Physics 505 - Autumn 2010

HW VI Solutions

11/10/10

Overview: Recall that solving physics problems is not (just) about solving differential equations. Use physical reasoning to help solve the following exercises and be certain to show your work. It is also important that you practice completely solving these exercises, checking for errors as you go along.

1) Fetter & Walecka – 4.13 (5 pts) This is an exercise in the transition to systems with large numbers of normal modes.

Solution: Now we want to consider a system of an arbitrarily large number of identical pendulums (*m* and *l*) spaced by *a* and coupled by springs *k*. We address this system by considering *N* such pendulums apply periodic boundary conditions. The corresponding Lagrangian looks like

$$L = \frac{m}{2}\sum_{j=1}^{N}\dot{\eta}_j^2 - \frac{mg}{2l}\sum_{j=1}^{N}\eta_j^2 - \frac{k}{2}\sum_{j=1}^{N}\left(\eta_j - \eta_{j-1}\right)^2,$$

η_j is the displacement of the j^{th} mass from equilibrium (the pendulum hanging straight down). This yields the expected equation of motion

$$m\ddot{\eta}_j = -\frac{mg}{l}\eta_j - 2k\eta_j + k\left(\eta_{j+1} + \eta_{j-1}\right).$$

We proceed by employing our favorite exponential Ansatz in the form $\eta_j = Ae^{iqaj - i\omega t}$ with wave number *q* and frequency *ω*. Substitution in the equation of motion yields

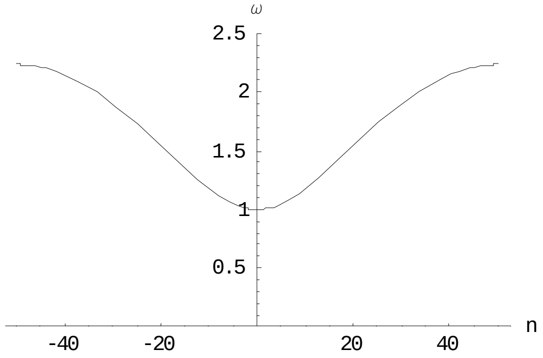
$$-\omega^2 mAe^{iqaj-i\omega t} = -\frac{mg}{l}Ae^{iqaj-i\omega t} - 2kAe^{iqaj-i\omega t} + kAe^{iqaj-i\omega t}\left(e^{iqa} + e^{-iqa}\right)$$

$$\Rightarrow \omega^2(q) = \frac{g}{l} + 2\frac{k}{m}\left(1 - \cos(qa)\right) = \frac{g}{l} + 4\frac{k}{m}\sin^2\left(\frac{qa}{2}\right).$$

This last result is the desired dispersion relation relating the frequency to the wave number. The eigenvalue constraint comes from the imposition of periodic boundary conditions

$$\eta_0 = \eta_N \Rightarrow Ae^{-i\omega t} = Ae^{iqaN - i\omega t}$$
$$\Rightarrow qaN = 2\pi n, \begin{cases} n = 0, \pm1, \pm2, \ldots, \pm(N-1)/2 : N \text{ odd} \\ n = 0, \pm1, \pm2, \ldots, \pm(N/2-1), N/2 : N \text{ even} \end{cases}$$
$$\Rightarrow \omega_n^2 = \frac{g}{l} + 4\frac{k}{m}\sin^2\left(\frac{\pi n}{N}\right), q_n = \frac{2\pi n}{Na}, \lambda_n = \frac{Na}{|n|}.$$

Hence the frequency of the lowest, *i.e.*, longest wavelength, mode is $\omega_0 = \sqrt{g/l}$ with corresponding wavelength $\lambda_0 \to \infty$. In this case all the pendulums swing together and the springs play no role. A plot of this dispersion relation for the case $g/l = k/m = 1/s^2, N = 100$ is shown in the figure to the right.



2) Fetter & Walecka – 4.16 (6 pts) Here we get the opportunity to consider what happens to normal modes in 2 spatial dimensions and in the continuum limit.

Solution: (a) (1 pt) We imagine describing a grid of identical masses *m*, spaced by a distance *a* in each of 2 directions, that are arrayed on a lattice of identical strings with tension τ. If the coordinate μ_{kl} (1 index for each of the 2 spatial dimensions) describes the (transverse) displacement from equilibrium of the mass at the lattice point *kl*, then we can write the kinetic energy, potential energy and Lagrangian as (recall the 1-D example discussed in class)

$$T = \frac{m}{2}\sum_{k,l=1}^{N} \dot{\mu}_{kl}^2, U = \frac{\tau}{2a}\sum_{k,l=0}^{N}\left[\left(\mu_{k+1\,l} - \mu_{kl}\right)^2 + \left(\mu_{k\,l+1} - \mu_{k,l}\right)^2\right],$$
$$L = \frac{m}{2}\sum_{k,l=1}^{N} \dot{\mu}_{kl}^2 - \frac{\tau}{2a}\sum_{k,l=0}^{N}\left[\left(\mu_{k+1\,l} - \mu_{kl}\right)^2 + \left(\mu_{k\,l+1} - \mu_{k,l}\right)^2\right].$$

Thus the (2-D) equation of motion becomes

$$m\ddot{\mu}_{kl}-\frac{\tau}{a}\Big[\big(\mu_{k+1\,l}-2\mu_{k\,l}+\mu_{k-1\,l}\big)+\big(\mu_{k\,l+1}-2\mu_{k\,l}+\mu_{k\,l-1}\big)\Big]=0.$$

In the now familiar fashion we try the Ansatz $\mu_{k\,l}=Ae^{i(q_xk+q_yl)a-i\omega t}$ for traveling waves, now with a 2-D wave number, $\vec{q}=q_x\hat{x}+q_y\hat{y}$. Substitution in the equation of motion leads to the dispersion relation,

$$-\omega^2mAe^{i(q_xk+q_yl)a-i\omega t}-\frac{\tau}{a}Ae^{i(q_xk+q_yl)a-i\omega t}\Big[\big(e^{iq_xa}-2+e^{-iq_xa}\big)+\big(e^{iq_ya}-2+e^{-iq_ya}\big)\Big]=0$$

$$\Rightarrow\omega^2(\vec{q})=\frac{2\tau}{am}\Big[\big(1-\cos(q_xa)\big)+\big(1-\cos(q_ya)\big)\Big]=\frac{4\tau}{am}\left[\sin^2\left(\frac{q_xa}{2}\right)+\sin^2\left(\frac{q_ya}{2}\right)\right].$$

(b) (2 pts) Now consider the continuum limit defined by

$$\begin{aligned}&N\to\infty,\,a\to 0,\,(N+1)a=l=\text{ constant},\\&m\to 0,\,\frac{mN^2}{l^2}\to\sigma=\text{ constant}=\frac{m}{a^2},\\&\frac{\tau}{a}=\kappa.\end{aligned}$$

Note that in the limit of 2 continuous spatial dimensions the mass density is now mass per unit area and the linear tension becomes the surface tension of a membrane with units of force per unit boundary or force per length. In this limit the dispersion relation becomes

$$\begin{aligned}\omega^2(\vec{q})&=\frac{4\tau}{am}\left[\sin^2\left(\frac{q_xa}{2}\right)+\sin^2\left(\frac{q_ya}{2}\right)\right]\to\frac{4\tau}{am}\left[\left(\frac{q_xa}{2}\right)^2+\left(\frac{q_ya}{2}\right)^2\right]\\&=\frac{4\kappa}{\sigma}\left[\left(\frac{q_x}{2}\right)^2+\left(\frac{q_y}{2}\right)^2\right]=\frac{\kappa}{\sigma}\vec{q}^{\,2}.\end{aligned}$$

As expected the coefficient κ/σ has units of (force/length)/(mass/area) =

$(\text{length})^2/(\text{time})^2$ or $(\text{velocity})^2$. So we define the wave velocity as $c=\sqrt{\kappa/\sigma}$. In a similar way we take the continuum limit of the equation of motion,

$$
\begin{aligned}
&m\ddot{\mu}_{kl}\to a^2\sigma\frac{\partial^2 u(x,y,t)}{\partial t^2},\\
&\frac{\tau}{a}\left[\left(\mu_{k+1\,l}-2\mu_{k\,l}+\mu_{k-1\,l}\right)+\left(\mu_{k\,l+1}-2\mu_{k\,l}+\mu_{k\,l-1}\right)\right]\to a^2\kappa\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]\\
&\Rightarrow \sigma\frac{\partial^2 u(x,y,t)}{\partial t^2}-\kappa\,\vec{\nabla}^2 u(x,y,t)=0,\\
&\frac{\partial^2 u(x,y,t)}{\partial t^2}-c^2\,\vec{\nabla}^2 u(x,y,t)=0,
\end{aligned}
$$

which is the 2-D wave equation. Likewise the Lagrangian becomes an integral over the 2-D Lagrangian density

$$
\begin{aligned}
&\frac{m}{2}\sum_{k,l=1}^{N}\dot{\mu}_{kl}^2\to\frac{\sigma}{2}\iint dxdy\,\dot{u}^2(x,y,t)\quad\left[a\Delta k\to dx, a\Delta l\to dy\right],\\
&\frac{\tau}{2a}\sum_{k,l=0}^{N}\left[\left(\mu_{k+1\,l}-\mu_{kl}\right)^2+\left(\mu_{k\,l+1}-\mu_{k,l}\right)^2\right]\to\frac{\kappa}{2}\iint dxdy\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]\\
&\Rightarrow L=\frac{1}{2}\iint dxdy\left[\sigma\dot{u}^2+\kappa\left(\vec{\nabla}u\right)^2\right]=\iint dxdy\,\mathcal{L},\\
&\mathcal{L}=\frac{1}{2}\left[\sigma\dot{u}^2+\kappa\left(\vec{\nabla}u\right)^2\right].
\end{aligned}
$$

(c) (1 pt) An important feature of physics on a lattice (a 2-D lattice in this case) is that the discreteness of the spatial structure breaks rotation symmetry. In the continuum case rotational symmetry is restored and the dispersion relation is expressed in terms of the usual scalar product $\vec{q}^{\,2}=q_x^2+q_y^2$, which is rotationally invariant by definition. This is to be compared to the right-hand-side of the *discrete* dispersion relation, $(4\tau/am)\left[\sin^2(q_x a/2)+\sin^2(q_y a/2)\right]$, which is not invariant under continuous rotations, $q_x'=\cos\theta q_x+\sin\theta q_y$, $q_y'=-\sin\theta q_x+\cos\theta q_y$ (because it is not in the form of powers of $q_x^2+q_y^2$), *i.e.*,

$$\vec{q}'^2 = q_x'^2 + q_y'^2 = q_x^2 + q_y^2 = \vec{q}^2,$$

$$\begin{aligned}
&\sin^2\left(q_x' a/2\right) + \sin^2\left(q_y' a/2\right) \\
&\quad = \sin^2\left(\left\{\cos\theta q_x + \sin\theta q_y\right\} a/2\right) + \sin^2\left(\left\{-\sin\theta q_x + \cos\theta q_y\right\} a/2\right) \\
&\quad \neq \sin^2\left(q_x a/2\right) + \sin^2\left(q_y a/2\right).
\end{aligned}$$

(d) (1 pt) Returning to the finite, discrete system, we consider the case of fixed boundaries, $\mu_{0l} = \mu_{k0} = \mu_{N+1\,l} = \mu_{k\,N+1} = 0$. As usual we want the Ansatz (using a product of the difference of the two oppositely moving waves) $\mu_{k\,l} = Ae^{-i\omega t}\left(e^{i(q_x k)a} - e^{-i(q_x k)a}\right)\left(e^{i(q_y l)a} - e^{-i(q_y l)a}\right)$, which is guaranteed to vanish at one boundary in each dimension, $k = 0, l = 0$. Requiring that it vanish at the other boundaries yields the eigenvalue conditions,

$$\begin{aligned}
&\left(e^{iq_x(N+1)a} - e^{-iq_x(N+1)a}\right) = 0 \Rightarrow q_x = \frac{\pi k}{a(N+1)}, k = 1,\ldots,N \\
&\left(e^{iq_y(N+1)a} - e^{-iq_y(N+1)a}\right) = 0 \Rightarrow q_y = \frac{\pi n}{a(N+1)}, n = 1,\ldots,N.
\end{aligned}$$

Thus the corresponding dispersion relation is

$$\omega_{kn}^2 = \frac{4\tau}{am}\left[\sin^2\left(\frac{\pi k}{2(N+1)}\right) + \sin^2\left(\frac{\pi n}{2(N+1)}\right)\right].$$

In the continuum limit this becomes

$$\begin{aligned}
\omega_{kn}^2 &\to \frac{4\kappa}{a^2\sigma}\left[\left(\frac{\pi k}{2(N+1)}\right)^2 + \left(\frac{\pi n}{2(N+1)}\right)^2\right] \to \frac{\kappa}{\sigma}\left(\frac{\pi}{l}\right)^2\left(k^2 + n^2\right) \\
&= \left(\frac{\pi c}{l}\right)^2\left(k^2 + n^2\right) : \vec{q}_{kn} = \frac{\pi}{l}\left(k\hat{x} + n\hat{y}\right), \lambda_{kn} = \frac{2l}{\sqrt{k^2 + n^2}}.
\end{aligned}$$

(e) (1 pt) Finally consider the continuum limit starting with *T* and *U*. The kinetic energy is easy in the continuum limit,

$$T = \frac{1}{2}\iint dm\dot{u}^2 = \frac{1}{2}\iint \sigma dxdy\,\dot{u}^2.$$

The potential energy takes a bit more thought. The energy stored pure unit area of a distorted membrane under tension is given by tension/length times the change in the area due to the distortion. The distorted area is given by $ds = dxdy\sqrt{1+\left(\vec{\nabla}u\right)^2}$, which is the 2-D analogue of the 1-D result $ds = dx\sqrt{1+\left(dy/dx\right)^2}$. Thus for small displacements from equilibrium we have

$$U = \kappa\iint dxdy\left[\sqrt{1+\left(\vec{\nabla}u\right)^2}-1\right] \simeq \frac{\kappa}{2}\iint dxdy\left(\vec{\nabla}u\right)^2.$$

Thus, as above, we have

$$\begin{aligned} L &= \iint dxdy\,\mathcal{L}, \\ \mathcal{L}\left(u,\dot{u},\partial u/\partial x,\partial u/\partial y\right) &= \frac{\sigma}{2}\dot{u}^2 - \frac{\kappa}{2}\left(\vec{\nabla}u\right)^2. \end{aligned}$$

The 2 space +1 time dimensional Lagrange equation (the generalized version of Eq. 25.59) yields

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\left(\partial u/\partial t\right)}\right)+\frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial\left(\partial u/\partial x\right)}\right)+\frac{\partial}{\partial y}\left(\frac{\partial\mathcal{L}}{\partial\left(\partial u/\partial y\right)}\right)-\frac{\partial\mathcal{L}}{\partial u}=0 \\ &\Rightarrow \sigma\frac{\partial^2 u}{\partial t^2}-\kappa\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]=0 \Rightarrow \frac{\partial^2 u}{\partial t^2}-c^2\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]=0, \end{aligned}$$

which is the expected 2-D continuum wave equation.

3) (8 pts) As practice thinking in phase space let us take one more look at our favorite problem of the rotating hoop, Fetter & Walecka – 3.1 and 4.4. Write the

Lagrangian *and* the Hamiltonian in terms of the canonical variables θ, p_θ. Also construct the effective potential as in

$$H = \frac{p_\theta^2}{2ma^2} + U_{eff}(\theta),$$

taking gravity to be the only external force (other than constraint forces). In terms of Hamilton's equations, *i.e.*, think of the flow in phase space, recalling the velocity field discussed in Lecture 7, again locate all equilibria and determine their stability, permitting Ω to vary. Observe that the Hamiltonian has a certain reflection symmetry (*i.e.*, think about the symmetry with respect to the point θ = 0). Show that for $\Omega < \omega_\theta = \sqrt{g/a}$, where ω_θ is the critical rotation speed, there is only one stable equilibrium point and it exhibits the reflection symmetry of the Hamiltonian. However, when $\Omega > \omega_\theta$, there are two stable equilibrium points and they 'break' the symmetry. To illustrate the symmetry make a sketch of the effective potential ($U_{eff}(\theta)\,\mathrm{vs}\,\theta$) in each case, where we allow plus and minus values of θ. Sketch the phase portraits, *i.e.*, the flow patterns in (θ, p_θ) phase space, for both $\Omega < \omega_\theta$ and $\Omega > \omega_\theta$. Plot the equilibrium solutions, *i.e.*, the value of θ_0, as a function of Ω. Sketch stable solutions as solid curves, unstable ones as dashed curves, and observe that the diagram has the form of a pitchfork. (This is an example of a symmetry-breaking pitchfork bifurcation and is analogous to the "spontaneous symmetry breaking" that occurs in the Higgs Phenomenon of particle physics fame.)

Solution: In our previous analyses of this system we found

$$L = \frac{m}{2}\left[\left(a\dot{\theta}\right)^2 + \left(a\Omega\sin\theta\right)^2\right] + mga\cos\theta.$$

Hence we have

$$
\begin{aligned}
p_\theta &= \frac{\partial L}{\partial \dot\theta} = ma^2\dot\theta,\\
\Rightarrow H &= \frac{p_\theta^2}{2ma^2} - \frac{a^2 m\Omega^2}{2}\sin^2\theta - mga\cos\theta\\
&\equiv \frac{p_\theta^2}{2ma^2} + U_{eff}\left(\theta\right).
\end{aligned}
$$

We note, in particular, that H and U_{eff} are invariant under the reflection transformation $\theta \Leftrightarrow -\theta$ (*i.e.*, $\phi \to \phi + \pi$). If we think of this system as a flow problem in $\left(\theta, p_\theta\right)$ phase space, we have

$$
\begin{aligned}
\dot\theta &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ma^2} = V_\theta,\\
\dot p_\theta &= -\frac{\partial H}{\partial \theta} = -\frac{\partial U_{eff}}{\partial\theta} = a^2 m\Omega^2 \sin\theta\cos\theta - mga\sin\theta\\
&= -mga\sin\theta\left(1 - \frac{a\Omega^2}{g}\cos\theta\right) = V_{p_\theta}.
\end{aligned}
$$

Since an equilibrium point corresponds to $\vec V = 0$, we have (as before) 4 points (at most) where V_{p_θ} vanishes

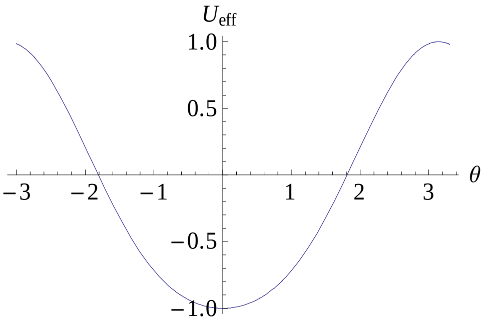
$$
\begin{aligned}
V_\theta &= 0 \Rightarrow p_\theta = 0 \Rightarrow \dot\theta_0 = 0,\\
V_{p_\theta} &= 0 \Rightarrow \begin{cases} \theta_0 = 0, \pi \\ \theta_0 = \pm\cos^{-1}\dfrac{g}{a\Omega^2}, \Omega^2 > \dfrac{g}{a} = \omega_\theta^2 \end{cases},
\end{aligned}
$$

where the ± signs correspond to the 2 branches of the hoop, *i.e.*, the 2 branches in ϕ. In the language of flows in phase space (recall the Appendix to Lecture 7) we want to test the stability of these equilibria by testing the behavior of the velocity field component V_{p_θ} near each of the equilibrium points, $\theta \to \theta_0 + \delta\theta$. Recall that $\vec V = \left(V_\theta, V_{p_\theta}\right) \propto \left(c_1 p_\theta, -c_2\delta\theta\right)$ (with c_1, c_2 positive constants) corresponds to elliptic or stable flow about θ_0, while $\vec V = \left(V_\theta, V_{p_\theta}\right) \propto \left(c_1 p_\theta, c_2\delta\theta\right)$ corresponds to hyperbolic or unstable flow. In the present problem we have

$$V_\theta = \frac{p_\theta}{ma^2},$$

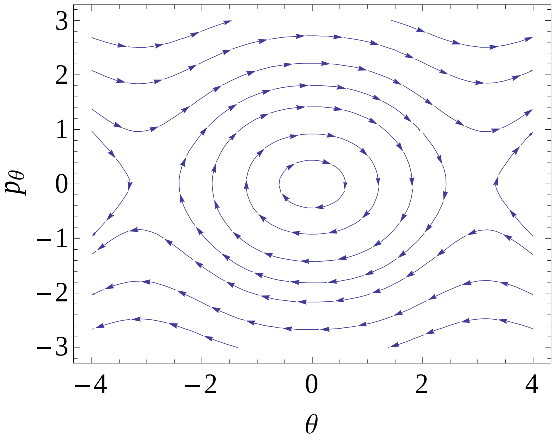
$$V_{p_\theta} \simeq \begin{cases} \theta_0 = 0 : V_{p_\theta} \simeq -mga\left(1 - \dfrac{\Omega^2}{\omega_\theta^2}\right)\delta\theta : \begin{matrix}\text{stable } \Omega \le \omega_\theta \\ \text{unstable } \Omega > \omega_\theta\end{matrix} \\ \theta_0 = \pi : V_{p_\theta} \simeq mga\left(1 + \dfrac{\Omega^2}{\omega_\theta^2}\right)\delta\theta : \text{unstable} \\ \theta_0 = \cos^{-1}\left(\dfrac{\omega_\theta^2}{\Omega^2}\right) : V_{p_\theta} \simeq -mga\dfrac{\Omega^2}{\omega_\theta^2}\left(1 - \dfrac{\omega_\theta^2}{\Omega^2}\right)\delta\theta : \begin{matrix}\text{no equilibrium } \Omega < \omega_\theta \\ \text{stable } \Omega \ge \omega_\theta\end{matrix} \\ \theta_0 = -\cos^{-1}\left(\dfrac{\omega_\theta^2}{\Omega^2}\right) : V_{p_\theta} \simeq -mga\dfrac{\Omega^2}{\omega_\theta^2}\left(1 - \dfrac{\omega_\theta^2}{\Omega^2}\right)\delta\theta : \begin{matrix}\text{no equilibrium } \Omega < \omega_\theta \\ \text{stable } \Omega \ge \omega_\theta\end{matrix} \end{cases}.$$

Thus, as we learned in our previous studies of this system, the point $\theta_0 = \pi$ exhibits only unstable, hyperbolic behavior. As previously advertised, for the other 3 equilibrium points the behavior depends on the ratio of the angular frequency Ω to the critical value of the angular frequency is $\omega_\theta = \sqrt{g/a}$.



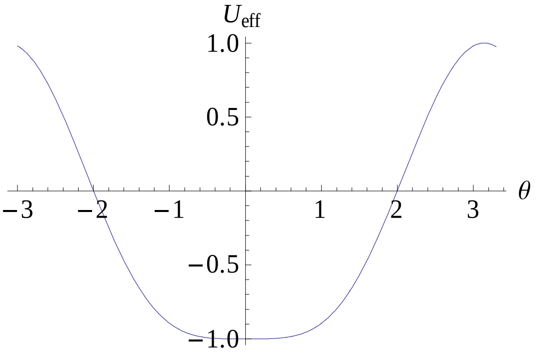
For $\Omega < \omega_\theta$ there is only the single stable equilibrium point at $\theta = 0$ ($\cos^{-1}\left(\omega_\theta^2/\Omega^2\right)$ has no real solutions). The symmetry of U_{eff} about this point is apparent in the plot of the effective potential to the right. Note, as expected, that the point at $\theta_0 = \pi$ is a maximum not a minimum.

The corresponding phase portrait, shown in the figure to the right, is characteristic of elliptic behavior near the origin. But note the hyperbolic flow about the points at $\theta(=x) = \pm\pi$, $p_\theta(=y) = 0$.

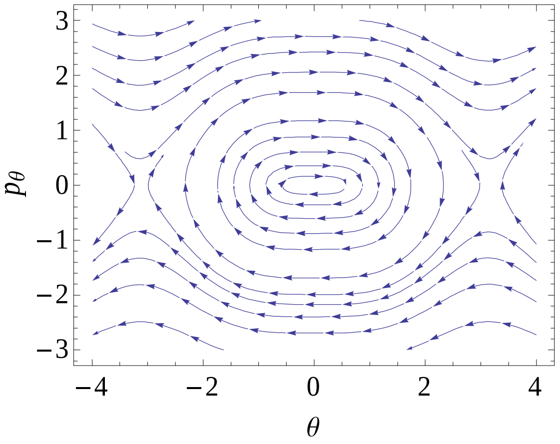


Phase Portrait (prepared in Mma with StreamPlot)

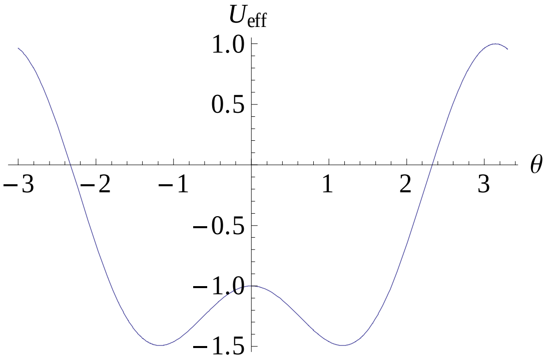
Precisely at the point $\Omega = \omega_\theta$ the three equilibria at $\theta_0 = 0, \pm\cos^{-1}\left(\omega_\theta^2/\Omega^2\right)$ coincide at $\theta_0 = 0$ and the linear term in the restoring force (the velocity field) vanishes. To understand the behavior of the system at this point we must expand the potential to higher powers. As indicated in the figure to the right the potential is now dominated by the quartic terms, *i.e.*, the potential is much flatter near the origin. While the behavior about thus equilibrium is no longer harmonic, it is still stable as suggested by the elliptic flow in the next figure.



Phase Portrait

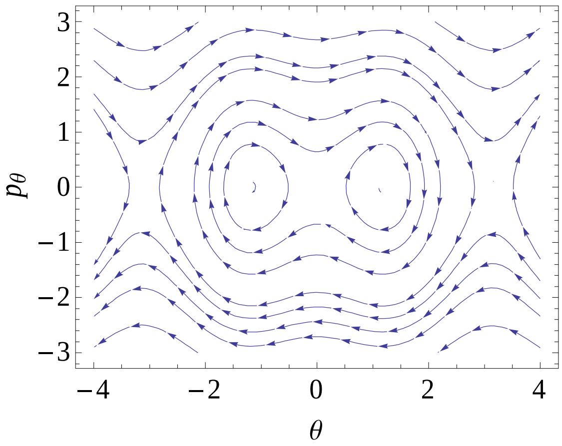


For $\Omega > \omega_\theta$ the system has 4 equilibria, of which the ones at $\theta = 0, \pi$ are now both hyperbolic (unstable) and the 2 at $\theta_0 = \pm\cos^{-1}\left(\omega_\theta^2/\Omega^2\right)$ are elliptic (stable). The figure at the right indicates the corresponding effective potential with the new minima displaced from the origin (but note that the symmetry about $\theta = 0$ is still present).



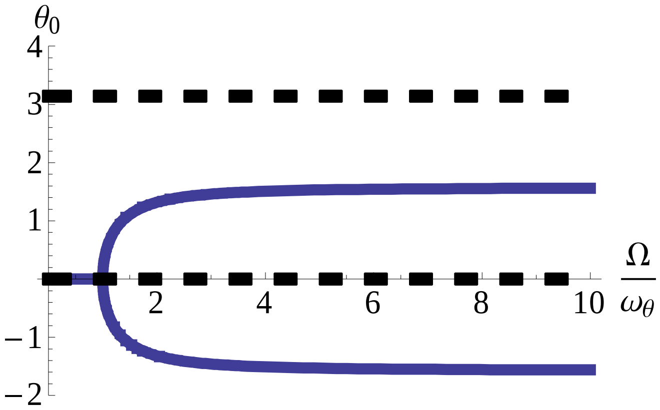
The corresponding phase portrait is shown in the next figure. Note the areas of elliptic flow and the areas of hyperbolic flow.

Phase Portrait



Once the system is at one of these stable equilibria the effective potential expanded around this point is no longer (fully) symmetric, and there is spontaneous symmetric breaking.

The equilibria as a function of Ω/ω_θ look like (solid = stable, dashed = unstable)



"pitchfork" diagram