Physics 505 - Autumn 2010

HW X

Due 12/8/10

Overview: Recall that solving physics problems is not (just) about solving differential equations. Use physical reasoning to help solve the following exercises and be certain to show your work. It is also important that you practice completely solving these exercises, checking for errors as you go along.

1) (8 pts) Consider a mechanical system described by the following equation of motion (often labeled the Duffing oscillator, including in the general case also a linear term)

$$\ddot{x} + \gamma \dot{x} + x^3 = f \cos t.$$

This system exhibits chaotic behavior in certain ranges of the parameters and we want to explore this behavior numerically (using Mathematica or the Runge-Kutta routine we have already practiced with). Use initial conditions $\left(x(0), \dot{x}(0)\right) = (0,0)$ and investigate the following two parameter regimes:

a) (5 pts) Take $\gamma = 0.2$ and $f = 10$. Evaluate and plot $x(t)$ in the range $0 \le t \le 30\pi$. You should find that the asymptotic behavior is periodic. What is the period of this periodic motion? For the same *t* range, make a 2-D phase space plot for the corresponding trajectory in $\left(x(t), \dot{x}(t)\right)$. Make another plot showing the trajectory in $\left(x(t), \dot{x}(t)\right)$ space after periodicity has set in and the limit cycle has been established. Note that this second phase space plot is (necessarily) a simple closed curve. Finally graph the appropriate Poincaré section and reconcile this plot with your answer about the period of the motion. (You will likely need 5000 data points with R-K.)

b) (3 pts) Now try the nearby point in parameter space $\gamma = 0.25$ and $f = 11$. This time you should observe chaotic behavior. Plot $x(t)$ in the range $0 \le t \le 60\pi$. Make a plot of an appropriate Poincaré section and describe what it tells us, contrasting with the results in part a).

2) (4 pts) Consider the bakers' transformation, a map defined by

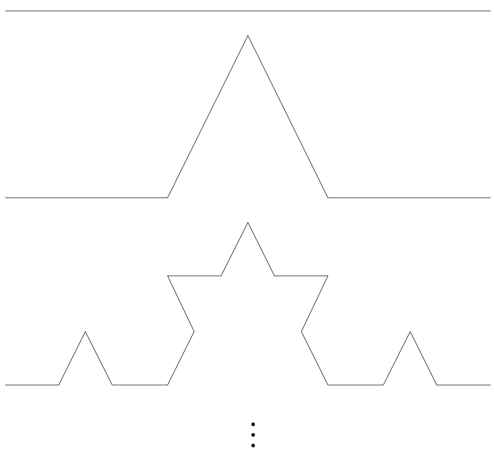
$$x_{n+1} = 2x_n \bmod 1$$
$$y_{n+1} = \begin{cases} ay_n, & 0 \le x_n \le \frac{1}{2} \\ ay_n + \frac{1}{2}, & \frac{1}{2} \le x_n \le 1 \end{cases},$$

which can exhibit a variety of behaviors due to the inherent stretching and folding implied by the formula (in analogy to the way a baker stretches and folds the bread dough). Compute the Liapunov exponents for this system. Is the system conservative or dissipative, *i.e.*, as a function of the parameter $a$?

3) (4 pts) For the bakers' transformation with $a$ = 1/2, draw a sequence of diagrams in phase space that shows how the block of initial conditions $0 \le x \le 1$, $0 \le y \le 1/2$ is transformed by the map (for n = 1,2,3,…). Identify graphically the effects of the two Liapunov exponents in the resulting figures.

4) (3 pts) The concept of fractals, "objects" with non-integer dimension, is important to understanding and characterizing chaotic behavior. As an example consider the Koch curve defined by

| <u>Segments</u> | <u>Lengths</u> |
|---|---|
| 1 | 1 |
| 4 | 1/3 |
| 16 | 1/9 |
| ⋮ | ⋮ |



Calculate its dimension.