Lecture 14: Appendix A The general central force problem

We can apply the ideas of Lecture 14 to the general central force problem and (as promised at the beginning of the course) ask which central potential can yield periodic (clock-like) behavior. We consider a very general form for the potential

$$U(\rho) = -\sum_j \frac{k_{-j}}{\rho^{\beta_j}} + k_0 \ln \rho + \sum_j k_j \rho^{\alpha_j}, \tag{14.A.1}$$

with $\alpha_j > 0$ and $\beta_j < 2 - \varepsilon$, $\varepsilon > 0$, to ensure bounded motion. The question is, which terms in this expression can yield periodic trajectories independently of the specific initial conditions and the specific force constants k_j, *i.e.*, we are interested in when stable periodic behavior arises. This question can be stated in terms of the following integral in terms of the effective potential V (recall Eqs. (14.15) and (14.16) in the lecture)

$$\begin{aligned} W &= \frac{\Phi}{2\pi} = \int_{\rho_{\min}}^{\rho_{\max}} \frac{L_z}{2\pi\rho^2} \frac{d\rho}{\sqrt{2\mu\left(E - V(\rho)\right)}}, \\ V(\rho) &= \frac{L_z^2}{2\mu\rho^2} + U(\rho), \end{aligned} \tag{14.A.2}$$

where we ask under what conditions does this integral yield a rational result independent of the specific values of E, L_z, k_j. We can study this problem by focusing first on circular orbits where $E = V(\rho_0), V'(\rho_0) = 0, V''(\rho_0) > 0$ and $p_\rho = 0$. This is a single frequency problem with $\omega_\phi = \dot{\phi} = L_z / \mu\rho_0^2$. Next we consider small perturbations around this solution and determine when these perturbed orbits are independent of the initial conditions and force constants and then when they yield rational values. As usual with such linearized perturbation problems we will start with a harmonic oscillator problem. We can simplify the expressions by choosing a rescaled variable $x = L_z / \rho$, setting $\mu = 1$ (*i.e.*, rescaling all energies by the reduced mass) and defining a new potential

$$w(x) = \frac{x^2}{2} + U\left(\frac{L_z}{x}\right). \tag{14.A.3}$$

Now we are interested in the quantity

$$\Phi = \int_{x_{\min}}^{x_{\max}} \frac{dx}{\sqrt{2\left(E - w\left(x\right)\right)}}, \tag{14.A.4}$$

which is the half period of a particle described by the 1-D (effective) Hamiltonian

$$h = \frac{\dot{x}^2}{2} + w\left(x\right) = E. \tag{14.A.5}$$

By construction, we are near a circular orbit at $x_0 = L_z / \rho_0$ where

$$w\left(x\right) \simeq w\left(x_0\right) + w''\left(x_0\right)\frac{\left(x - x_0\right)^2}{2}. \tag{14.A.6}$$

Thus the frequency of the perturbed radial motion is

$$\Phi = \frac{\pi}{\omega_0} = \frac{\pi}{\sqrt{w''\left(x_0\right)}} \tag{14.A.7}$$

and we want to know when this quantity is independent of initial conditions and force constants. Using the expression for the potential in Eq. (14.A.6) we have

$$\begin{aligned}
w'\left(x\right) &= x - \frac{L_z}{x^2}U'\left(\frac{L_z}{x}\right) \Rightarrow 1 = \frac{L_z U'\left(\frac{L_z}{x_0}\right)}{x_0^3}, \\
w''\left(x\right) &= 1 + \left(\frac{L_z}{x^2}\right)^2 U''\left(\frac{L_z}{x}\right) + 2\frac{L_z}{x^3}U'\left(\frac{L_z}{x}\right) \\
\Rightarrow w''\left(x_0\right) &= \frac{3x_0 L_z U'\left(\frac{L_z}{x_0}\right) + L_z^2 U''\left(\frac{L_z}{x_0}\right)}{x_0^4}
\end{aligned}$$

$$\Rightarrow w''(x_0) = \frac{3x_0 L_z U'\left(\frac{L_z}{x_0}\right) + L_z^2 U''\left(\frac{L_z}{x_0}\right)}{x_0 L_z U'\left(\frac{L_z}{x_0}\right)}$$

$$= \frac{3U'\left(\frac{L_z}{x_0}\right) + \frac{L_z}{x_0} U''\left(\frac{L_z}{x_0}\right)}{U'\left(\frac{L_z}{x_0}\right)}. \tag{14.A.8}$$

So finally we have

$$W = \frac{\Phi}{2\pi} = \frac{1}{2\sqrt{w''(x_0)}} = \frac{1}{2}\sqrt{\frac{U'(\rho_0)}{3U'(\rho_0) + \rho_0 U''(\rho_0)}} \tag{14.A.9}$$

and we want to know when this expression is independent of the initial conditions and the force constants, and then when is it rational. The dependence on the angular momentum (an initial condition) and on the force constant appears through the dependence on ρ_0. So we first ask how the square root in Eq. (14.A.9) can be independent of ρ_0. This can clearly only occur when the potential is just one of the terms in Eq. (14.A.1) (*i.e.*, a single power of ρ). Since in this case the above expression is independent of the force constant and its sign, we need only consider $U \propto \rho^\alpha$ ($-2 < \alpha$ for bounded motion) and find

$$U' \propto \alpha\rho^{\alpha-1}, U'' \propto \alpha(\alpha-1)\rho^{\alpha-2}$$
$$\Rightarrow \frac{U'}{3U' + \rho U''} = \frac{\alpha}{3\alpha + \alpha(\alpha-1)} = \frac{1}{\alpha+2} \tag{14.A.10}$$
$$\Rightarrow W = \frac{1}{2\sqrt{\alpha+2}}.$$

So a potential with a single power can produce a winding number that is independent of the initial conditions and the force constant. However, the result is periodic only if W is rational. Thus the allowed powers ($-2 < \alpha$) are -1, 2, 7, 14, ($\alpha + 2 = n^2$, with n an integer).

Next we ask what happens for larger perturbations around the circular orbit. Since the result in Eq. (14.A.10) is required to be stable with respect to changes in the energy *E*, we might as well consider large changes such that the resulting expressions are relatively simple. For the case $U = k\rho^{\alpha}$, $k,\alpha > 0$ we consider the limit $E \to \infty$. In this limit we expect that the $x^2/2$ term in $w(x)$ will dominate at the maximum turning point $x \to x_{\max} \to \infty$, $E = w(x_{\max}) \simeq x_{\max}^2/2$. We can scale out this behavior with the change of variables $y = x/x_{\max}$ ($\rho = L_z/x = L_z/yx_{\max}$),

$$\frac{w(x)}{E} = y^2 + 2k\frac{L_z^{\alpha}}{y^{\alpha}x_{\max}^{2+\alpha}}, \qquad (14.A.11)$$

where we expect, in the $E \to \infty$ limit, that $y_{\min} = x_{\min}/x_{\max} \to 0$. With these definitions of the variables for the $E \to \infty$ limit and assuming that the y^2 term dominates (*y* fixed, $x_{\max} \to \infty$), we have from Eq. (14.A.4) that

$$\begin{aligned} W_{E\to\infty} &= \frac{1}{2\pi}\int_{x_{\min}}^{x_{\max}} \frac{dx}{\sqrt{2\left(E - w(x)\right)}}\Bigg|_{E\to\infty} \\ &= \frac{1}{2\pi}\int_0^1 \frac{dy}{\sqrt{1-y^2}} = \frac{1}{2\pi}\sin^{-1} y\Big|_0^1 \qquad (14.A.12) \\ &= \frac{1}{2\pi}\frac{\pi}{2} = \frac{1}{4}. \end{aligned}$$

From Eq. (14.A.10) we see that this corresponds to $\alpha = 2$ and means that this is the only positive exponent that gives the *same* winding number both at the energy of the circular orbit and at arbitrarily large energy. Thus we conclude that the only potential with a positive exponent that yields periodic motion *independent* of the initial conditions, *i.e.*, the value of *E*, is the isotropic harmonic oscillator.

To study the negative exponent case, $U = -k\rho^{-\beta}$, $0 < \beta < 2$ we consider the opposite limit $E \to 0$. We have

$$
\begin{aligned}
&w(x) = \frac{x^2}{2} - k\left(\frac{x}{L_z}\right)^{\beta}, \\
&w(x_{\max}) = w(x_{\min}) = E \to 0, \\
&\Rightarrow \begin{cases} x_{\min} = 0 \\ x_{\max} = \left(\dfrac{2k}{L_z^{\beta}}\right)^{1/(2-\beta)} \end{cases}.
\end{aligned}
$$

Note that here $x_{\min} = 0$ implies that $\rho \to \infty$ where we can consider that the orbits close. Again we use $y = x/x_{\max}$ and, noting that $x_{\max}^2/2 = kx_{\max}^{\beta}/L_z^{\beta}$, we have

$$
-w(x) = \frac{k}{L_z^{\beta}} x^{\beta} - \frac{x^2}{2} \Rightarrow \frac{x_{\max}^2}{2}\left(y^{\beta} - y^2\right). \qquad (14.A.13)
$$

Thus the winding number is given by

$$
\begin{aligned}
W_{E\to 0} &= \frac{1}{2\pi} \int_{x_{\min}}^{x_{\max}} \frac{dx}{\sqrt{2\left(E - w(x)\right)}}\Bigg|_{E\to 0} \\
&= \frac{1}{2\pi} \int_0^1 \frac{dy}{\sqrt{y^{\beta} - y^2}} = \frac{1}{2\pi} \int_0^1 \frac{dy y^{-\beta/2}}{\sqrt{1 - y^{2-\beta}}} \\
&= \frac{1}{2\pi(2-\beta)} \int_0^1 \frac{dz z^{\frac{-\beta}{2(2-\beta)} + \frac{1}{2-\beta} - 1}}{\sqrt{1-z}} \\
&\qquad \left[z = y^{2-\beta}, dy = \frac{dz z^{\frac{1}{2-\beta} - 1}}{2-\beta} \right]
\end{aligned}
$$

$$
\begin{aligned}
W_{E\to 0} &= \frac{1}{2\pi\left(2-\beta\right)}\int_0^1 \frac{dz}{\sqrt{z}\sqrt{1-z}} = \frac{B\left(\frac{1}{2},\frac{1}{2}\right)}{2\pi\left(2-\beta\right)} \\
&= \frac{1}{2\pi\left(2-\beta\right)}\frac{\Gamma^2\left(\frac{1}{2}\right)}{\Gamma\left(1\right)} = \frac{\pi}{2\pi\left(2-\beta\right)} \\
&= \frac{1}{2\left(2-\beta\right)}.
\end{aligned}
\tag{14.A.14}
$$

Again we demand that this result match that in Eq. (14.A.10), which yields $\beta = -\alpha = 1$. The only negative power potential that yields periodic motion for all initial conditions is the $1/r$ (Kepler) potential. As we have claimed since the beginning of the quarter, only two central potentials yield periodic behavior independent of the details of the initial conditions and independent of the force constant (k), $U_{\text{Kepler}} = -k/r$ and $U_{\text{HO}} = kr^2/2$. This result is typically called the Bertrand-Königs theorem.

To summarize, for W = irrational we find quasiperiodic behavior and the function for $\phi\left(\rho\right)$, or better it's inverse $\rho\left(\phi\right)$, must be an infinitely valued function. For periodic behavior we have $W = m/n$ and the orbit function is an m-valued function of ϕ if n is even and a $2m$-valued function of ϕ if n is odd. For the Kepler (and harmonic oscillator) problem m = 1 with n even and the orbit function is single-valued, *i.e.*, analytic. This analyticity arises from the existence of another conserved quantity, which allows the problem to be solved algebraically (and analytically) rather than via an integral. Here we discuss the Kepler problem. A similar strategy works for the oscillator problem.

Quite generally for the central force problem we have

$$\begin{aligned}
&\vec{p} = \mu\dot{\vec{r}},\ \dot{\vec{p}} = -U'(r)\hat{r} \\
&\Rightarrow \dot{\vec{p}}\times\vec{L} = -U'(r)\hat{r}\times\vec{L} = -U'(r)\hat{r}\times\left(\vec{r}\times\mu\dot{\vec{r}}\right) \\
&\qquad\qquad = -\mu U'(r)\left(\vec{r}\hat{r}\cdot\dot{\vec{r}} - \hat{r}\cdot\vec{r}\dot{\vec{r}}\right) \\
&\qquad\qquad = -\mu U'(r)\left(\vec{r}\dot{r} - r\dot{\vec{r}}\right) \\
&\qquad\qquad = \mu U'(r) r^2 \frac{d}{dt}\left(\frac{\vec{r}}{r}\right) \\
&\qquad\qquad = \mu U'(r) r^2 \frac{d}{dt}\hat{r}.
\end{aligned} \tag{14.A.15}$$

Since the total angular momentum is conserved for a central force problem, we can write $d\left(\vec{p}\times\vec{L}\right)/dt = \dot{\vec{p}}\times\vec{L}$. For the Kepler potential we have $U' = k/r^2$ so that we can rewrite Eq. (14.A.15) as

$$\begin{aligned}
&\frac{d}{dt}\left(\vec{p}\times\vec{L}\right) = \mu k\frac{d}{dt}\hat{r} \\
&\Rightarrow \frac{d}{dt}\left(\vec{p}\times\vec{L} - \mu k\hat{r}\right) = 0.
\end{aligned} \tag{14.A.16}$$

Thus we have another conserved vector quantity,

$$\vec{A} \equiv \vec{p}\times\vec{L} - \mu k\hat{r}, \tag{14.A.17}$$

called the Laplace-Runge-Lenz, or just Runge-Lenz, vector. Note that, since the angular momentum is orthogonal to the radius vector, we have

$$\vec{A}\cdot\vec{L} = 0. \tag{14.A.18}$$

Thus the Runge-Lenz vector lies in the plane orthogonal to the angular momentum. It has a magnitude defined by (using helpful vector identities)

$$
\begin{aligned}
A^2 &= \left(\vec{p}\times\vec{L}\right)\cdot\left(\vec{p}\times\vec{L}\right) - 2\mu k\hat{r}\cdot\left(\vec{p}\times\vec{L}\right) + \mu^2 k^2 \\
&= p^2L^2 - \left(\vec{p}\cdot\vec{L}\right)^2 - 2\mu k\vec{L}\cdot\left(\hat{r}\times\vec{p}\right) + \mu^2k^2 \\
&= p^2L^2 - 2\mu k\frac{L^2}{r} + \mu^2k^2 \\
&= 2\mu L^2\left[\frac{p^2}{2\mu} - \frac{k}{r}\right] + \mu^2k^2 \\
&= 2\mu E L^2 + \mu^2 k^2.
\end{aligned}
\tag{14.A.19}
$$

It is now easy to find the orbit function in terms of this conserved quantity and the angular momentum (using the form of the scalar triple product – symmetric under cyclic permutations),

$$
\begin{aligned}
\vec{r}\cdot\vec{A} = \vec{r}\cdot\left(\vec{p}\times\vec{L}\right) - \mu kr = \vec{L}\cdot\left(\vec{r}\times\vec{p}\right) - \mu kr \\
= L^2 - \mu kr.
\end{aligned}
\tag{14.A.20}
$$

As before we choose to define cylindrical coordinates with the *z*-axis aligned with $\vec{L}$ and the *x*-axis along the vector $\vec{A}$ in the $\left(x,y\right)$ plane. Then, for motion in that plane as required by angular momentum conservation, Eq. (14.A.20) reads

$$
\begin{aligned}
&\rho A\cos\phi = L_z^2 - \mu k\rho \\
&\Rightarrow \rho = \frac{L_z^2}{A\cos\phi + \mu k},
\end{aligned}
\tag{14.A.21}
$$

an analytic orbit function. We can further simplify this expression by defining $\varepsilon = \sqrt{1+2EL_z^2/k^2} = A/\mu k$ and $p' = L_z^2/\mu k = 1/C$ so that

$$
\begin{aligned}
&\rho = \frac{p'}{\varepsilon\cos\phi + 1}, \\
&\frac{1}{\rho} = C\left(\varepsilon\cos\phi + 1\right).
\end{aligned}
\tag{14.A.22}
$$

These familiar expressions allow us to identify ε as the eccentricity of the corresponding orbit: $E < 0 \Rightarrow \varepsilon < 1 \Rightarrow$ an ellipse, $E = 0 \Rightarrow \varepsilon = 1 \Rightarrow$ a circle, and $E > 0 \Rightarrow \varepsilon > 1 \Rightarrow$ a hyperbola. More formally the 3 components of $\vec{L}$ and the 3 components of the (properly normalized) vector $\vec{D} = \vec{A}\big/\sqrt{2\mu E}$ are all conserved and, viewed as operators, provide a representation of the 6 generators of the group SO(4) (rotations in a Euclidean 4-D space), which is the true symmetry of the Kepler problem (this is a formal extension of the more familiar 3-D configuration space).

The extra conserved quantity in the isotropic harmonic oscillator problem is a symmetric (3x3) tensor, which has 6 independent components. However, the trace of this tensor (correctly normalized) is just the Hamiltonian. So besides the Hamiltonian, we have 3 ($\vec{L}$) plus 5 (the components of the traceless symmetric tensor), *i.e.*, 8, conserved quantities. These 8 quantities, viewed as operators, provide a representation of the group SU(3), the true symmetry of the 3-D isotopic harmonic oscillator.