Lecture 1 - Introduction and Review: Newton's Laws, Conserved Quantities and N-body Systems (Chapter 1 in F&W, Sections 1 and 2).

This class is about "Classical Mechanics". For our purposes the word "classical" implies two points. The first point is that classical means that this subject has a long history. In particular, classical mechanics was one of the first areas of research to illustrate the real strength of physics, *i.e.*, it's predictive power. The many successes of mechanics in describing the behavior of the physical world help to illustrate that physics is often simpler at more fundamental levels. The study of classical mechanics also helps to illustrate the power of symmetries, some examples of which we will discuss in the coming weeks.

The second meaning to "classical" is that we will almost always consider the limit in which ħ goes to zero. At least initially, we will also focus on small velocities, *i.e.*, non-relativistic physics (c goes to infinity, $\beta = v/c \to 0$).

As our starting point let us return to freshman physics. What we first learn about in mechanics are the laws of Sir Isaac Newton. Newton's first law of motion states that, if the vector sum of the forces acting on an object is zero, then the object will remain at rest or remain moving with constant momentum (as observed in an inertial frame, *i.e.*, historically a frame moving with constant velocity with respect to the distant stars). If the force exerted on an object is zero, the object does not necessarily have zero velocity. Without any forces acting on it, including friction, an object in motion will continue to travel with constant momentum. For example, consider this law as applied to a point particle (*i.e.*, finite mass m and no spatial extent) in one spatial dimension (1-D) with coordinate labeled *x*,

$$\frac{d\left(m\dot{x}\right)}{dt} = 0, \tag{1.1}$$

where the velocity is

$$\dot{x} \equiv \frac{dx}{dt}. \tag{1.2}$$

Note that for such a point particle the coordinate $x(t)$ provides a complete description of the motion of this particle as a function of the time t. This relation implies that the quantity $m\dot{x}$, which we call the momentum and which will be derived more carefully shortly, is a constant as time evolves (when the force vanishes). Such a quantity is

referred to (in an obvious notation) as a constant of the motion. Said another way, the momentum is conserved. For the usual case of a constant mass, this simply implies zero acceleration,

$$\ddot{x} = 0. \tag{1.3}$$

It is also useful to point out, as we will make more use of later, the invariance properties of Eq. (1.1). In particular, if we simply translate our particle (*i.e.*, change our coordinate system) by a *constant* displacement $x \to x + x_0$, the form of the equation remains unchanged (since $\dot{x}_0 = 0$). The essential point is that the invariance properties of the dynamics lead in a predictable way (Noether's Theorem) to the form of the constants of motion (*i.e.*, the conserved quantities). In this particular case, translational invariance leads to conserved, *i.e.*, constant, (rectilinear) momentum. A related point is that the (quantum mechanical) operator that generates an infinitesimal translation is precisely the momentum operator!

The second law of Newton is simply the case for nonzero force. Again in one dimension we have

$$\frac{d\left(m\dot{x}\right)}{dt} = F\left(x,t\right). \tag{1.4}$$

Once more, assuming a constant mass, we have the usual relationship between acceleration, mass and force,

$$m\ddot{x} = ma = F\left(x,t\right). \tag{1.5}$$

This universally true law of motion (in 1-D for a constant mass point particle) serves to define the concepts of both inertial mass, *i.e.*, the coefficient of resistance to changes in motion, and force, *i.e.*, the actuator of changes in motion.

As a particularly simple and familiar example consider a linear restoring force (*i.e.*, a force with a linear dependence on x),

$$m\ddot{x} = -kx. \tag{1.6}$$

As we all recognize, this is just Newton for a harmonic oscillator. It is a particularly important special case, since we are very often interested in the properties of systems near their (stable) equilibrium conditions. Almost always equilibrium is defined by

the presence of (only) a linear restoring force, at least for small deviations from equilibrium. If the force had a dramatically different form, it would not correspond to an equilibrium point. This second order *linear* differential equation can be solved precisely by the (hopefully) familiar methods. If we define a frequency,

$$\omega = \sqrt{\frac{k}{m}}, \tag{1.7}$$

then the general solution to Eq. (1.6), in terms of two arbitrary constants, takes the form

$$x(t) = a\cos(\omega t) + b\sin(\omega t). \tag{1.8}$$

The constants are typically determined in terms of the initial conditions,

$$a = x(0),\, b = \frac{\dot{x}(0)}{\omega}. \tag{1.9}$$

This illustrates an important property of classical mechanics. For many systems of interest the dynamics (*i.e.*, Newton's equations) can be solved exactly (simply by integration) and the properties of the system as a function of time are completely determined by the (two) initial conditions. This was particularly important for the early studies of the motion of the "heavenly bodies".

Note that in the harmonic oscillator case the original equation of motion, Eq. (1.6), is not translationally invariant. since the origin is implicitly defined. As a (necessary) result the momentum is no longer a constant of the motion,

$$m\dot{x}(t) = -\omega m x(0)\sin(\omega t) + m\dot{x}(0)\cos(\omega t). \tag{1.10}$$

Let us generalize to the case of three spatial dimensions. Now the case of a linear restoring force looks like

$$\frac{d\left(m\dot{\vec{r}}\right)}{dt} = -k\vec{r}, \tag{1.11}$$

where, as usual (in Cartesian coordinates with unit, orthogonal coordinate vectors, $\hat{x}, \hat{y}, \hat{z}, \hat{x}\cdot\hat{x} = \hat{y}\cdot\hat{y} = \hat{z}\cdot\hat{z} = 1, \hat{x}\cdot\hat{y} = \hat{y}\cdot\hat{z} = \hat{z}\cdot\hat{x} = 0$),

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}. \tag{1.12}$$

This form for the force, where it depends only on the separation from the origin, the center of force, and the only direction defined is $\hat{r} = \vec{r}/|\vec{r}|$, is characteristic of a class of problems called central force problems,

$$\vec{F}_{\text{central}}(x, y, z) = f(r)\hat{r}. \tag{1.13}$$

In this example we can think of the origin as the point where the "spring" is mounted. Although the dynamics of such a system is not invariant under translations and therefore linear momentum is not conserved, this system does have an invariance that is relevant. In particular, consider rotations around the origin. These transformations will not change the length of the radius vector $r = |\vec{r}|$. Rotations will, however, change the direction of the unit vector $\hat{r}$, but it will be changed in the same way on both sides of the equation of motion. Thus the *form* of the equation of motion for central force problems, $d\left(m\dot{\vec{r}}\right)/dt = f(r)\hat{r}$, is invariant under rotations. Another way to look at this situation is to note that the central force does not select any special direction, except the direction defined by the point particle itself, *i.e.*, the radial direction. Hence the problem formally looks the same if we rotate the system (an active transformation), or rotate the Cartesian coordinate system $\hat{x}, \hat{y}, \hat{z}$ (a passive transformation). Formally the rotations are described by a Lie group, SO(3), a connection that we will pursue later. The conserved quantity that is associated with this invariance is the angular momentum defined by (here we introduce the symbol $\vec{p}$ for the linear momentum $m\dot{\vec{r}}$)

$$\vec{L} = \vec{r}\times\vec{p} = \vec{r}\times m\dot{\vec{r}}. \tag{1.14}$$

<u>ASIDE</u>: Recall that such vector (cross) products can be easily represented in terms of components if we use the fully antisymmetric tensor ε_{ijk}, $\varepsilon_{ijk} = -\varepsilon_{ijk} = -\varepsilon_{ikj}$, *etc.*, $\left(\vec{a}\times\vec{b}\right)_k = \varepsilon_{klm}a_l b_m$. This tensor also provides the structure constants for the algebra of the group SO(3).

The time derivative of the angular momentum is then

$$\begin{aligned}\frac{d\vec{L}}{dt} &= \frac{d}{dt}\left\{\vec{r}\times m\dot{\vec{r}}\right\} = \dot{\vec{r}}\times m\dot{\vec{r}} + \vec{r}\times\frac{d\left(m\dot{\vec{r}}\right)}{dt} \\ &= \vec{r}\times\frac{d\left(m\dot{\vec{r}}\right)}{dt} = \vec{r}\times\dot{\vec{p}},\end{aligned} \tag{1.15}$$

using an obvious vector identity. The corresponding quantity on the RHS of Newton's equation is the torque

$$\vec{N} = \vec{r}\times\vec{F}. \tag{1.16}$$

With these definitions the three-dimensional version of Newton's second law crossed with $\vec{r}$ becomes

$$\frac{d\vec{L}}{dt} = \vec{r}\times\frac{d\vec{p}}{dt} = \vec{r}\times\vec{F} = \vec{N}. \tag{1.17}$$

Now focus again on the central force problem. We find immediately that there can be no torque,

$$\vec{N} = \vec{r}\times\vec{F}_{\text{central}} = \vec{r}\times f\left(r\right)\hat{r} = 0, \tag{1.18}$$

and in this case the total angular momentum cannot change with time. Thus it must be a constant of the motion. In particular the vector direction of $\vec{L}$ is fixed and all motion must in the plane perpendicular to $\vec{L}$, *i.e.*, by construction both $\vec{r}$ and $\dot{\vec{r}}$ are orthogonal to $\vec{L}$ and thus, by Newton, $\ddot{\vec{r}}$ will be also.

Just as in the case of translational invariance and (linear) momentum conservation, the (quantum mechanical) operator that generates an infinitesimal rotation is the angular momentum operator, the conserved quantity when there is rotational invariance.

Note the following important distinction. Although the equation of motion, Eq. (1.11), exhibits rotational invariance (its form is not changed by rotations), a specific

solution to the equation (a specific trajectory) may (and usually does) exhibit lower symmetry (just rotations in the plane perpendicular to $\vec{L}$). (This is the idea behind the Higgs mechanism at work.) In the current case the full 3-D rotational symmetry is gone when the initial conditions, $\vec{r}(0)$ and $\dot{\vec{r}}(0)$, define the specific (2D) plane of the subsequent motion (via $\vec{r}(0)\times\dot{\vec{r}}(0)$). In the degenerate limit that $\dot{\vec{r}}(0)$ is along $\vec{r}(0)$ the problem reduces to being 1-dimensional ($\vec{L}=0$).

Along with linear and angular momentum, we want to consider a third possible conserved quantity, the total energy, which is typically associated with the concept of so-called conservative forces. The corresponding invariance is with respect to translations in time, *i.e.*, we will (eventually) associate energy conservation with the absence of explicit time dependence in the dynamics. In quantum mechanics the generator of an infinitesimal time translation is just the Hamiltonian, whose eigenvalues are the energy. For the present consider a nonrelativistic point particle with constant mass (we will relax these restrictions later). If we multiply Newton's second law,

$$\frac{d\left(m\dot{\vec{r}}\right)}{dt}=\vec{F} \tag{1.19}$$

by the velocity (using the usual scalar product) we find

$$\frac{d\left(m\dot{\vec{r}}\right)}{dt}\cdot\dot{\vec{r}}=\frac{d\left(\frac{1}{2}m\dot{r}^2\right)}{dt}=\frac{dT}{dt}=\vec{F}\cdot\dot{\vec{r}}, \tag{1.20}$$

where T is the (nonrelativistic) kinetic energy. Next we integrate with respect to time,

$$\begin{aligned}\int_{t_1}^{t_2}\frac{dT}{dt}dt&=T\left(t_2\right)-T\left(t_1\right)=\Delta T\\&=\int_{t_1}^{t_2}\vec{F}\cdot\dot{\vec{r}}\,dt=\int_{\vec{r}_1}^{\vec{r}_2}\vec{F}\cdot d\vec{r}=W.\end{aligned} \tag{1.21}$$

We identify the quantity W with the work done by the force on the point particle as it moves along the specified trajectory between the initial and finals points. Now the relevant question – is the work so defined described by an ordinary point function that depends only on the endpoints ($\vec{r}_1,\vec{r}_2$), or is it described by a function*al* that is

dependent on the specific path between the points, *i.e.*, is the quantity $\vec{F}\cdot d\vec{r}$ an exact differential? If it is an exact differential, we refer to the force as a conservative force and associate with it a scalar function, $U(\vec{r})$, the potential energy. We can express this path *independence* mathematically by the constraint that the integral around all *closed* paths must vanish,

$$\oint \vec{F}\cdot d\vec{r} = 0 \tag{1.22}$$

By Stoke's theorem this means that the force is curl free,

$$\oint \vec{F}\cdot d\vec{r} = \iint \vec{\nabla}\times\vec{F}\cdot d\vec{S} = 0 \Rightarrow \vec{\nabla}\times\vec{F} = 0, \tag{1.23}$$

where the last step, yielding a local equation (no integral), follows from the fact that the first equation holds for *all* possible closed paths. Thus we must be able to express such conservative (curl-free) forces as the gradient of a scalar function, the potential energy (the choice of sign is by convention),

$$\vec{F}(\vec{r}) = -\vec{\nabla}U(\vec{r}). \tag{1.24}$$

Returning to Eq. (1.21) we find

$$T(t_2) - T(t_1) = W = U(\vec{r}_1) - U(\vec{r}_2), \tag{1.25}$$

or

$$T(t_2) + U(\vec{r}_2) = T(t_1) + U(\vec{r}_1) = E. \tag{1.26}$$

The total energy $E = T + U$ is a constant along the trajectory; (total) energy is conserved in conservative systems. Particularly interesting examples of (central) potentials are $-k/r$ for the inverse square problem and $+\frac{1}{2}kr^2$ for the linear restoring force (the harmonic oscillator).

Finally consider the 3rd Law of Mr. Newton, which states that an object experiences a force because it is interacting with some other object, and that the force exerted by object 1 on object 2 must be equal to the force exerted by object 2 on object 1, but in the opposite direction. This implies that we were missing something earlier when we

considered the central force problem. We explicitly ignored the second body that is positioned at (or near) the origin, which is either providing the gravitational field in the inverse square problem or is where we mount the other end of the spring in the linear restoring force problem. This is justified only if the second body is much more massive than the point particle we were focused on (as we will see shortly). Thus mechanics problems are really 2, or more, body problems and we should work out the formalism for the general N-body problem. The solution of such a problem corresponds to finding the N functions of time, $\vec{r}_j(t)$, describing the trajectories of N particles, where the $\vec{r}_j(t)$ are measured with respect to some *arbitrary* origin.

Consider a set of *N* point particles labeled by an index *k*. For now we take the masses of the individual particles to be constant. These particles interact with each other pair-wise. If the force experienced by particle *j due* to particle *k* is represented by $\vec{F}_{jk}$, then the 3rd law says

$$\vec{F}_{jk} = -\vec{F}_{kj}. \tag{1.27}$$

If we include the possibility of an *external* force, *e.g.*, due to interactions with particles not included in the *N*-bodies on our list, Newton's second law for particle *j* looks like

$$m\ddot{\vec{r}}_j = \dot{\vec{p}}_j = \vec{F}_j^{ext} + \sum_{\substack{k=1 \\ k\neq j}}^{N} \vec{F}_{jk}. \tag{1.28}$$

Thus, if we sum over all of the particles and invoke the 3rd law, we find

$$\begin{aligned} \sum_{j=1}^{N} \dot{\vec{p}}_j &= \sum_{j=1}^{N} \vec{F}_j^{ext} + \sum_{\substack{j,k=1 \\ k\neq j}}^{N} \vec{F}_{jk} \\ &= \vec{F}_{TOT}^{ext} + \frac{1}{2}\sum_{\substack{j,k=1 \\ k\neq j}}^{N} \left(\vec{F}_{jk} + \vec{F}_{kj}\right) \\ &= \vec{F}_{TOT}^{ext}, \end{aligned} \tag{1.29}$$

where we used the symmetry of the second sum to write it in a form that clearly vanishes due to the asymmetry of $\vec{F}_{jk}$. This simplification suggests that it would be helpful to use summed or center-of-mass (CM) coordinates as defined by

$$\begin{aligned} M &= \sum_{k=1}^{N} m_k, \\ \vec{R} &= \frac{1}{M}\sum_{k=1}^{N} m_k \vec{r}_k, \\ \vec{P} &= \sum_{k=1}^{N} m_k \dot{\vec{r}}_k = \sum_{k=1}^{N} \vec{p}_k = M\dot{\vec{R}}. \end{aligned} \tag{1.30}$$

Thus the system as a whole (the motion of the CM system) obeys Newton in the form

$$\dot{\vec{P}} = \vec{F}_{TOT}^{\,ext}. \tag{1.31}$$

The motion of the CM of the N-body system is determined by the net external force, *independent* of the internal forces. If the net external force is zero, the total momentum $\vec{P}$ is a constant of the motion.

Likewise we can define the total angular momentum (with respect to some origin),

$$\vec{L} = \sum_{k=1}^{N} \vec{r}_k \times \vec{p}_k, \tag{1.32}$$

which satisfies (again we attempt to symmetrize the double sum)

$$
\begin{aligned}
\frac{d\vec{L}}{dt} &= \sum_{k=1}^{N} \vec{r}_k \times \vec{F}_k^{ext} + \sum_{k=1}^{N} \vec{r}_k \times \sum_{\substack{j=1 \\ j\neq k}}^{N} \vec{F}_{kj} \\
&= \sum_{k=1}^{N} \vec{r}_k \times \vec{F}_k^{ext} + \frac{1}{2} \sum_{\substack{k,j=1 \\ j\neq k}}^{N} \left(\vec{r}_k \times \vec{F}_{kj} + \vec{r}_j \times \vec{F}_{jk} \right) \\
&= \sum_{k=1}^{N} \vec{r}_k \times \vec{F}_k^{ext} + \frac{1}{2} \sum_{\substack{k,j=1 \\ j\neq k}}^{N} \left(\vec{r}_k - \vec{r}_j \right) \times \vec{F}_{kj} \\
&= \sum_{k=1}^{N} \vec{r}_k \times \vec{F}_k^{ext} \equiv \vec{N}^{ext},
\end{aligned}
\tag{1.33}
$$

where the final quantity is the (net) external torque. Note that we have used the fact that the vector direction of the pair-wise force $\vec{F}_{jk}$ points in the direction of (or opposite to) $\vec{r}_{jk} = \vec{r}_j - \vec{r}_k$ so that the cross product vanishes.

> <u>ASIDE</u>: This is an example of the very powerful "what else can it be?" theorem. This direction is the only direction defined by the locations of the two particles, and satisfies Newton #3. Unless the force knows about something besides the locations of the 2 particles, and we are assuming that it does not, it must point along this direction (up to a sign). An interesting apparent violation of this simple picture arises from magnetic forces, where the force knows about the direction of the velocities of the particles. Think about it!

We conclude that, if the net external torque vanishes, the total angular momentum of the system is a constant of the motion.

Next consider the issue of energy. The total (non-relativistic) kinetic energy of the system is given by

$$
T = \sum_{k=1}^{N} \frac{m_k \dot{\vec{r}}_k^{\,2}}{2}. \tag{1.34}
$$

The work performed on the system (by all forces) as the system moves along its trajectory from point 1 to point 2 is given by

$$
\begin{aligned}
W_{12} &= \sum_{j=1}^{N} \int_1^2 \left(\vec{F}_j^{ext} + \sum_{\substack{k=1 \\ k\neq j}}^{N} \vec{F}_{jk} \right) \cdot d\vec{r}_j \\
&= \sum_{j=1}^{N} \int_1^2 \left(m_j \ddot{\vec{r}}_j \right) \cdot d\vec{r}_j = \sum_{j=1}^{N} \int_1^2 \left(m_j \ddot{\vec{r}}_j \cdot \dot{\vec{r}}_j \right) dt \\
&= \frac{1}{2} \sum_{j=1}^{N} \int_1^2 \frac{d}{dt} \left(m_j \dot{\vec{r}}_j^{\,2} \right) dt \\
&= \frac{1}{2} \sum_{j=1}^{N} m_j \dot{\vec{r}}_j^{\,2} \Big|_1^2 = T_2 - T_1 .
\end{aligned}
\tag{1.35}
$$

So the net work performed by all forces on the system serves to increase the total kinetic energy.

At this point it is convenient to transform to the CM (primed) variables defined by

$$
\begin{aligned}
&\vec{r}_k = \vec{R} + \vec{r}_k' = \frac{1}{M} \sum_{j=1}^{N} m_j \vec{r}_j + \vec{r}_k', \\
&\sum_{k=1}^{N} m_k \vec{r}_k' = \sum_{k=1}^{N} m_k \vec{r}_k - \sum_{k=1}^{N} m_k \frac{1}{M} \sum_{j=1}^{N} m_j \vec{r}_j = 0 \\
&\dot{\vec{r}}_k = \dot{\vec{R}} + \dot{\vec{r}}_k', \\
&\sum_{k=1}^{N} m_k \dot{\vec{r}}_k' = \sum_{k=1}^{N} m_k \dot{\vec{r}}_k - \sum_{k=1}^{N} m_k \frac{1}{M} \sum_{j=1}^{N} m_j \dot{\vec{r}}_j = 0.
\end{aligned}
\tag{1.36}
$$

Thus we have that the total kinetic energy splits into two components (since Eq. (1.36) says that the cross term vanishes)

$$
\begin{aligned}
T &= \frac{1}{2} \sum_{j=1}^{N} m_j \left(\dot{\vec{R}} + \dot{\vec{r}}_k' \right)^2 = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \sum_{j=1}^{N} m_j \dot{\vec{r}}_k'^2 \\
&= T_{CM} + T_{\text{int}} .
\end{aligned}
\tag{1.37}
$$

The first term describes the motion *of* the CM, while the second term accounts for the kinetic energy of the particles moving *with respect to* the CM (labeled as T_{int} for *int*ernal, but labeled T_{rel} in the text). A similar decomposition occurs when we consider the total angular momentum in these coordinates. We write

$$\begin{aligned}\vec{L} &= \sum_{j=1}^{N} \vec{r}_j \times m_j \dot{\vec{r}}_j = \sum_{j=1}^{N} \left(\vec{R} + \vec{r}_j'\right) \times m_j \left(\dot{\vec{R}} + \dot{\vec{r}}_j'\right) \\ &= \vec{R} \times M\dot{\vec{R}} + \sum_{j=1}^{N} \vec{r}_j' \times m_j \dot{\vec{r}}_j' + \vec{R} \times \sum_{j=1}^{N} m_j \dot{\vec{r}}_j' + \sum_{j=1}^{N} m_j \vec{r}_j' \times \dot{\vec{R}} \\ &= \vec{R} \times M\dot{\vec{R}} + \sum_{j=1}^{N} \vec{r}_j' \times m_j \dot{\vec{r}}_j' = \vec{L}_{CM} + \vec{L}_{\text{int}}.\end{aligned} \tag{1.38}$$

Again the cross terms vanish and the total angular momentum has two components, the angular momentum of the entire system (the CM) with respect to the externally defined origin and the "internal" angular momentum corresponding to the motion of the particles with respect to the CM. Note that, since the coordinates in the CM system ($\vec{r}_j', \dot{\vec{r}}_j'$) are constrained to sum to zero, the internal kinetic energy and the internal angular momentum are constructed from only 3N-3 internal (unconstrained) coordinates (and not 3N). These "lost" coordinates are accounted for by the motion of the overall CM ($\vec{R}, \dot{\vec{R}}$). The equation of motion for the angular momentum also separates. Clearly we have

$$\frac{d\vec{L}_{CM}}{dt} = \vec{R} \times \dot{\vec{P}} = \vec{R} \times \vec{F}_{TOT}^{ext}, \tag{1.39}$$

as we would expect as the CM (by itself) acts as a single point particle. On the other hand we have

$$\frac{d\vec{L}_{\text{int}}}{dt} = \frac{d\vec{L}}{dt} - \frac{d\vec{L}_{CM}}{dt} = \sum_{j=1}^{N} \left(\vec{r}_j - \vec{R}\right) \times \vec{F}_j^{ext} = \sum_{j=1}^{N} \vec{r}_j' \times \vec{F}_j^{ext}. \tag{1.40}$$

The time rate of change of the internal angular momentum *with respect to* the CM responds to the external torque *with respect to* the CM, independent of the motion of the CM itself (even if it is accelerating!).

Now let us return to the issue of the forces. We proceed initially by assuming that they are conservative forces, both internal and external, so that

$$\vec{F}_j^{ext}(\vec{r}_j) = -\vec{\nabla}_j U_j^{ext}(\vec{r}_j) \Rightarrow \sum_{j=1}^{N} \int_1^2 \vec{F}_j^{ext}(\vec{r}_j) \cdot d\vec{r}_j = -\sum_{j=1}^{N} U_j^{ext}\Big|_1^2. \quad (1.41)$$

With $\vec{r}_{jk} = \vec{r}_j - \vec{r}_k$, we find (as desired)

$$\vec{F}_{jk} = -\vec{\nabla}_j U_{jk}(r_{jk}) = \vec{\nabla}_k U_{jk}(r_{jk}) = -\vec{F}_{kj} : \left[U_{jk} = U_{kj}\right], \quad (1.42)$$

and also that

$$\vec{F}_{jk} = -\vec{\nabla}_{jk} U_{jk}(r_{jk}) = \vec{\nabla}_{kj} U_{jk}(r_{jk}) : \left[\vec{\nabla}_{jk} \equiv \frac{\partial}{\partial(\vec{r}_j - \vec{r}_k)}\right]. \quad (1.43)$$

Thus the net work performed by (all of) the internal forces is

$$\begin{aligned}
\sum_{\substack{j,k=1\\k\neq j}}^{N} \int_1^2 \vec{F}_{jk} \cdot d\vec{r}_j &= \frac{1}{2} \sum_{\substack{j,k=1\\k\neq j}}^{N} \int_1^2 \left(\vec{F}_{jk} \cdot d\vec{r}_j + \vec{F}_{kj} \cdot d\vec{r}_k\right) \\
&= \frac{1}{2} \sum_{\substack{j,k=1\\k\neq j}}^{N} \int_1^2 \left(\vec{F}_{jk} \cdot d\vec{r}_{jk}\right) \\
&= -\frac{1}{2} \sum_{\substack{j,k=1\\k\neq j}}^{N} \int_1^2 \left(\vec{\nabla}_{jk} U_{jk}(r_{jk})\right) \cdot d\vec{r}_{jk} \\
&= -\frac{1}{2} \sum_{\substack{j,k=1\\k\neq j}}^{N} U_{jk}\Big|_1^2 = -\sum_{\substack{j,k=1\\j<k}}^{N} U_{jk}\Big|_1^2,
\end{aligned} \quad (1.44)$$

where the last sum is over all distinct pairs (as expected).

Putting these results together we have

$$
\begin{aligned}
&T_2 - T_1 = -\sum_{j=1}^{N} U_j^{ext}\Big|_1^2 - \sum_{\substack{j,k=1\\ j<k}}^{N} U_{jk}\Big|_1^2 \Rightarrow \\
&T_2 + U_2^{ext} + U_2^{\text{int}} = T_1 + U_1^{ext} + U_1^{\text{int}} = E^{TOT}.
\end{aligned}
\tag{1.45}
$$

The total energy, *i.e.*, the kinetic energy summed over particles plus the external potential energy summed over particles plus the internal potential energy summed over pairs of particles, is a constant of the motion. These relations form the basis of our understanding of thermodynamics and statistical mechanics.

We will close this lecture with a brief look at the simply but still interesting (*i.e*, common) case $N = 2$. For this case it is often useful to re-express the internal (or relative) kinetic energy in terms of the reduced mass and the relative velocity of the two point particles. We define the reduced mass as

$$
\mu = \frac{m_1 m_2}{m_1 + m_2} \tag{1.46}
$$

and note that (subtracting a useful expression for zero)

$$
\begin{aligned}
T^{\text{int}} &= \frac{1}{2}\left(m_1 \dot{\vec{r}}_1'^2 + m_2 \dot{\vec{r}}_2'^2\right) = \frac{1}{2}\left(m_1 \dot{\vec{r}}_1'^2 + m_2 \dot{\vec{r}}_2'^2\right) - \frac{\left(m_1 \dot{\vec{r}}_1' + m_2 \dot{\vec{r}}_2'\right)^2}{m_1 + m_2} \\
&= \frac{1}{2}\mu \dot{\vec{r}}_{12}^{\,2} : \left[\dot{\vec{r}}_{12} \equiv \dot{\vec{r}}_1' - \dot{\vec{r}}_2' = \dot{\vec{r}}_1 - \dot{\vec{r}}_2\right].
\end{aligned}
\tag{1.47}
$$

Thus the internal kinetic energy depends on just the one (reduced) mass and the single (relative) velocity. For pairs with very asymmetric masses the reduced mass is essentially the lighter mass and the relative velocity is essentially the CM velocity of the lighter particle (the heavier particle is essentially at rest), *i.e.*, the CM (approximately) coincides with the rest frame of the heavier particle.

If we consider the collision of two particles (no external forces) and characterize any inelasticity in the collision in terms of Newton's coefficient of restitution, $0 < \alpha < 1$, we have (now prime means after the collision)

$$
\begin{aligned}
T^{\text{int}\prime} &= \alpha^2 T^{\text{int}}, \\
\dot{r}'_{12} &= \alpha \dot{r}_{12.}
\end{aligned}
\tag{1.48}
$$