Physics 505 - Autumn 2010

HW X - Solutions

12/8/10

Overview: Recall that solving physics problems is not (just) about solving differential equations. Use physical reasoning to help solve the following exercises and be certain to show your work. It is also important that you practice completely solving these exercises, checking for errors as you go along.

1) (8 pts) Consider a mechanical system described by the following equation of motion (often labeled the Duffing oscillator, including in the general case also a linear term)

$$\ddot{x} + \gamma \dot{x} + x^3 = f \cos t.$$

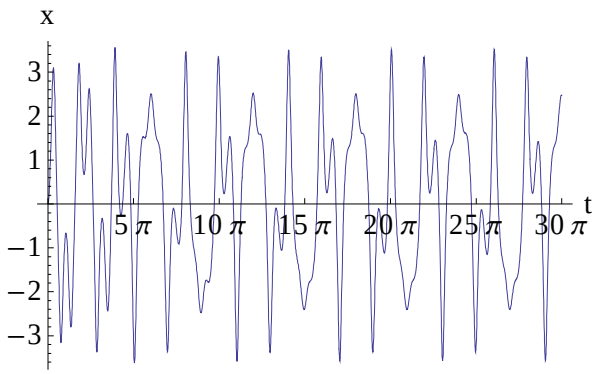
This system exhibits chaotic behavior in certain ranges of the parameters and we want to explore this behavior numerically (using Mathematica or the Runge-Kutta routine we have already practiced with). Use initial conditions $\left(x(0), \dot{x}(0)\right) = (0,0)$ and investigate the following two parameter regimes:

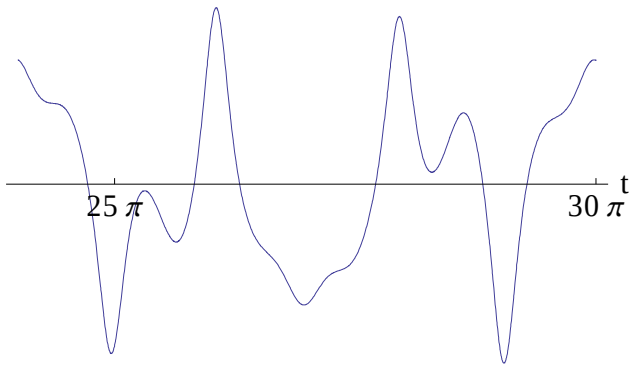
Solution: See the attached *Mathematica* notebook.

a) (5 pts) Take $\gamma = 0.2$ and $f = 10$. Evaluate and plot $x(t)$ in the range $0 \le t \le 30\pi$. You should find that the asymptotic behavior is periodic. What is the period of this periodic motion? For the same t range, make a 2-D phase space plot for the corresponding trajectory in $\left(x(t), \dot{x}(t)\right)$. Make another plot showing the trajectory in $\left(x(t), \dot{x}(t)\right)$ space after periodicity has set in and the limit cycle has been established. Note that this second phase space plot is (necessarily) a simple closed curve. Finally graph the appropriate Poincaré section and reconcile this plot with your answer about the period of the motion. (You will likely need 5000 data points if using R-K.)

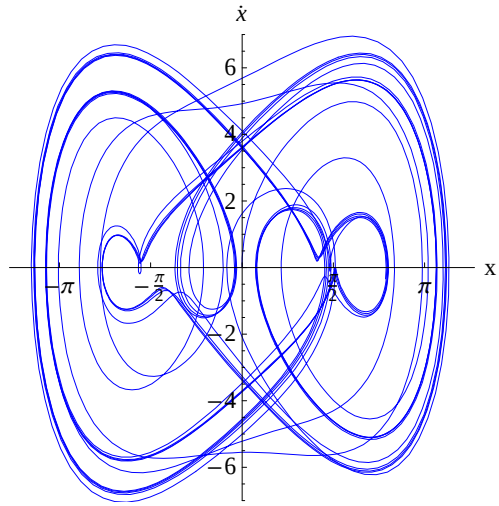
<u>Solution:</u> The plot of $x(t)$ has the form indicated in the next figure with periodic

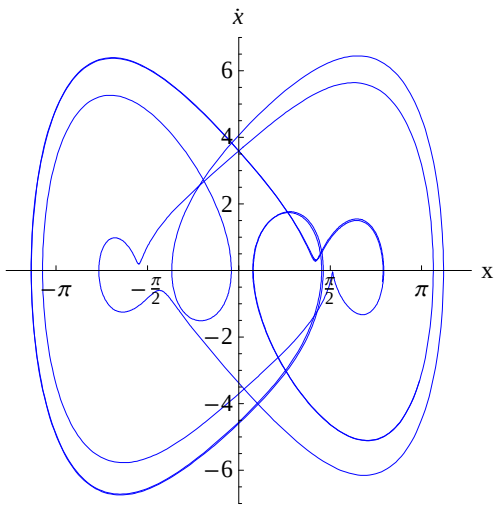
motion of period 6π appearing at large times as in the second figure. (Note that you may want to check at much larger times to verify the stability.)



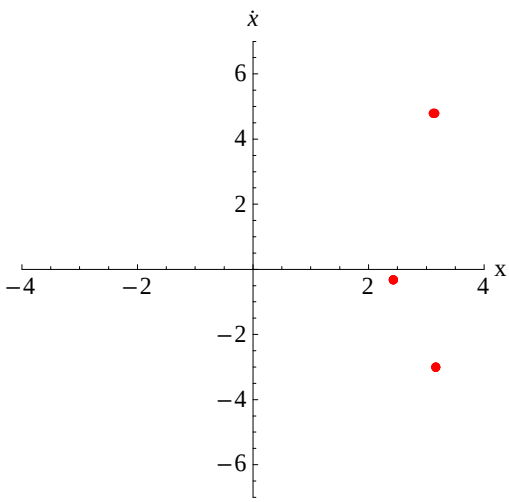


The corresponding (full) trajectory in phase space looks like the (complicated curve) in the next figure. The figure just below shows the phase space trajectory corresponding to the attractor (the late time behavior).

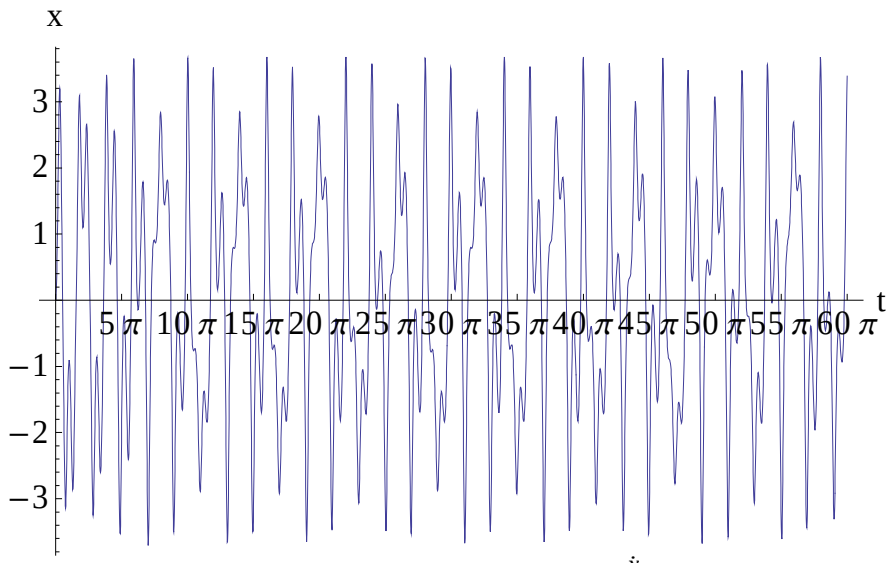




Finally we can verify the 6π period by looking at sections with a strobing period of 2π. We see the expected 3 points in the next figure. (Note there is some drift in the location of the points at a level below 1 part in a thousand that I believe is just the numerical integration error.)

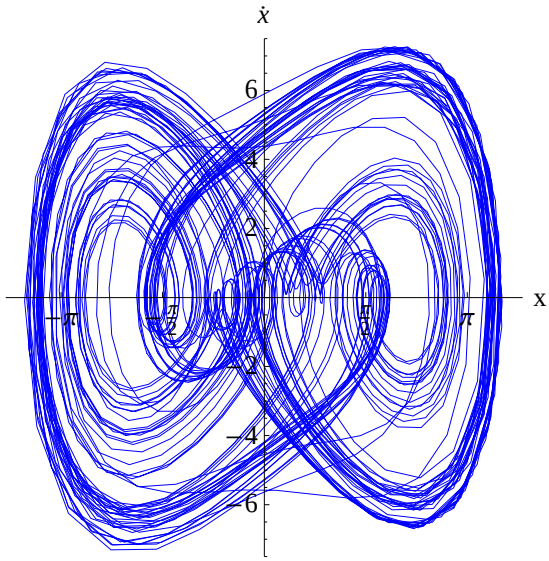


b) (3 pts) Now try the nearby point in parameter space $\gamma = 0.25$ and $f = 11$. This time you should observe chaotic behavior. Plot $x(t)$ in the range $0 \le t \le 60\pi$. Make a plot of an appropriate Poincaré section and describe what it tells us, contrasting with the results in part a).

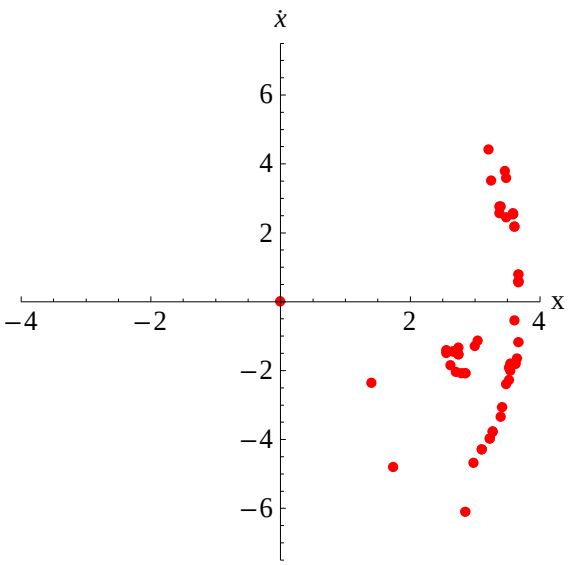


Solution: Now the behavior is clearly not periodic, even out to large times.

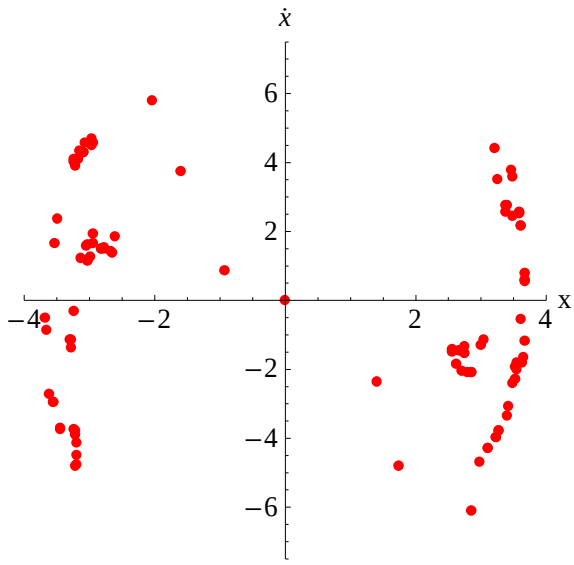
The phase space trajectory in the figure to the right is now very complicated, suggesting no periodic behavior (for t up to 100 π) with no evidence of an attractor, although the general shape is as in part a).



If we strobe at the driving frequency, we now see a Poincaré section that shows a complex curve.



If we strobe at twice the driving frequency we see much more of that curve with complexity indicative of chaotic behavior.



2) (4 pts) Consider the bakers' transformation, a map defined by

$$x_{n+1} = 2x_n \bmod 1$$

$$y_{n+1} = \begin{cases} ay_n, & 0 \le x_n \le \frac{1}{2} \\ ay_n + \frac{1}{2}, & \frac{1}{2} \le x_n \le 1 \end{cases},$$

which can exhibit a variety of behaviors due to the inherent stretching and folding implied by the formula (in analogy to the way a baker stretches and folds the bread dough). Compute the Liapunov exponents for this system.

<u>Solution</u>: (2 pts) The map is a combination of the binary Bernoulli shift and something like the tent map. The map's equilibrium point is the origin $x_0 = y_0 = 0$. For small perturbations about this point we have

$$\delta X_n \simeq A^n \delta X_0,$$
$$A = \begin{pmatrix} 2 & 0 \\ 0 & a \end{pmatrix} \Rightarrow \det \begin{bmatrix} 2-\mu & 0 \\ 0 & a-\mu \end{bmatrix} = 0$$
$$\Rightarrow \mu_\pm = \frac{2+a \pm \sqrt{(2+a)^2 - 8}}{2} = 2, a$$
$$\Rightarrow \lambda_\pm = \ln \mu_\pm = \frac{\ln 2}{\ln a}.$$

Is the system conservative or dissipative, *i.e.*, as a function of the parameter *a*?

<u>Solution</u>: (2 pts) The answer to this question clearly depends on the value of *a*. If $a = 1/2$, we have $\lambda_+ + \lambda_- = 0$. Thus the map is area preserving (det *A* = 1) and conservative.

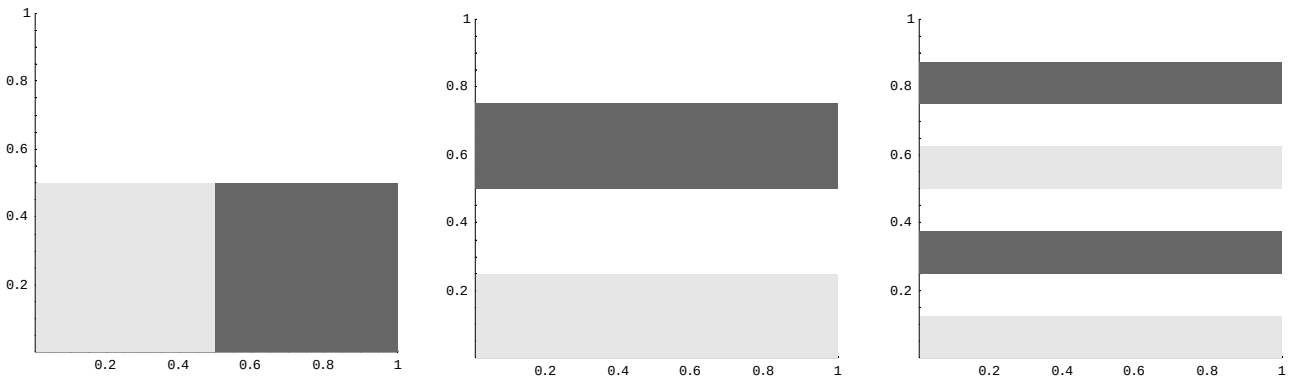
For $a < 1/2$ the area shrinks and the flow corresponds to dissipation, while the opposite is true for $a > 1/2$.

3) (4 pts) For the bakers' transformation with *a* = 1/2, draw a sequence of diagrams in phase space that shows how the block of initial conditions $0 \le x \le 1$, $0 \le y \le 1/2$ is transformed by the map (for n = 1,2,3,…). Identify graphically the effects of the two Liapunov exponents in the resulting figures.

<u>Solution</u>: (4 pts) We want to consider the area preserving version of the baker's map. We look at the flow starting with initial conditions in the regime $0 \le x \le 1$, $0 \le y \le ½$. First consider the "block flow" during the first few iterations. The first application of the map will take initial conditions in the block $0 \le x \le 0.5, 0 \le y \le 0.5$ and spread them out over the block $0 \le x \le 1.0, 0 \le y \le 0.25$ (preserving the area). On the other hand the initial block $0.5 \le x \le 1.0, 0 \le y \le 0.5$ will be mapped into $0 \le x \le 1.0, 0.5 \le y \le 0.75$. Hence the blocks $0 \le x \le 1.0, 0.25 \le y \le 0.5$ and $0 \le x \le 1.0, 0.75 \le y \le 1.0$ are empty after the first iteration. The second iteration takes the block $0 \le x \le 0.5, 0 \le y \le 0.25$ into $0 \le x \le 1.0, 0 \le y \le 0.125$ and the block $0 \le x \le 0.5, 0.5 \le y \le 0.75$ into the block $0 \le x \le 1.0, 0.25 \le y \le 0.375$. On the other hand the blocks $0.5 \le x \le 1.0, 0 \le y \le 0.25$ and $0.5 \le x \le 1.0, 0.5 \le y \le 0.75$ are mapped into $0 \le x \le 1.0, 0.5 \le y \le 0.625$ and $0 \le x \le 1.0, 0.75 \le y \le 0.875$, respectively. Thus each

iteration yields turns the initial blocks into horizontal strips with progressively finer separation and larger maximum *y* values. At all times the actual area occupied is just one half of the unit square as expected from an area preserving map.

n = 0 n = 1 n = 2



(You might want to consider this map for $a < 1/2$, which is "dissipative" with a strange attractor of dimension $d = 1 + \ln(1/2)/\ln(a) < 2$.)

The second issue is to see if we can detect the presence of the positive and negative exponents we found in Exercise 2).

We expect the positive exponent to lead to chaotic behavior, *i.e.*, nearby starting points are expected to lead to quite different trajectories.

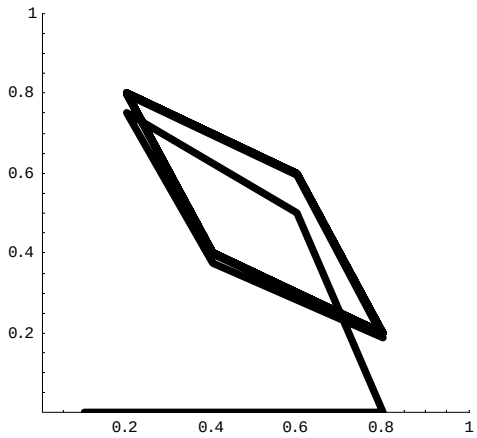
The negative exponent, on the other hand, should allow limit cycles to appear. Both types of behavior are seen to be present.

We can illustrate this with some examples, each of which we label with the starting point (x_0, y_0). The plots below typically are for 40 iterations.

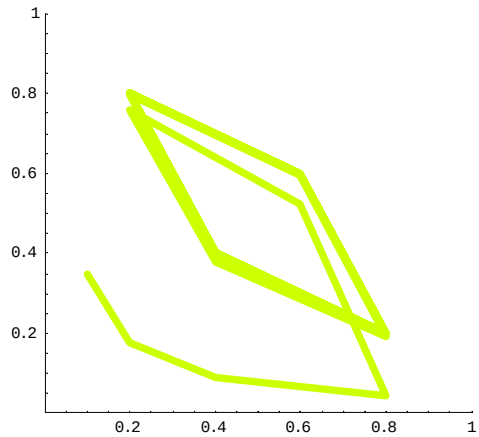
2 Examples – 2 points for some figures like the following – (grade very forgivingly).

The starting point $(0.1, 0)$ leads to the 4-point limit cycle. If we start at the same *x* value but a (any) nonzero *y* value, we end up in the same limit cycle.

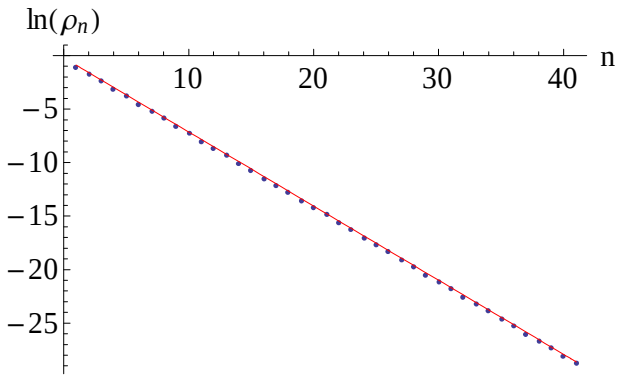
$(0.1, 0)$



$(0.1, 0.35)$

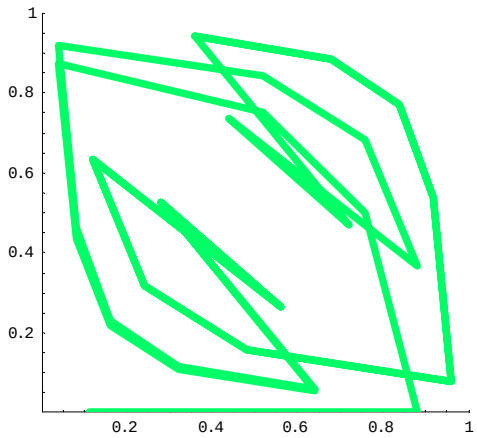


The 2-D distance $\left(\rho_n = \sqrt{\Delta x_n^2 + \Delta y_n^2}\right)$ between these 2 trajectories compared to the expected exponential damping (the negative exponent) is displayed in the figure to the right.

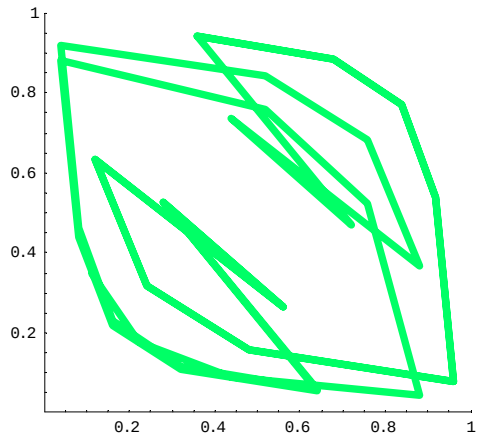


If, on the other hand, we start at a slightly different x value a very different trajectory arises, again with limit cycle behavior in evidence, but with a different cycle from above. Thus these are not attractors in the sense of attracting all trajectories with similar initial conditions (*i.e.*, no well-defined basins of attractions).

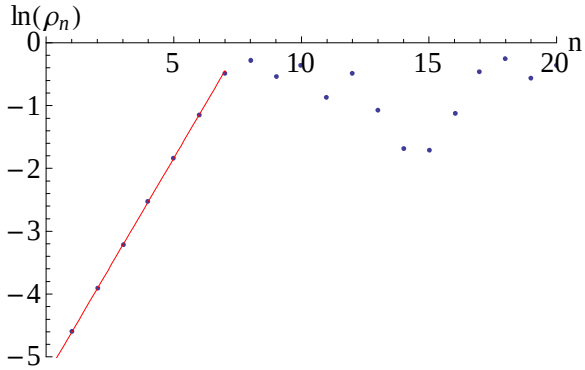
$(0.11, 0)$



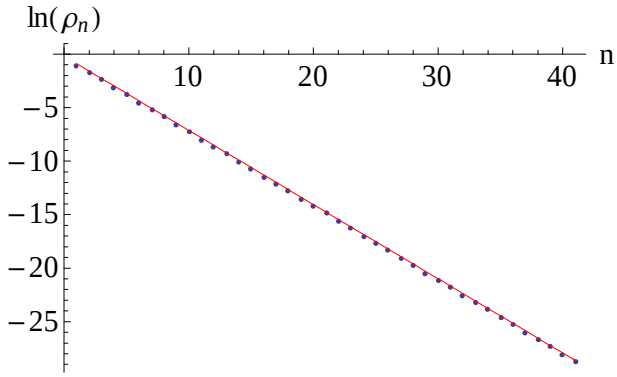
$(0.11, 0.35)$



The 2-D distance between the trajectories corresponding to (0.10, 0.35) and (0.11, 0.35) is displayed in the next figure indicating the presence of the positive exponent (the red line).

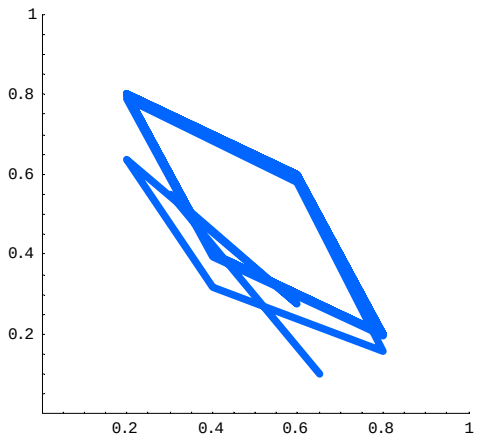


The next figure instead compares (0.11, 0.35) with (0.11,0) and again indicates the negative exponent.

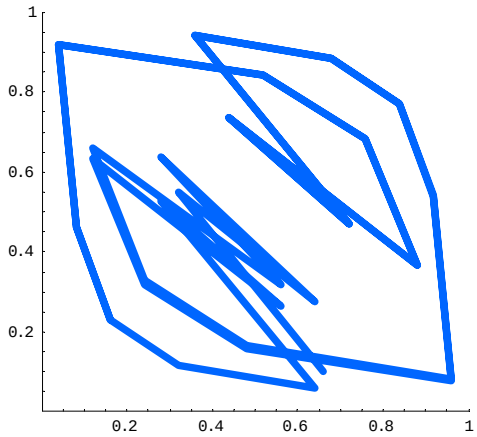


Similar pairs of behavior occur elsewhere in the plane. This is a somewhat different kind of chaos.

$(0.65, 0.1)$

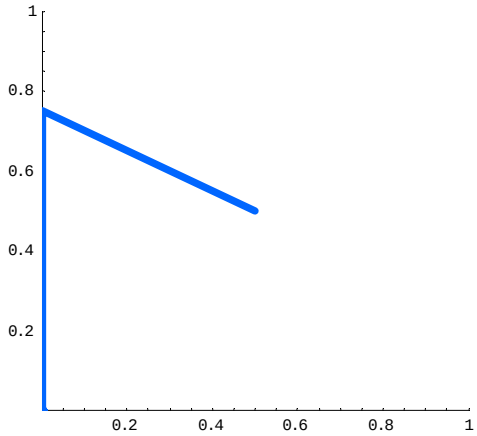


$(0.66, 0.1)$

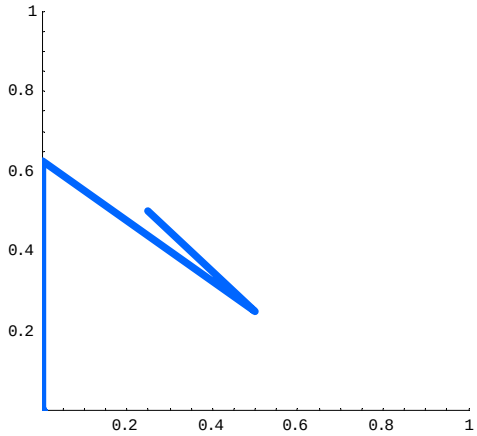


The other distinctive trajectories are those that can be mapped into $x = 0$, which all end at the origin. As examples consider the following

$(0.5, 0.5)$



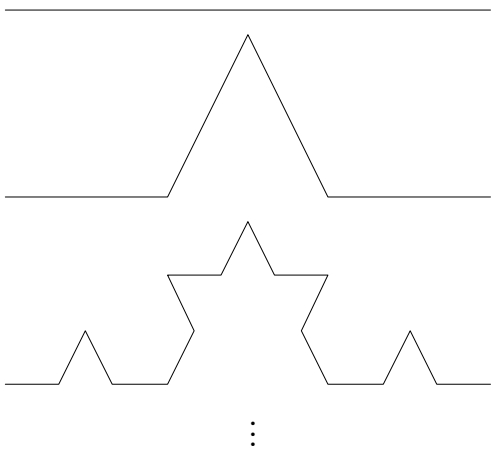
$(0.25, 0.5)$



Thus we have both chaotic behavior (strong dependence on the initial conditions, positive exponent) and limit cycles (1-D attractors, negative exponent), but in a map that explicitly preserves areas. There is also a range of initial values that all map to the origin.

4) (3 pts) The concept of fractals, "objects" with non-integer dimension, is important to understanding and characterizing chaotic behavior. As an example consider the Koch curve defined by

# of Segments	Length of Segment
1	1
4	1/3
16	1/9
⋮	⋮



Calculate its dimension.

Solution: This issue is, in fact, addressed in B&G on page 112. The general form of the number of segments is 4^n, where n is the step number, while the length of the segments at step n is $(1/3)^n$ (this is the parameter ε in B&G). Thus the (capacity) dimension of this "fractal line" is given by

$$N(n) = 4^n = \left(\frac{L_0}{L_n}\right)^d = (3)^{nd},$$

$$d = \frac{\ln(N(n))}{L_0/L_n} \Rightarrow d_C = \lim_{n\to\infty} \frac{n\ln 4}{n\ln 3} = 2\frac{\ln 2}{\ln 3} = 1.2619\ldots > 1.$$

As expected, since the line is "fuzzed out" by this definition, its dimension is larger than 1. The Koch curve has larger "area' than an ordinary $d = 1$ line.

You might enjoy watching the PBS/NOVA segment on fractal behavior, "Hunting the Hidden Dimension", originally shown 10/28/08 and available on the web at

http://www.pbs.org/wgbh/nova/physics/hunting-hidden-dimension.html