Physics 505 - Autumn 2010

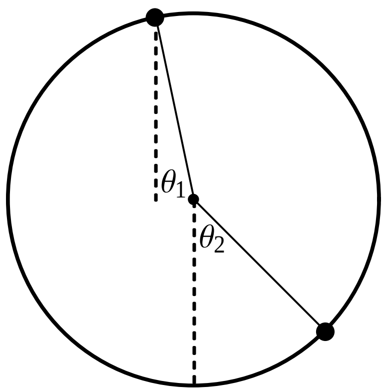
HW V Solutions

11/3/10

NOTE: The MidTerm Exam is Friday 11/5/10

Overview: Recall that solving physics problems is not (just) about solving differential equations. Use physical reasoning to help solve the following exercises and be certain to show your work. It is also important that you practice completely solving these exercises, checking for errors as you go along.

1) Fetter & Walecka – 4.1 (9 pts) Here we expand our study of small oscillations to include the full normal mode formalism.



a) (3 pts) We use the two angles in the figure as the two coordinates, where we assume that the angles remain small, *i.e.*, that we only need to express the kinetic and potential energies to second order in the angles. (Note that both angles are measured from the vertical.) The kinetic energy of the hoop is easily expressed in terms of the motion of the CM and the motion about the CM, $I = MR^2$,

$$T_{hoop} = \frac{M}{2}\left(R\dot{\theta}_1\right)^2 + \frac{1}{2}\left(MR^2\right)\dot{\theta}_1^2 = MR^2\dot{\theta}_1^2.$$

For the motion of the mass on the hoop (the bead) the small angle limit is important as the vertical motion is of order (small angle)^4 and higher and can be ignored. So to quadratic order we have just the horizontal motion that can be expressed as

$$T_{mass} \simeq \frac{M}{2}\left[\left(R\dot{\theta}_1\right) + \left(R\dot{\theta}_2\right)\right]^2 = \frac{MR^2}{2}\left[\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\right].$$

So in this system the coupling of the 2 harmonic oscillators comes from the form of the kinetic energy. With the gravitational energy chosen to be zero at the lowest point we have (recall $\cos\theta \simeq 1-\theta^2/2$ for $\theta \ll 1$)

$$U_{hoop} \simeq \frac{MgR}{2}\theta_1^2,$$
$$U_{mass} \simeq \frac{MgR}{2}\left(\theta_1^2+\theta_2^2\right).$$

In terms of the notation for small oscillations we define $\eta_k = R\theta_k$ and write

$$L = T - U = \frac{M}{2}\left(3\dot{\eta}_1^2 + 2\dot{\eta}_1\dot{\eta}_2 + \dot{\eta}_2^2\right) - \frac{Mg}{2R}\left(2\eta_1^2+\eta_2^2\right)$$
$$\Rightarrow \begin{cases} \eta_1 : M\left(3\ddot{\eta}_1 + \ddot{\eta}_2\right) + 2\dfrac{Mg}{R}\eta_1 = 0 \Rightarrow 3\ddot{\eta}_1 + \ddot{\eta}_2 + 2\dfrac{g}{R}\eta_1 = 0 \\ \eta_2 : M\left(\ddot{\eta}_1 + \ddot{\eta}_2\right) + \dfrac{Mg}{R}\eta_2 = 0 \Rightarrow \ddot{\eta}_1 + \ddot{\eta}_2 + \dfrac{g}{R}\eta_2 = 0 \end{cases}.$$

These equations of motion verify that the point $\eta_1 = \eta_2 = 0$ is an equilibrium point and further that it is a stable equilibrium point, since the first order forces (in η_k) are restoring forces. However, since the η_k modes are coupled (by the kinetic energy), we want to use the general normal mode formalism. From the above equations of motion we can identify the two matrices describing this coupling of modes,

$$m_{kl} = M\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}_{kl}, v_{kl} = \frac{gM}{R}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}_{kl}.$$

Thus the standard eigenvalue equation yields the desired frequencies $\left(\ddot{\eta} = -\omega^2\eta = -\lambda\eta\right)$

$$\det\left[v-\lambda m\right]=\det\begin{pmatrix}\frac{2gM}{R}-3\lambda M & 0-\lambda M\\ 0-\lambda M & \frac{gM}{R}-\lambda M\end{pmatrix}=0$$

$$\Rightarrow 2\frac{g^2M^2}{R^2}-5\lambda\frac{gM^2}{R}+3\lambda^2M^2-\lambda^2M^2=0\Rightarrow\lambda_\pm=\omega_\pm^2=\frac{g}{R}\left(\frac{5}{4}\pm\frac{\sqrt{25-16}}{4}\right)$$

$$\Rightarrow\omega_+=\omega_2=\sqrt{\frac{2g}{R}},\quad\omega_-=\omega_1=\sqrt{\frac{g}{2R}}.$$

b) (3 pts) We can construct the normal eigenvectors in the usual way. We find

$$(v-\lambda_k m)\eta_k=0\Rightarrow\frac{gM}{2R}\begin{pmatrix}4-3 & -1\\ -1 & 2-1\end{pmatrix}\begin{pmatrix}\eta_1^{(1)}\\ \eta_2^{(1)}\end{pmatrix}=0\Rightarrow\begin{pmatrix}\eta_1^{(1)}\\ \eta_2^{(1)}\end{pmatrix}\propto\begin{pmatrix}1\\ 1\end{pmatrix}$$

$$\Rightarrow\frac{gM}{R}\begin{pmatrix}2-6 & -2\\ -2 & 1-2\end{pmatrix}\begin{pmatrix}\eta_1^{(2)}\\ \eta_2^{(2)}\end{pmatrix}=0\Rightarrow\begin{pmatrix}\eta_1^{(2)}\\ \eta_2^{(2)}\end{pmatrix}\propto\begin{pmatrix}1\\ -2\end{pmatrix}.$$

The standard normalization, $\eta_j^\dagger m\eta_j=1$, is

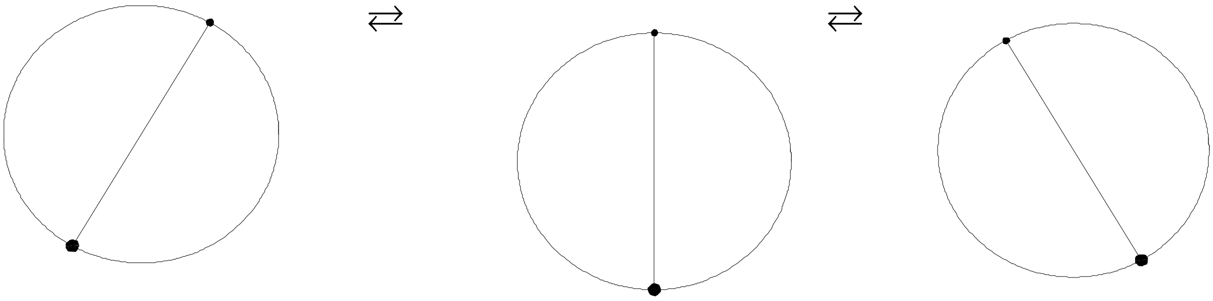
$$\begin{pmatrix}\eta_1^{(1)}\\ \eta_2^{(1)}\end{pmatrix}=\alpha_1\begin{pmatrix}1\\ 1\end{pmatrix}\Rightarrow\alpha_1^2M\begin{pmatrix}1\\ 1\end{pmatrix}^T\begin{pmatrix}3 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix}=\alpha_1^2M\begin{pmatrix}1\\ 1\end{pmatrix}^T\begin{pmatrix}4\\ 2\end{pmatrix}=6\alpha_1^2M=1$$

$$\Rightarrow\begin{pmatrix}\eta_1^{(1)}\\ \eta_2^{(1)}\end{pmatrix}=\frac{1}{\sqrt{6M}}\begin{pmatrix}1\\ 1\end{pmatrix},$$

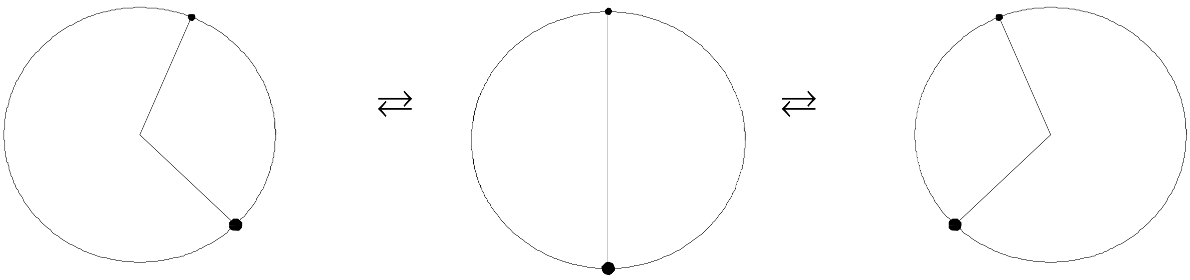
$$\begin{pmatrix}\eta_1^{(2)}\\ \eta_2^{(2)}\end{pmatrix}=\alpha_2\begin{pmatrix}1\\ 2\end{pmatrix}\Rightarrow\alpha_2^2M\begin{pmatrix}1\\ -2\end{pmatrix}^T\begin{pmatrix}3 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1\\ -2\end{pmatrix}=\alpha_2^2M\begin{pmatrix}1\\ -2\end{pmatrix}^T\begin{pmatrix}1\\ -1\end{pmatrix}=3\alpha_2^2M=1$$

$$\Rightarrow\begin{pmatrix}\eta_1^{(2)}\\ \eta_2^{(2)}\end{pmatrix}=\frac{1}{\sqrt{3M}}\begin{pmatrix}1\\ -2\end{pmatrix}.$$

Thus the first eigenmode, with the smaller frequency, is characterized by η_1 and η_2 (and thus θ_1 and θ_2) being of the same magnitude and in phase. As indicated in the set of figures, this means that the point mass stays at the point on the hoop antipodal to the suspension point and the hoop/mass system oscillates as a single object.



For the higher frequency mode η_2 (θ_2) has twice the magnitude of η_1 (θ_1) and is exactly out of phase as indicated in the second set of figures.



c) (1 pt) The corresponding modal matrix has the form

$$N = \begin{pmatrix} \eta_1^{(1)} \eta_1^{(2)} \\ \eta_2^{(1)} \eta_2^{(2)} \end{pmatrix} = \frac{1}{\sqrt{6M}} \begin{pmatrix} 1 & \sqrt{2} \\ 1 & -2\sqrt{2} \end{pmatrix},$$

$$N^T = \frac{1}{\sqrt{6M}} \begin{pmatrix} 1 & 1 \\ \sqrt{2} & -2\sqrt{2} \end{pmatrix},$$

$$N^{-1} = N^T m = \sqrt{\frac{M}{6}} \begin{pmatrix} 1 & 1 \\ \sqrt{2} & -2\sqrt{2} \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} = \sqrt{\frac{M}{3}} \begin{pmatrix} 2\sqrt{2} & \sqrt{2} \\ 1 & -1 \end{pmatrix}.$$

d) (2 pts) The normal coordinates can be obtained from

$$\varsigma = N^{-1}\eta = N^T m\eta \Rightarrow \varsigma = \begin{pmatrix} \varsigma_1 \\ \varsigma_2 \end{pmatrix} = \sqrt{\frac{M}{3}} \begin{pmatrix} 2\sqrt{2} & \sqrt{2} \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$
$$= \sqrt{\frac{M}{3}} \begin{pmatrix} \sqrt{2}\left(2\eta_1 + \eta_2\right) \\ \eta_1 - \eta_2 \end{pmatrix}.$$

Note in particular that ς_1 vanishes if $\eta_1 = -2\eta_2$, *i.e.*, the second eigenmode defined above, and ς_2 vanishes if $\eta_1 = \eta_2$, *i.e.*, the first eigenmode. Expressed the other way around we have

$$\eta = N\varsigma \Rightarrow \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \frac{1}{\sqrt{6M}} \begin{pmatrix} 1 & \sqrt{2} \\ 1 & -2\sqrt{2} \end{pmatrix} \begin{pmatrix} \varsigma_1 \\ \varsigma_2 \end{pmatrix} = \frac{1}{\sqrt{6M}} \begin{pmatrix} \varsigma_1 + \sqrt{2}\varsigma_2 \\ \varsigma_1 - 2\sqrt{2}\varsigma_2 \end{pmatrix}.$$

We can substitute this result in the Lagrangian and verify that it is diagonal,

$$\frac{M}{2}\left(3\dot{\eta}_1^2 + 2\dot{\eta}_1\dot{\eta}_2 + \dot{\eta}_2^2\right) - \frac{Mg}{2R}\left(2\eta_1^2 + \eta_2^2\right) \Rightarrow$$
$$\frac{M}{2}\frac{1}{6M}\left[3\left(\dot{\varsigma}_1 + \sqrt{2}\dot{\varsigma}_2\right)^2 + 2\left(\dot{\varsigma}_1 + \sqrt{2}\dot{\varsigma}_2\right)\left(\dot{\varsigma}_1 - 2\sqrt{2}\dot{\varsigma}_2\right) + \left(\dot{\varsigma}_1 - 2\sqrt{2}\dot{\varsigma}_2\right)^2\right]$$
$$-\frac{Mg}{2R}\frac{1}{6M}\left[2\left(\varsigma_1 + \sqrt{2}\varsigma_2\right)^2 + \left(\varsigma_1 - 2\sqrt{2}\varsigma_2\right)^2\right]$$
$$= \frac{1}{12}\left[6\dot{\varsigma}_1^2 + 6\dot{\varsigma}_2^2\right] - \frac{g}{12R}\left[3\varsigma_1^2 + 12\varsigma_2^2\right]$$
$$= \frac{1}{2}\left[\dot{\varsigma}_1^2 - \omega_1^2\varsigma_1^2\right] + \frac{1}{2}\left[\dot{\varsigma}_2^2 - \omega_2^2\varsigma_2^2\right].$$

Clearly the problem has split into 2 independent harmonic oscillators at the expected frequencies. If we assume initial conditions of the form $\theta_1(0) = \dot{\theta}_1(0) = \dot{\theta}_2(0) = 0,\ \theta_2(0) = \theta_0$, we have

$$\varsigma(t)=\sqrt{\frac{M}{3}}R\begin{pmatrix}\sqrt{2}\theta_0\cos\sqrt{\frac{g}{2R}}t\\ -\theta_0\cos\sqrt{\frac{2g}{R}}t\end{pmatrix},$$

$$\eta(t)=\frac{R\theta_0}{3}\begin{pmatrix}\cos\sqrt{\frac{g}{2R}}t-\cos\sqrt{\frac{2g}{R}}t\\ \cos\sqrt{\frac{g}{2R}}t+2\cos\sqrt{\frac{2g}{R}}t\end{pmatrix}.$$

2) Fetter & Walecka – 4.3 (9 pts) The double pendulum problem is a classic normal mode study and we should solve it.

Solution: We consider the situation in Fig 13.1 c with different masses on the pendulums, m_1 and m_2 in the small angle limit. The transverse displacement of the first mass is $\eta_1 = l\theta_1$ and that of the second with respect to the first is $\eta_2 = l\theta_2$, for pendulums of equal length *l*. Thus the second mass is displaced by a total distance $\eta_1+\eta_2$.

a) (1 pt) Thus we can write down the kinetic and potential energies and the Lagrangian, ($U = mgl(1-\cos\theta)\simeq(mgl/2)\theta^2=(mg/2l)\eta^2$)

$$T=\frac{m_1}{2}\dot{\eta}_1^2+\frac{m_2}{2}\left(\dot{\eta}_1+\dot{\eta}_2\right)^2, U=\frac{m_1 g}{2l}\eta_1^2+\frac{m_2 g}{2l}\left(\eta_1^2+\eta_2^2\right)$$
$$\Rightarrow L=T=\frac{m_1}{2}\dot{\eta}_1^2+\frac{m_2}{2}\left(\dot{\eta}_1+\dot{\eta}_2\right)^2-\frac{m_1 g}{2l}\eta_1^2-\frac{m_2 g}{2l}\left(\eta_1^2+\eta_2^2\right).$$

b) (1 pt) Thus we can the metric and the potential matrix as

$$m_{kl}=\begin{pmatrix}m_1+m_2 & m_2\\ m_2 & m_2\end{pmatrix}_{kl}, v_{kl}=\frac{g}{l}\begin{pmatrix}m_1+m_2 & 0\\ 0 & m_2\end{pmatrix}.$$

We obtain the desired eigenvalues from the standard eigenvalue equation

$$\det\left[v-\lambda m\right]=\det\begin{pmatrix}\frac{g\left(m_1+m_2\right)}{l}-\lambda\left(m_1+m_2\right) & -\lambda m_2 \\ -\lambda m_2 & \frac{gm_2}{l}-\lambda m_2\end{pmatrix}=0$$

$$\Rightarrow \frac{g^2 m_2\left(m_1+m_2\right)}{l^2}-2\lambda\frac{gm_2\left(m_1+m_2\right)}{l}+\lambda^2 m_2\left(m_1+m_2\right)-\lambda^2 m_2^2=0$$

$$\Rightarrow \lambda\ =\omega^2=\frac{g}{l}\left(\frac{2m_2\left(m_1+m_2\right)}{2m_1m_2}\pm\frac{\sqrt{4\left(m_2\left(m_1+m_2\right)\right)^2-4m_1m_2^2\left(m_1+m_2\right)}}{2m_1m_2}\right)$$

$$=\frac{g}{lm_1}\left(m_1+m_2\pm\sqrt{m_2^2+m_1m_2}\right)=\frac{g\left(m_1+m_2\right)}{lm_1}\left(1\pm\sqrt{\frac{m_2}{m_1+m_2}}\right)$$

$$\Rightarrow \omega_{\mp}^2\equiv\frac{gm_2}{lm_1}\gamma^{-2}\left(1\pm\gamma\right)=\frac{gm_2}{lm_1}\gamma^{-2}\frac{1-\gamma^2}{1\mp\gamma}=\frac{g}{l}\frac{1}{1\mp\gamma}:\gamma=\sqrt{\frac{m_2}{m_1+m_2}}.$$

c) (2 pts) We obtain the normal mode eigenvectors from the usual equations

$$\left(v-\lambda_k m\right)\eta_k=0\Rightarrow\frac{g}{l}\begin{pmatrix}\left(m_1+m_2\right)-\frac{m_1+m_2}{1\pm\gamma} & -\frac{m_2}{1\pm\gamma} \\ -\frac{m_2}{1\pm\gamma} & m_2-\frac{m_2}{1\pm\gamma}\end{pmatrix}\begin{pmatrix}\eta_1^{(\pm)} \\ \eta_2^{(\pm)}\end{pmatrix}$$

$$=\frac{g}{l\left(1\pm\gamma\right)}\begin{pmatrix}\pm\gamma\left(m_1+m_2\right) & -m_2 \\ -m_2 & \pm\gamma m_2\end{pmatrix}\begin{pmatrix}\eta_1^{(\pm)} \\ \eta_2^{(\pm)}\end{pmatrix}=0\Rightarrow\begin{pmatrix}\eta_1^{(\pm)} \\ \eta_2^{(\pm)}\end{pmatrix}\propto\begin{pmatrix}\pm\gamma \\ 1\end{pmatrix}.$$

To test this result consider the limit $m_1 \ll m_2, \gamma \to 1$. In this limit we have essentially just one pendulum of length 2*l* and, as expected $\omega_+ \to \sqrt{g/2l}$ and $\omega_- \to \infty$. The former eigenfunction corresponds to the 2 pendulums having the same phase and amplitude (as expected for essentially a single pendulum), while in the second mode the two pendulums are out of phase. In the opposite limit $m_1 \gg m_2, \gamma \to 0$ the two eigenfrequencies are nearly degenerate at the naive value $\sqrt{g/l}$ and the motion of mass 2 is much larger than mass 1, either in or out of phase.

d) (2 pts) We normalize in the usual way, $\eta_j^\dagger m \eta_j = 1$, to find

$$\begin{pmatrix}\eta_1^{(\pm)}\\ \eta_2^{(\pm)}\end{pmatrix} = \alpha_\pm \begin{pmatrix}\pm\gamma\\ 1\end{pmatrix} \Rightarrow \alpha_\pm^2 \begin{pmatrix}\pm\gamma\\ 1\end{pmatrix}^T \begin{pmatrix} m_1+m_2 & m_2\\ m_2 & m_2\end{pmatrix}\begin{pmatrix}\pm\gamma\\ 1\end{pmatrix}$$

$$= \alpha_\pm^2 \begin{pmatrix}\pm\gamma\\ 1\end{pmatrix}^T \begin{pmatrix}\pm\gamma\left(m_1+m_2\right)+m_2\\ \pm\gamma m_2 + m_2\end{pmatrix} = \alpha_\pm^2 m_2\left(2\pm 2\gamma\right) = 1$$

$$\Rightarrow \begin{pmatrix}\eta_1^{(\pm)}\\ \eta_2^{(\pm)}\end{pmatrix} = \frac{1}{\sqrt{2m_2\left(1\pm\gamma\right)}}\begin{pmatrix}\pm\gamma\\ 1\end{pmatrix}.$$

Hence the modal matrix takes the form (using $\sqrt{m_2/m_1} = \gamma\big/\sqrt{1-\gamma^2}$)

$$N = \begin{pmatrix}\eta_1^{(+)}\,\eta_1^{(-)}\\ \eta_2^{(+)}\,\eta_2^{(-)}\end{pmatrix} = \frac{1}{\sqrt{2m_2}}\begin{pmatrix}\frac{\gamma}{\sqrt{1+\gamma}} & \frac{-\gamma}{\sqrt{1-\gamma}}\\ \frac{1}{\sqrt{1+\gamma}} & \frac{1}{\sqrt{1-\gamma}}\end{pmatrix} = \frac{1}{\sqrt{2m_1}}\begin{pmatrix}\sqrt{1-\gamma} & -\sqrt{1+\gamma}\\ \frac{\sqrt{1-\gamma}}{\gamma} & \frac{\sqrt{1+\gamma}}{\gamma}\end{pmatrix},$$

$$N^T = \frac{1}{\sqrt{2m_1}}\begin{pmatrix}\sqrt{1-\gamma} & \frac{\sqrt{1-\gamma}}{\gamma}\\ -\sqrt{1+\gamma} & \frac{\sqrt{1+\gamma}}{\gamma}\end{pmatrix},$$

$$N^{-1} = N^T m = \frac{1}{\sqrt{2m_1}}\begin{pmatrix}\sqrt{1-\gamma} & \frac{\sqrt{1-\gamma}}{\gamma}\\ -\sqrt{1+\gamma} & \frac{\sqrt{1+\gamma}}{\gamma}\end{pmatrix}\begin{pmatrix}m_1+m_2 & m_2\\ m_2 & m_2\end{pmatrix}$$

$$= \sqrt{\frac{m_1}{2}}\begin{pmatrix}\sqrt{1-\gamma} & \frac{\sqrt{1-\gamma}}{\gamma}\\ -\sqrt{1+\gamma} & \frac{\sqrt{1+\gamma}}{\gamma}\end{pmatrix}\begin{pmatrix}\frac{1}{1-\gamma^2} & \frac{\gamma^2}{1-\gamma^2}\\ \frac{\gamma^2}{1-\gamma^2} & \frac{\gamma^2}{1-\gamma^2}\end{pmatrix} = \sqrt{\frac{m_1}{2}}\begin{pmatrix}\frac{1}{\sqrt{1-\gamma}} & \frac{\gamma}{\sqrt{1-\gamma}}\\ \frac{-1}{\sqrt{1+\gamma}} & \frac{\gamma}{\sqrt{1+\gamma}}\end{pmatrix}.$$

Thus the rotated versions of the initial matrices are

$$N^T m N = \sqrt{\frac{m_1}{2}} \begin{pmatrix} \frac{1}{\sqrt{1-\gamma}} & \frac{\gamma}{\sqrt{1-\gamma}} \\ \frac{-1}{\sqrt{1+\gamma}} & \frac{\gamma}{\sqrt{1+\gamma}} \end{pmatrix} \frac{1}{\sqrt{2m_1}} \begin{pmatrix} \sqrt{1-\gamma} & -\sqrt{1+\gamma} \\ \frac{\sqrt{1-\gamma}}{\gamma} & \frac{\sqrt{1+\gamma}}{\gamma} \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$N^T v N = N^T = \frac{1}{\sqrt{2m_1}} \begin{pmatrix} \sqrt{1-\gamma} & \frac{\sqrt{1-\gamma}}{\gamma} \\ -\sqrt{1+\gamma} & \frac{\sqrt{1+\gamma}}{\gamma} \end{pmatrix} \frac{g}{l} \begin{pmatrix} m_1+m_2 & 0 \\ 0 & m_2 \end{pmatrix} \frac{1}{\sqrt{2m_1}} \begin{pmatrix} \sqrt{1-\gamma} & -\sqrt{1+\gamma} \\ \frac{\sqrt{1-\gamma}}{\gamma} & \frac{\sqrt{1+\gamma}}{\gamma} \end{pmatrix}$$
$$= \frac{g}{2l} \begin{pmatrix} \sqrt{1-\gamma} & \frac{\sqrt{1-\gamma}}{\gamma} \\ -\sqrt{1+\gamma} & \frac{\sqrt{1+\gamma}}{\gamma} \end{pmatrix} \begin{pmatrix} \frac{1}{1-\gamma^2} & 0 \\ 0 & \frac{\gamma^2}{1-\gamma^2} \end{pmatrix} \begin{pmatrix} \sqrt{1-\gamma} & -\sqrt{1+\gamma} \\ \frac{\sqrt{1-\gamma}}{\gamma} & \frac{\sqrt{1+\gamma}}{\gamma} \end{pmatrix}$$
$$= \frac{g}{2l} \begin{pmatrix} \sqrt{1-\gamma} & \frac{\sqrt{1-\gamma}}{\gamma} \\ -\sqrt{1+\gamma} & \frac{\sqrt{1+\gamma}}{\gamma} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{1-\gamma}}{1-\gamma^2} & \frac{-\sqrt{1+\gamma}}{1-\gamma^2} \\ \frac{\gamma\sqrt{1-\gamma}}{1-\gamma^2} & \frac{\gamma\sqrt{1+\gamma}}{1-\gamma^2} \end{pmatrix} = \frac{g}{2l} \begin{pmatrix} \frac{2}{1+\gamma} & 0 \\ 0 & \frac{2}{1-\gamma} \end{pmatrix}$$
$$= \begin{pmatrix} \omega_+^2 & 0 \\ 0 & \omega_-^2 \end{pmatrix}.$$

As desired, the matrices are diagonal.

e) (1 pt) The normal coordinates have the form

$$\varsigma = N^{-1}\eta = N^T m\eta \Rightarrow \varsigma = \begin{pmatrix} \varsigma_+ \\ \varsigma_- \end{pmatrix} = \sqrt{\frac{m_1}{2}} \begin{pmatrix} \frac{1}{\sqrt{1-\gamma}} & \frac{\gamma}{\sqrt{1-\gamma}} \\ \frac{-1}{\sqrt{1+\gamma}} & \frac{\gamma}{\sqrt{1+\gamma}} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

$$= \sqrt{\frac{m_1}{2}} \begin{pmatrix} \frac{\eta_1 + \gamma\eta_2}{\sqrt{1-\gamma}} \\ \frac{-\eta_1 + \gamma\eta_2}{\sqrt{1+\gamma}} \end{pmatrix}.$$

f) (2 pts) Finally for $m_1 \gg m_2, \gamma \to 0$ and initial conditions $\eta_1(0) = \delta$, $\dot{\eta}_1(0) = \eta_2(0) = \dot{\eta}_2(0) = 0$ we see that the motion of the normal coordinates is given by (keeping terms to order γ)

$$\begin{pmatrix} \varsigma_+(t) \\ \varsigma_-(t) \end{pmatrix} = \sqrt{\frac{m_1}{2}} \begin{pmatrix} \frac{\delta \cos\omega_+ t}{\sqrt{1-\gamma}} \\ \frac{-\delta \cos\omega_- t}{\sqrt{1+\gamma}} \end{pmatrix} \simeq \sqrt{\frac{m_1}{2}} \begin{pmatrix} \delta\left(1+\frac{\gamma}{2}\right)\cos\sqrt{\frac{g}{l}}\left(1-\frac{\gamma}{2}\right)t \\ -\delta\left(1-\frac{\gamma}{2}\right)\cos\sqrt{\frac{g}{l}}\left(1+\frac{\gamma}{2}\right)t \end{pmatrix},$$

and

$$\eta = N\varsigma \Rightarrow \begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} = \frac{1}{\sqrt{2m_1}}\begin{pmatrix}\sqrt{1-\gamma} & -\sqrt{1+\gamma}\\ \dfrac{\sqrt{1-\gamma}}{\gamma} & \dfrac{\sqrt{1+\gamma}}{\gamma}\end{pmatrix}\begin{pmatrix}\varsigma_+\\ \varsigma_-\end{pmatrix} =$$

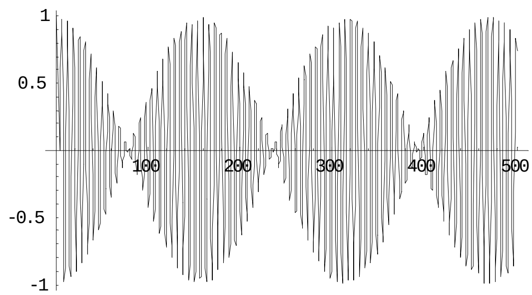
$$= \frac{1}{\sqrt{2m_1}}\begin{pmatrix}\left(1-\dfrac{\gamma}{2}\right)\varsigma_+ - \left(1+\dfrac{\gamma}{2}\right)\varsigma_-\\ \left(\dfrac{1}{\gamma}-\dfrac{1}{2}\right)\varsigma_+ + \left(\dfrac{1}{\gamma}+\dfrac{1}{2}\right)\varsigma_-\end{pmatrix}$$

$$\simeq \frac{\delta}{2}\begin{pmatrix}\left(1-\dfrac{\gamma}{2}\right)\left(1+\dfrac{\gamma}{2}\right)\cos\sqrt{\dfrac{g}{l}}\left(1-\dfrac{\gamma}{2}\right)t + \left(1+\dfrac{\gamma}{2}\right)\left(1-\dfrac{\gamma}{2}\right)\cos\sqrt{\dfrac{g}{l}}\left(1+\dfrac{\gamma}{2}\right)t\\ \left(\dfrac{1}{\gamma}-\dfrac{1}{2}\right)\left(1+\dfrac{\gamma}{2}\right)\cos\sqrt{\dfrac{g}{l}}\left(1-\dfrac{\gamma}{2}\right)t - \left(\dfrac{1}{\gamma}+\dfrac{1}{2}\right)\left(1-\dfrac{\gamma}{2}\right)\cos\sqrt{\dfrac{g}{l}}\left(1+\dfrac{\gamma}{2}\right)t\end{pmatrix}$$

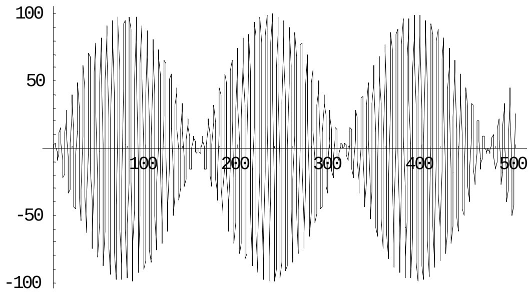
$$\simeq \delta\begin{pmatrix}\cos\sqrt{\dfrac{g}{l}}t\cos\sqrt{\dfrac{g}{l}}\dfrac{\gamma t}{2}\\ \dfrac{1}{\gamma}\sin\sqrt{\dfrac{g}{l}}t\sin\sqrt{\dfrac{g}{l}}\dfrac{\gamma t}{2}\end{pmatrix}.$$

The structure of this solution is very similar to the example discussed in Lecture 8. There is a fast oscillation ($\omega_F = \sqrt{g/l}$) modulated (*i.e.*, there are beats) by a slow oscillation ($\omega_S = \gamma\sqrt{g/l}/2$) with both oscillations being out of phase for the two modes. The energy in the system is passed back and forth between the two pendulums. Due to the large difference in the masses the amplitudes of the two oscillations are quite different. For $\gamma = 1/100$ this looks like

$\eta_1(t)$



$\eta_2(t)$



3) Fetter & Walecka – 4.4 (7 pts) Here is a chance to return to a system we have already analyzed and study it more thoroughly.

<u>Solution:</u> (a) (2 pts) From our previous consideration of exercise 3.1 in F&W we have

$$T = \frac{m}{2}\left[\left(a\dot{\theta}\right)^2 + \left(a\sin\theta\,\Omega\right)^2\right], V = -mga\cos\theta,$$
$$L = \frac{m}{2}\left[\left(a\dot{\theta}\right)^2 + \left(a\sin\theta\,\Omega\right)^2\right] + mga\cos\theta.$$

Lagrange gives the equation of motion as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = ma^2\ddot{\theta} - ma^2\Omega^2\sin\theta\cos\theta + mga\sin\theta = 0.$$

With the usual integrating factor $\dot{\theta}$ the equation of motion can be integrated once to yield the following first integral

$$\int_0^t dt\dot{\theta}\left[ma^2\ddot{\theta} - ma^2\Omega^2\sin\theta\cos\theta + mga\sin\theta\right]$$
$$= \frac{ma^2}{2}\dot{\theta}^2 - \frac{ma^2\sin^2\theta}{2}\Omega^2 - mga\cos\theta + \text{constant} = 0$$
$$\Rightarrow \frac{ma^2}{2}\dot{\theta}^2 - \frac{ma^2\sin^2\theta}{2}\Omega^2 - mga\cos\theta = -\text{constant}.$$

(b) (3 pts) To find the positions of dynamic equilibrium we look for where $\ddot{\theta} = 0$.

From the equation of motion we want

$$ma^2\Omega^2\sin\theta\cos\theta = mga\sin\theta \Rightarrow \begin{cases} \sin\theta_0 = 0 \\ \cos\theta_0 = \dfrac{g}{a\Omega^2} \end{cases}.$$

Thus we have equilibrium at $\theta_0 = 0, \pi, \pm\cos^{-1}\left(g/a\Omega^2\right)$. You may find the use of the plus/minus signs here peculiar. In that case think of these two equilibrium points in terms of true spherical coordinates, $\theta_0 = \cos^{-1}\left(g/a\Omega^2\right), \phi_0 = 0, \pi$. There are two equilibrium points on opposite sides of the hoop. Note that the quantity $\sqrt{g/a}$ defines the natural frequency of this system and we might as well give it a name $\omega_\theta \equiv \sqrt{g/a}$ as we did the first time we analyzed this system.

We can determine the stability of the equilibrium points by considering the second derivative of the Lagrangian at these points, *i.e.*, the coefficient of the linear term in the expansion of the right hand side of the equation of motion with respect to small oscillation around the equilibrium point. We have

$$\begin{aligned} v &\propto \frac{\partial^2 L}{\partial\theta^2} = ma^2\Omega^2\left[\cos^2\theta - \sin^2\theta\right] - mga\cos\theta, \\ &\Rightarrow \ddot{\theta} \simeq \frac{v}{ma^2}\theta. \end{aligned}$$

The question of stability than corresponds to the question of the sign of the right-hand-side of this equation. A negative sign implies stability while a positive sign means instability. For the 3 equilibrium points we have, where the oscillation frequency is indicated for the stable cases,

$$\begin{aligned} &\theta_0 = 0 : ma^2\Omega^2 - mga = ma^2\Omega^2\left(1 - \frac{\omega_\theta^2}{\Omega^2}\right) \\ &\Rightarrow \begin{cases} \text{stable} : \Omega < \omega_\theta,\ \omega_{osc} = \omega_\theta\sqrt{1 - \dfrac{\Omega^2}{\omega_\theta^2}}, \\ \text{unstable} : \Omega > \omega_\theta \end{cases} \end{aligned}$$

$$\theta_0 = \pi : ma^2\Omega^2 + mga = ma^2\Omega^2\left(1 + \frac{\omega_\theta^2}{\Omega^2}\right) > 0$$
$$\Rightarrow \text{unstable},$$

and

$$\begin{matrix}\theta_0 = \cos^{-1}\left(\omega_\theta^2/\Omega^2\right) \\ \phi_0 = 0, \pi\end{matrix} : ma^2\Omega^2\left(2\frac{\omega_\theta^4}{\Omega^4} - 1\right) - \frac{mg^2}{\Omega^2} = ma^2\Omega^2\left(\frac{\omega_\theta^4}{\Omega^4} - 1\right)$$
$$\Rightarrow \begin{cases} \text{stable} : \Omega > \omega_\theta, \, \omega_{osc} = \Omega\sqrt{1 - \frac{\omega_\theta^4}{\Omega^4}}. \\ \text{unstable} : \Omega < \omega_\theta \end{cases}$$

We see that in the limit $\Omega \ll \omega_\theta$ only the case $\theta_0 = 0$ is stable, which is just the usual pendulum $\omega_{osc} = \omega_\theta$ and the spinning does not matter. In the other limit $\Omega \gg \omega_\theta$ only the nontrivial solution $\theta_0 = \cos^{-1}\left(\omega_\theta^2/\Omega^2\right) \to \cos^{-1}(0) = \pi/2$ is stable with $\omega_{osc} = \Omega$.

As we saw in our study of exercise 3.1 the occurrence of an equilibrium point, as seen in the rotating frame, corresponds to the vanishing of the force tangential to the loop (in which direction there can be no constraint force). The two contributions to this force (in the $\hat{\theta}$ direction) arise from gravity ($mg\left(-\sin\theta\hat{\theta} + \cos\theta\hat{r}\right)$) downward and the upward tangential component of the centrifugal force ($m\vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right) = m\Omega^2 a\sin\theta\left(\cos\theta\hat{\theta} + \sin\theta\hat{r}\right)$). Thus the tangential force vanishes when $m\Omega^2 a\sin\theta\cos\theta - mg\sin\theta = 0$ as noted above.

(c) (2 pts) From part (a) we find the Hamiltonian via

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = ma^2\dot{\theta}$$
$$\Rightarrow H = p_\theta\dot{\theta} - L = \frac{ma^2}{2}\dot{\theta}^2 - \frac{ma^2\sin^2\theta}{2} - mga\cos\theta.$$

This is the first integral of part (a) and thus is a constant of the motion. However, it is not the total energy,

$$E = \frac{ma^2}{2}\dot{\theta}^2 + \frac{ma^2 \sin^2\theta}{2}\Omega^2 - mga\cos\theta,$$

which is not a constant of the motion due to the work done by the constraint forces, *i.e.*, the constraint is *time dependent*.