocity $\dot{\theta}$, which vanishes here. Thus the instantaneous angular velocity of the body-associated frame as seen in the inertia frame, but expressed in terms of the body-associated unit vectors, is

$$\vec{\Omega} = \dot{\theta}\hat{e}_1 + \dot{\phi}\cos\theta\hat{e}_2 - \dot{\phi}\sin\theta\hat{e}_3.$$

The total angular momentum in the inertial (CM) frame includes that due to both the motion of the body-associated frame and the motion in that frame. We have

$$\begin{aligned}\vec{I} &= \hat{e}_1 I_1 \hat{e}_1 + \hat{e}_2 I_1 \hat{e}_2 + \hat{e}_3 I_3 \hat{e}_3, \\ \vec{L}_{CM} &= \vec{I} \cdot \vec{\Omega} + \gamma \vec{I} \cdot \hat{e}_3 = I_1 \dot{\theta} \hat{e}_1 + I_1 \dot{\phi} \cos \theta \hat{e}_2 + I_3 (\dot{\gamma} - \dot{\phi} \sin \theta) \hat{e}_3.\end{aligned}$$

(c) (3 pts) Finally we consider the relation between the motion in the inertial frame and the body-associated frame. We know from our studies of rotating frames that

$$\left. \frac{d\vec{L}}{dt} \right|_{CM} = \left. \frac{d\vec{L}}{dt} \right|_{body} + \vec{\Omega} \times \vec{L} = \vec{\Omega} \times \vec{L},$$

where we have used the fact that the angular momentum as viewed in the body-associated frame is constant (for $\ddot{\gamma} = \ddot{\phi} = \ddot{\theta} = 0$). Working out the right-hand-side (using the fact that \vec{L} has a component parallel to $\vec{\Omega}$) and taking the 1-component yields (using the non-slip constraint)

$$\begin{aligned}\vec{L} &= I_1 \dot{\theta} \hat{e}_1 + I_1 \dot{\phi} \cos \theta \hat{e}_2 + I_3 (\dot{\gamma} - \dot{\phi} \sin \theta) \hat{e}_3 \\ &= I_1 (\dot{\theta} \hat{e}_1 + \dot{\phi} \cos \theta \hat{e}_2 - \dot{\phi} \sin \theta \hat{e}_3) + [I_3 \dot{\gamma} - (I_3 - I_1) \dot{\phi} \sin \theta] \hat{e}_3 \\ \Rightarrow \vec{\Omega} \times \vec{L} &= (\dot{\theta} \hat{e}_1 + \dot{\phi} \cos \theta \hat{e}_2 - \dot{\phi} \sin \theta \hat{e}_3) \times [I_3 \dot{\gamma} - (I_3 - I_1) \dot{\phi} \sin \theta] \hat{e}_3 \\ &= [I_3 \dot{\gamma} - (I_3 - I_1) \dot{\phi} \sin \theta] (-\dot{\theta} \hat{e}_2 + \dot{\phi} \cos \theta \hat{e}_1) \\ \Rightarrow (\vec{\Omega} \times \vec{L}) \cdot \hat{e}_1 &= \dot{\phi} \cos \theta [I_3 \dot{\gamma} - (I_3 - I_1) \dot{\phi} \sin \theta] = \frac{Ma^2}{4} \dot{\phi}^2 \cos \theta \left[\frac{2b}{a} - \sin \theta \right].\end{aligned}$$

The corresponding component of the left-hand-side is

$$\begin{aligned}\hat{e}_1 \cdot \left(\frac{d\vec{L}_{CM}}{dt} \right) &= Ma \left[g \sin \theta - \dot{\phi}^2 \cos \theta (b - a \sin \theta) \right] \\ &= Ma^2 \left[\frac{g}{a} \sin \theta - \dot{\phi}^2 \cos \theta \left(\frac{b}{a} - \sin \theta \right) \right].\end{aligned}$$

Substituting in the earlier equation we have

$$\begin{aligned}
Ma^2 \left[\frac{g}{a} \sin \theta - \dot{\phi}^2 \cos \theta \left(\frac{b}{a} - \sin \theta \right) \right] &= \frac{Ma^2}{4} \dot{\phi}^2 \cos \theta \left[\frac{2b}{a} - \sin \theta \right] \\
\Rightarrow \frac{g}{a} \sin \theta = \dot{\phi}^2 \cos \theta \left[\left(\frac{b}{a} - \sin \theta \right) + \frac{1}{4} \left(\frac{2b}{a} - \sin \theta \right) \right] &= \frac{\dot{\phi}^2}{4} \cos \theta \left[\frac{6b}{a} - 5 \sin \theta \right] \\
\Rightarrow \frac{1}{\dot{\phi}^2} = \frac{\cos \theta (6b - 5a \sin \theta)}{4g \sin \theta}.
\end{aligned}$$

If we identify this last quantity with the period of rolling around the circle divided by 2π (squared), we obtain the desired relation

$$\left(\frac{\tau}{2\pi} \right)^2 = \frac{1}{\dot{\phi}^2} = \frac{\cos \theta (6b - 5a \sin \theta)}{4g \sin \theta}.$$

2) (5 pts) As another rigid body analysis (with clear application to your spare time pursuits and sometimes on Qual Exams) we want to work through the problem illustrated in Fig. 28.3 in Fetter & Walecka. A homogeneous billiard ball of mass M and radius a is struck impulsively with a horizontal force (*i.e.*, a cue stick), with the force applied a distance h above the center of the ball. The subsequent motion is determined by the force of friction that retards motion, with the frictional acceleration given in units of g (the gravitational acceleration),

$$\ddot{x} = -\mu g.$$

Here μ is assumed to be constant (independent of the ball's velocity). Let v_0 be the ball's initial velocity (after been struck). For h positive (as in Fig. 28.3), but otherwise arbitrary, determine the time and velocity at which the ball begins to roll without sliding. As noted in F&W, there are 3 distinct regimes in h to consider.

Solution: In the inertial frame described by Fig. 28.3 we have

$$\begin{aligned}
\ddot{x} = -\mu g, \vec{F}_f = -\mu g M \hat{e}_1^0, \vec{N}_f = \mu g a M \hat{e}_3^0 \\
\Rightarrow \dot{L}_3 = \mu g a M = I_3 \ddot{\phi} = \frac{2Ma^2}{5} \ddot{\phi} \Rightarrow \ddot{\phi} = \frac{5\mu g}{2a}.
\end{aligned}$$

The initial conditions are determined by the impulse provided by the initial stroke. The corresponding impulse and initial angular momentum are

$$\text{Impulse} = \int F_x dt = Mv_0,$$

$$L_0 = Mv_0 h = I_3 \dot{\phi}(0) \Rightarrow \dot{\phi}(0) = \frac{5v_0 h}{2a^2}.$$

The solutions of the equations of motion are

$$\dot{x} = v_0 - \mu g t, \dot{\phi} = \frac{5v_0 h}{2a^2} + \frac{5\mu g}{2a} t.$$

The issue we are concerned about is whether the billiard ball is slipping or not. To that end we want to compare \dot{x} with $a\dot{\phi}$,

$$v_0 - \mu g t \overset{?}{\longleftrightarrow} \frac{5v_0 h}{2a} + \frac{5\mu g}{2} t.$$

Motion without slipping sets in when the two sides of this expression become equal, *i.e.*, when

$$t_{NS} = v_0 \frac{(2a - 5h)}{7a\mu g}.$$

So we see immediately that there are 3 distinct possibilities depending on the value of h .

- a) For $h < 2a/5$, the ball is initially moving faster than it is rotating and it takes the time t_{NS} until the two motions become commensurate, after which it rolls without slipping. The velocity at this time is

$$t = t_{NS} = v_0 \frac{(2a - 5h)}{7a\mu g},$$

$$v(t_{NS}) = v_0 - \mu g t_{NS} = v_0 \left(1 - \frac{(2a - 5h)}{7a} \right) = \frac{5}{7} v_0 \left(\frac{a + h}{a} \right)$$

$$\Rightarrow \frac{5}{7} v_0 < v(t_{NS}) < v_0.$$

- b) For $h = 2a/5$ the rolling and linear motion are commensurate from the beginning and the ball just rolls without slipping ($t_{NS} = 0$). The corresponding velocity is just v_0 .
- c) For $h > 2a/5$ the ball is initially rolling faster than it is sliding and the frictional force acts in the opposite direction,

$$\ddot{x} = \mu g, \vec{F}_f = \mu g M \hat{e}_1^0, \vec{N}_f = -\mu g a M \hat{e}_3^0$$

$$\Rightarrow \dot{L}_3 = -\mu g a M = I_3 \ddot{\phi} = \frac{2Ma^2}{5} \ddot{\phi} \Rightarrow \ddot{\phi} = -\frac{5\mu g}{2a}.$$

In this case the linear motion speeds up, while the rotational motion slows down until the two are commensurate at

$$t'_{NS} = v_0 \frac{(5h - 2a)}{7a\mu g}.$$

The corresponding velocity at the non-slipping time is

$$t = t'_{NS} = v_0 \frac{(5h - 2a)}{7a\mu g},$$

$$v(t'_{NS}) = v_0 + \mu g t'_{NS} = v_0 \left(1 + \frac{(5h - 2a)}{7a} \right) = \frac{5}{7} v_0 \left(\frac{a + h}{a} \right)$$

$$\Rightarrow v_0 < v(t'_{NS}) < \frac{10}{7} v_0.$$

3) (5 pts) As practice with Euler consider the following system. A uniform right circular cone of height h , half angle α , and density ρ rolls on its side without slipping on a uniform horizontal plane in such a way that it returns to its original position in a time τ . Find expressions for the kinetic energy and the components of the angular momentum of the cone. (This is another exercise to be found on past Qual exams.)

Solution: As the cone rolls without slipping the tip of the cone does not move while the circular end does move. Trigonometry tells us that the radius of the circular end is $r = h \tan \alpha$ while the length of the side of the cone is $l = h / \cos \alpha$. If we define γ to be the angle of rotation about the cone's axis of symmetry (as in our definition of the Euler angles), then in a rotation of $\Delta\gamma$ the moving end of the cone (traveling on a circle) advances by $\Delta\gamma r = \Delta\gamma h \tan \alpha$. This corresponds to a rotation (precession) of the cone about its tip of $|\Delta\phi| = |\Delta\gamma| r / l = |\Delta\gamma| \sin \alpha$ (where we use ϕ in its usual meaning since we have already defined a role for α in this problem). Note that, if we define ϕ in the RH sense about the vertical (up) axis and γ in the RH sense about the symmetry axis of the cone pointing out through the wide end of the cone, the constraint of rolling without slipping means that $\Delta\phi$ and $\Delta\gamma$ will have the opposite sign. Thus in the period τ we have $\Delta\phi = 2\pi$ (by choice) and $\Delta\gamma = -2\pi / \sin \alpha$, or $\dot{\phi} = 2\pi / \tau$ and $\dot{\gamma} = -2\pi / \tau \sin \alpha$. To find the energy and angular momentum we need to know the components of the moment of inertia for the cone. The volume of such a cone is given by $V = \pi r^2 h / 3 = \pi h^3 \tan^2 \alpha / 3$. Thus, if the mass of the cone is M , we have a (uniform) density of $\rho = 3M / \pi h^3 \tan^2 \alpha$. Take the axis of symmetry to be the z' axis. The easy component of the inertia tensor is

$$I_{zz} = \rho \int_0^h dz' \int_0^{z' \tan \alpha} \rho' d\rho' \rho'^2 \oint d\phi' = \frac{\pi\rho}{2} \tan^4 \alpha \int_0^h z'^4 dz'$$

$$= \frac{\pi\rho}{10} h^5 \tan^4 \alpha = \frac{3}{10} M h^2 \tan^2 \alpha = \frac{3}{10} M r^2.$$

This allows us to account for the kinetic energy and angular momentum due to $\dot{\gamma}$, the rotation about the symmetry axis. To account for that due to $\dot{\phi}$ we need the $I_{xx} = I_{yy}$ components with respect to the tip of the cone. We find, for example,

$$\begin{aligned}
I_{xx} &= \rho \int_0^h dz' \int_0^{z' \tan \alpha} \rho' d\rho' \oint d\phi' (z'^2 + y'^2) = \rho \int_0^h dz' \int_0^{z' \tan \alpha} \rho' d\rho' \oint d\phi' (z'^2 + \rho'^2 \sin^2 \phi') \\
&= 2\pi\rho \int_0^h dz' \int_0^{z' \tan \alpha} \rho' d\rho' \left(z'^2 + \frac{\rho'^2}{2} \right) = 2\pi\rho \int_0^h dz' z'^4 \left(\frac{\tan^2 \alpha}{2} + \frac{\tan^4 \alpha}{8} \right) \\
&= \pi\rho h^5 \left(\frac{\tan^2 \alpha}{5} + \frac{\tan^4 \alpha}{20} \right) = 3M \left(\frac{h^2}{5} + \frac{r^2}{20} \right) = \frac{3Mh^2}{20} (4 + \tan^2 \alpha).
\end{aligned}$$

We can think of the cone as a top at fixed polar angle $\beta (= \theta) = \pi/2 - \alpha$ and use our expressions for kinetic energy and angular momentum in terms of the Euler angles. We have for T

$$\begin{aligned}
T &= \frac{I_{xx}}{2} \dot{\phi}^2 \sin^2 \beta + \frac{I_{zz}}{2} (\dot{\phi} \cos \beta + \dot{\gamma})^2 \\
&= \frac{3Mh^2}{40} (4 + \tan^2 \alpha) \cos^2 \alpha \left(\frac{2\pi}{\tau} \right)^2 + \frac{3Mh^2 \tan^2 \alpha}{20} \left(\frac{2\pi}{\tau} \right)^2 \left(\sin \alpha - \frac{1}{\sin \alpha} \right)^2 \\
&= \frac{3Mh^2}{40} \left(\frac{2\pi}{\tau} \right)^2 \left(1 + 3\cos^2 \alpha + 2 \frac{(\cos^2 \alpha)^2}{\cos^2 \alpha} \right) \\
&= \frac{3Mh^2}{40} \left(\frac{2\pi}{\tau} \right)^2 (1 + 5\cos^2 \alpha).
\end{aligned}$$

The angular momentum about the vertical axis is

$$\begin{aligned}
P_\phi &= I_{xx} \dot{\phi} \sin^2 \beta + I_{zz} \cos \beta (\dot{\phi} \cos \beta + \dot{\gamma}) \\
&= \frac{3Mh^2}{20} (4 + \tan^2 \alpha) \cos^2 \alpha \left(\frac{2\pi}{\tau} \right) + \frac{3Mh^2 \tan^2 \alpha}{10} \left(\frac{2\pi}{\tau} \right) \left(\sin \alpha - \frac{1}{\sin \alpha} \right) \sin \alpha \\
&= \frac{3Mh^2}{20} \left(\frac{2\pi}{\tau} \right) \left(4\cos^2 \alpha + \sin^2 \alpha + 2 \frac{\sin^2 \alpha (-\cos^2 \alpha)}{\cos^2 \alpha} \right) \\
&= \frac{3Mh^2}{20} \left(\frac{2\pi}{\tau} \right) (5\cos^2 \alpha - 1).
\end{aligned}$$

The other component of angular momentum that we can easily evaluate is the component along the symmetry axis of the cone, the z' component in the body-

fixed frame,

$$\begin{aligned}
 P_\gamma &= I_{zz} (\dot{\phi} \cos \beta + \dot{\gamma}) = \frac{3Mh^2 \tan^2 \alpha}{10} \left(\frac{2\pi}{\tau} \right) \left(\sin \alpha - \frac{1}{\sin \alpha} \right) \\
 &= \frac{3Mh^2}{10} \left(\frac{2\pi}{\tau} \right) \frac{\sin \alpha (-\cos^2 \alpha)}{\cos^2 \alpha} = -\frac{3Mh^2}{10} \left(\frac{2\pi}{\tau} \right) \sin \alpha.
 \end{aligned}$$

4) Fetter & Walecka – 6.2 (5 pts) This exercise allows us to practice with Hamiltonian techniques.

Solution: Give the geometry described in the problem the Lagrangian looks like (this is a spherical pendulum)

$$L = \frac{m}{2} (l^2 \dot{\theta}^2 + l^2 \dot{\phi}^2 \sin^2 \theta) + mgl \cos \theta.$$

We can find the canonical momenta and the Hamiltonian from

$$\begin{aligned}
 p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = ml^2 \dot{\theta}, \quad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m \sin^2 \theta l^2 \dot{\phi} \\
 \Rightarrow H &= [\dot{\theta} p_\theta + \dot{\phi} p_\phi] - L \\
 &= ml^2 \dot{\theta}^2 + m \sin^2 \theta l^2 \dot{\phi}^2 - \frac{m}{2} (l^2 \dot{\theta}^2 + l^2 \dot{\phi}^2 \sin^2 \theta) - mgl \cos \theta \\
 &= \frac{m}{2} (l^2 \dot{\theta}^2 + l^2 \dot{\phi}^2 \sin^2 \theta) - mgl \cos \theta \\
 &= \frac{p_\theta^2}{2ml^2} + \frac{p_\phi^2}{2m \sin^2 \theta l^2} - mgl \cos \theta.
 \end{aligned}$$

Thus Hamilton's equations are

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{\cos \theta p_\phi^2}{ml^2 \sin^3 \theta} - mgl \sin \theta \quad \left[\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2} \right],$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0 \quad \left[\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{ml^2 \sin^2 \theta} \right].$$

We can easily “solve” these to find

$$p_\phi = ml^2 \sin^2 \theta \dot{\phi} = L_z = \text{constant},$$

$$\dot{p}_\theta = ml^2 \ddot{\theta} = \frac{\cos \theta L_z^2}{ml^2 \sin^3 \theta} - mgl \sin \theta,$$

$$H = \frac{m}{2} l^2 \dot{\theta}^2 + \frac{L_z^2}{2ml^2 \sin^2 \theta} - mgl \cos \theta.$$

Now consider perturbations about uniform circular motion at $\theta = \theta_0$, i.e., $\theta = \theta_0 + \delta$. Thus we have

$$\dot{\theta}^2 \rightarrow \dot{\delta}^2,$$

$$\frac{1}{\sin^2 \theta} \rightarrow \frac{1}{\sin^2(\theta_0 + \delta)} \approx \frac{1}{\sin^2 \theta_0} \left(1 + \frac{\cos \theta_0}{\sin \theta_0} \delta - \frac{\delta^2}{2} \right)^{-2}$$

$$\approx \frac{1}{\sin^2 \theta_0} \left(1 - 2\delta \cot \theta_0 + \delta^2 [1 + 3\cot^2 \theta_0] \right),$$

$$\cos \theta \rightarrow \cos(\theta_0 + \delta) \approx \cos \theta_0 \left(1 - \delta \tan \theta_0 - \frac{\delta^2}{2} \right),$$

$$\Rightarrow H \approx \frac{m}{2} l^2 \dot{\delta}^2 + \frac{L_z^2}{2ml^2 \sin^2 \theta_0} \left(1 - 2\delta \cot \theta_0 + \delta^2 [1 + 3\cot^2 \theta_0] \right)$$

$$- mgl \cos \theta_0 \left(1 - \delta \tan \theta_0 - \frac{\delta^2}{2} \right).$$

Hamilton’s equations to order δ yield

$$\begin{aligned}
ml^2\ddot{\delta} &= -\frac{\partial H}{\partial \delta} = \frac{L_z^2}{ml^2 \sin^2 \theta_0} \left(\cot \theta_0 - \delta [1 + 3 \cot^2 \theta_0] \right) - mgl \cos \theta_0 (\tan \theta_0 + \delta) \\
&= \frac{\cos \theta_0 L_z^2}{ml^2 \sin^3 \theta_0} - mgl \sin \theta_0 - \delta \left(\frac{L_z^2 [1 + 3 \cot^2 \theta_0]}{ml^2 \sin^2 \theta_0} + mgl \cos \theta_0 \right).
\end{aligned}$$

Thus, in order that θ_0 corresponds to uniform circular motion with $\dot{\theta} = \ddot{\theta} = 0$, we must have

$$\begin{aligned}
\frac{\cos \theta_0 L_z^2}{ml^2 \sin^3 \theta_0} &= mgl \sin \theta_0 \\
\Rightarrow L_z &= ml^2 \sin^2 \theta_0 \sqrt{\frac{g}{l \cos \theta_0}},
\end{aligned}$$

and small perturbations, δ , lead to oscillations described by

$$\begin{aligned}
\ddot{\delta} &\approx -\frac{\delta}{ml^2} \left(\frac{L_z^2 [1 + 3 \cot^2 \theta_0]}{ml^2 \sin^2 \theta_0} + mgl \cos \theta_0 \right) \\
&= -\frac{\delta}{ml^2} mgl \left(\tan \theta_0 \sin \theta_0 [1 + 3 \cot^2 \theta_0] + \cos \theta_0 \right) \\
&= -\delta \frac{g}{l \cos \theta_0} (\sin^2 \theta_0 + 4 \cos^2 \theta_0) \\
&= -\delta \frac{g}{l \cos \theta_0} (1 + 3 \cos^2 \theta_0) \equiv -\omega^2 \delta.
\end{aligned}$$

Thus we have, in this approximation of small oscillations, a simple harmonic oscillator with a frequency of oscillation given by $\omega^2 = (g/l \cos \theta_0)(1 + 3 \cos^2 \theta_0)$.