Lecture 16 - More About Chaos

Let us try to summarize what we know about systems that can exhibit chaotic behavior.

- First we note that chaotic dynamics can arise in *fully deterministic* systems, which nevertheless exhibit unpredictable, non-repeating behavior.

- Non-chaotic behavior is characterized by trajectories in phase space with slightly different initial conditions exhibiting separations that grow only linearly with the evolution parameter, *e.g.*, time for Hamiltonian systems. In contrast chaotic behavior is characterized by separations of trajectories with slightly different initial conditions growing exponentially with the evolution parameter, at least initially (*i.e.*, the separation may eventually be constrained by the size of the allowed phase space). The coefficient in the exponential behavior is the Liapunov exponent λ, separation ~ $e^{\lambda t}$. As a result any small (but nonzero, and it is always nonzero in real life) uncertainty in our knowledge of the initial conditions will eventually become an uncertainty of order 1 (*i.e.*, a 100% uncertainty). In a finite time, our ability to reliably predict the future motion of the system has completely vanished.

- The necessary conditions for chaos in mechanical systems include:

    - Nonlinearity in the dynamics, yielding computational complexity and the typically essential role of numerical solutions.

    - ≥ 3 independent dynamical variables (*i.e.*, the phase space is at least 3-D; recall that in 2-D the motion is either bounded, and necessarily periodic, or unbounded, effectively corresponding to constant momentum in the appropriate generalized coordinates. The dynamics (in Hamilton language) looks like

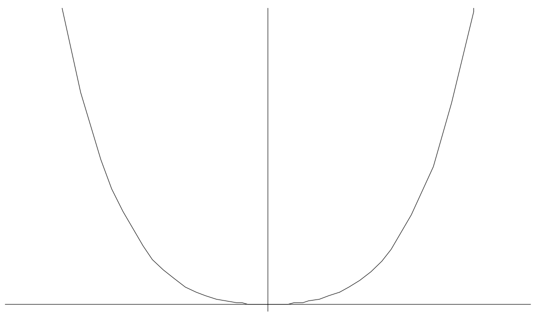
$$
\begin{aligned}
\frac{dx_1}{d\tau} &= V_1\left(x_1,\ldots x_n\right) \\
\frac{dx_2}{d\tau} &= V_2\left(x_1,\ldots x_n\right) \\
\vdots \qquad & \qquad \vdots \\
\frac{dx_n}{d\tau} &= V_n\left(x_1,\ldots x_n\right).
\end{aligned}
\tag{16.1}
$$

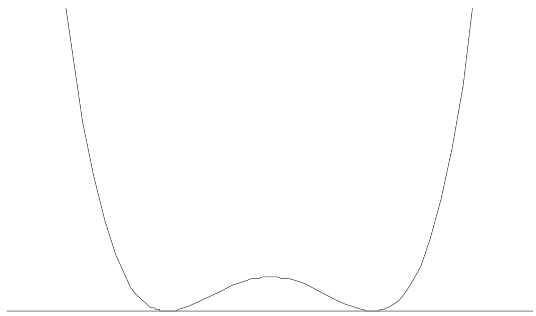
Recall that for 1-D systems, *i.e.*, 1-spatial coordinate, we obtained a 3-D phase space by including a driving force and using the phase of the driving force as the third dynamical variable.

- Characteristic features of chaotic behavior include:

  - Bifurcations in parameter space – small changes in the parameters yield "transitions" to qualitatively different behavior (*e.g.*, the Higgs phenomenon). Recall our multiple studies of the spinning hoop and its "pitchfork" transition described by the following transition in the potential.

$\Rightarrow$





  - A continuum of spectral (Fourier) components – not just harmonics and subharmonics

  - Extreme sensitivity to the initial conditions.

- Fractal structure in phase space – the separatrix separating regions of different dynamic behaviors become fuzzy or fractal as illustrated in Fig 3.12 in B&G.

We often see chaotic behavior develop out of periodic (or quasi-periodic) behavior as the parameters are varied. Consider our favorite example of the 1-D, damped, driven pendulum (a non-linear oscillator).

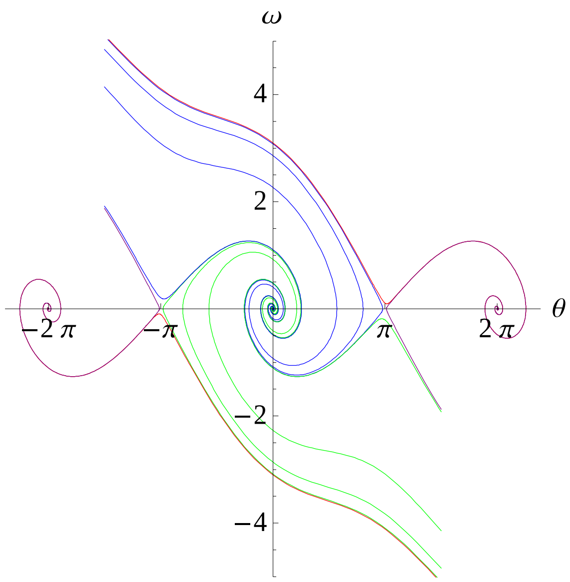
$$\ddot{\theta} + \gamma\dot{\theta} + \sin\theta = g\cos\omega_D t, \tag{16.2}$$

where, for simplicity, we have chosen to take unit natural frequency, $\omega_0 = 1$ (essentially a choice of time units). Thus the 3-D flow problem looks like ($\phi = \omega_D t$)

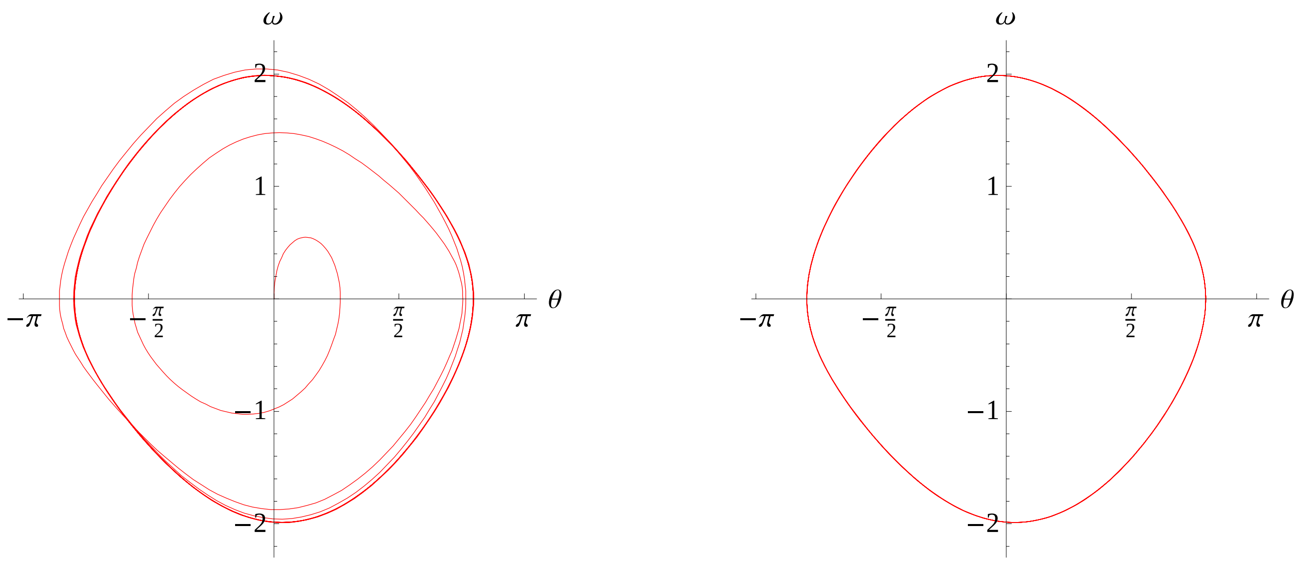
$$\begin{aligned} \dot{\theta} &= \omega, \\ \dot{\omega} &= -\gamma\omega - \sin\theta + g\cos\phi, \\ \dot{\phi} &= \omega_D, \end{aligned} \tag{16.3}$$

where, due to the dissipation, the flow does not conserve the volume in phase space, $\vec{\nabla}\cdot\vec{V} = -\gamma$, or in the Hamiltonian language the energy is not conserved. Recall that without the damping the flow volume would be conserved (a conservative system). The damping also allows attractors. Local nonzero divergence of the flow field corresponds to sources and sinks.

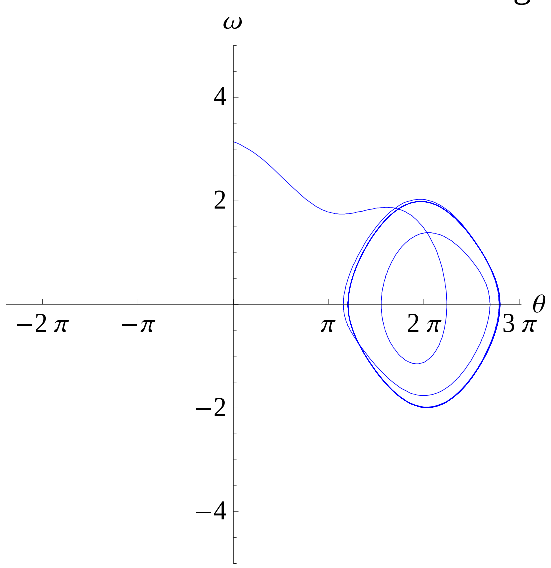
Recall that in the un-driven case we had the structure indicated in the figure to the right (see HW IX.3) with stable (elliptic) equilibria (or fixed points) at $\left(\theta = 2n\pi, \omega = \dot{\theta} = 0\right)$ and unstable (hyperbolic) equilibria (saddle points) at $\left((2n+1)\pi, 0\right)$. The trajectories passing through the unstable equilibria lie on the separatrix that separate the basins of attraction surrounding each of the 0-D attractors at the stable equilibria. (See the associated *Mathematica* notebook for how the figures here were created.)



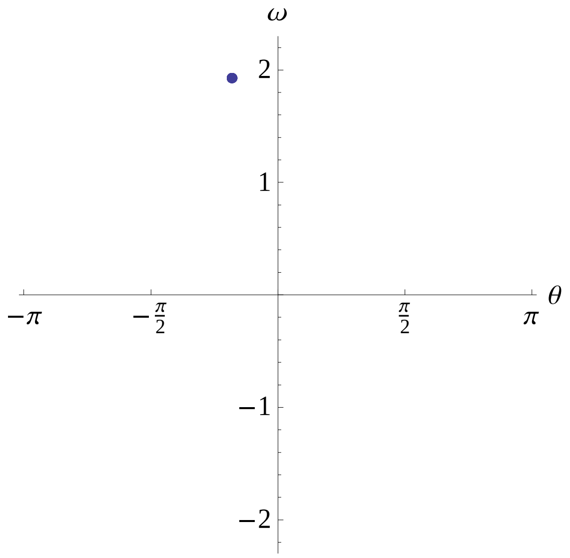
Now consider driving the pendulum. We take, for example, $\gamma = 0.5$, $\omega_D = 2/3$ as in B&G. Then for a driving force $g = 0.9$ we have periodic motion with after the transients have damped out. The indicated limit cycle trajectory is a 1-D attractor. We see this by displaying the trajectory for all times (with initial conditions $\dot{\theta}(0) = \theta(0) = 0$) to the left and then only for late times to the right.

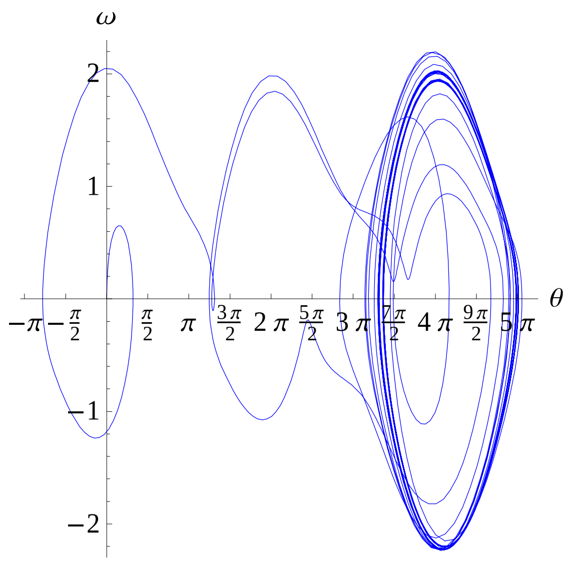


ASIDE: Note that, if we start well away from the origin, we may flow to other attractors as indicated in the next figure.

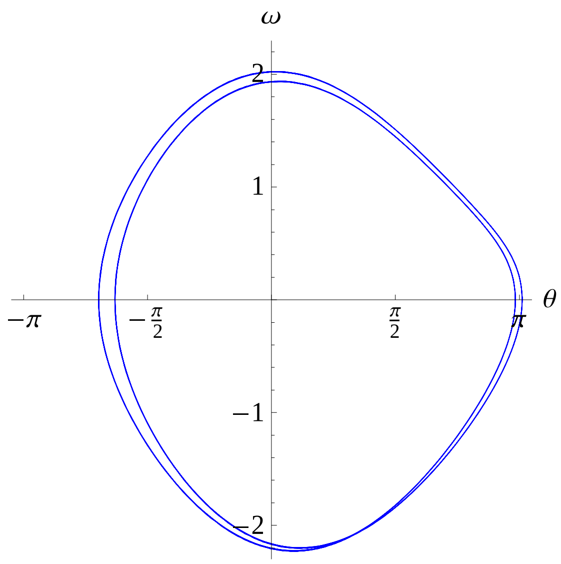


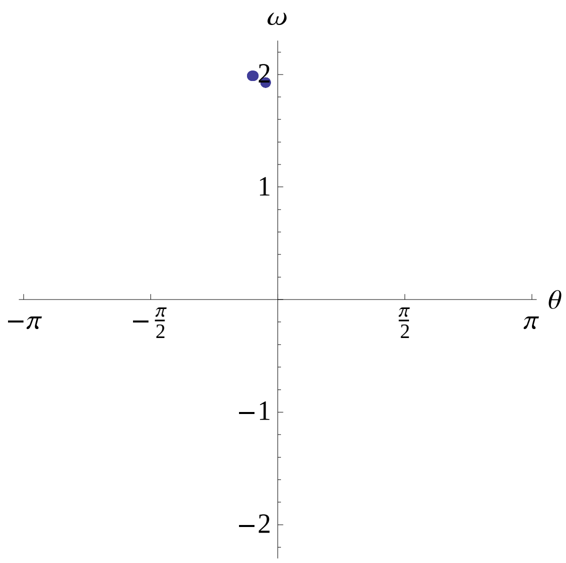
Returning to the original initial conditions and the periodic behavior above, we look at the Poincaré section (strobed at the driving frequency). As expected the frequency of the periodic motion is the driving frequency and we see a single point in the section to the right (the strobing and the motion are synchronized).



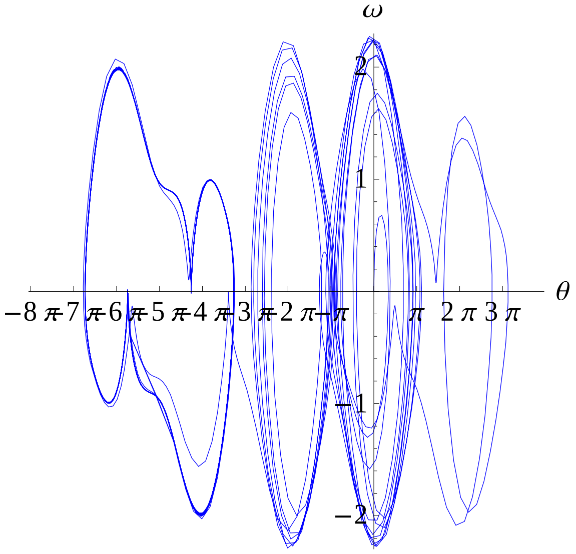


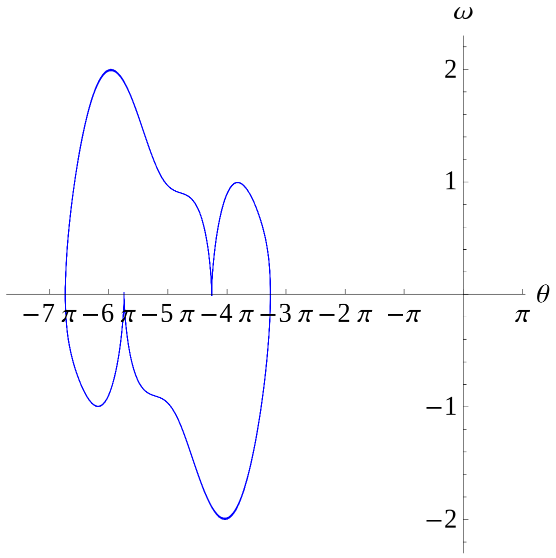
If we increase the driving force to $g = 1.07$, we observe many changes in the figure to the right. First we note that the long time motion is around the attractor at 4π (with the same initial conditions). To simplify the discussion we focus on long times and map back onto the range $-\pi$ to $+\pi$, *i.e.*, Mod 2π (as is done in B&G). The trajectory at the attractor is indicated in the next figure (just below). The essential new feature is that, while the motion is still periodic, the period has doubled (both ω_D and $\omega_D/2$ are present). This is suggested by the double curve and verified by the 2 points in the Poincaré section (to the right) corresponding to *two* points of crossing.



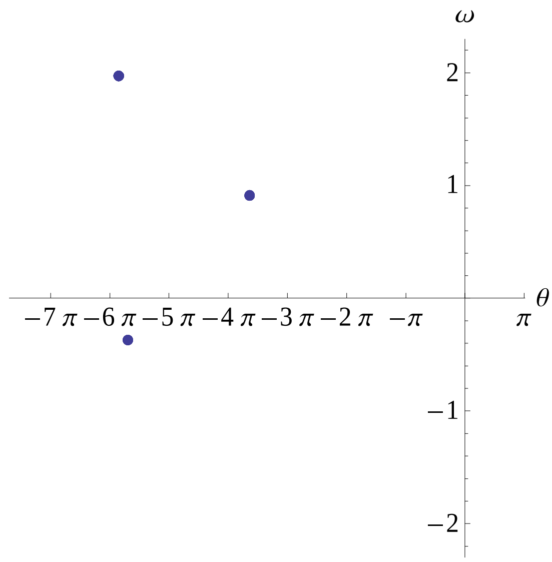


Next we consider increasing the driving term to $g = 1.12$, which leads to a more complicated attractor located at large negative angles (after considerable time spent exploring the region around the origin) - the next figure (left) includes the early times, while the second (right) is for $t > 300$.

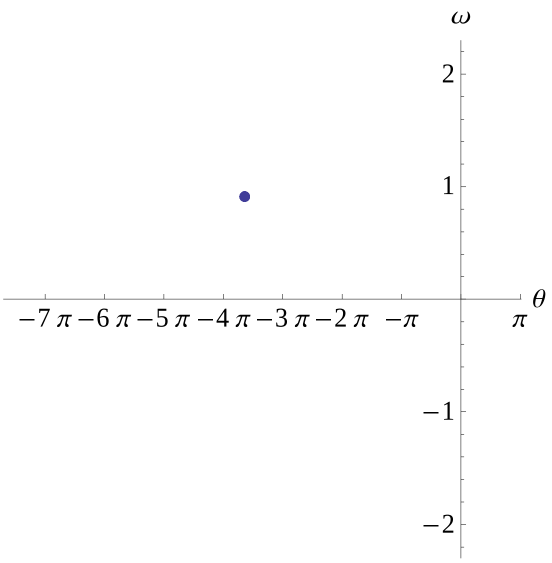




The interesting point here is that the period has now tripled as indicated by the 3 points in the next Poincaré section to the right. To fully verify this point consider the section when strobing at 1/3 the driving frequency as in the next figure - just a single point!

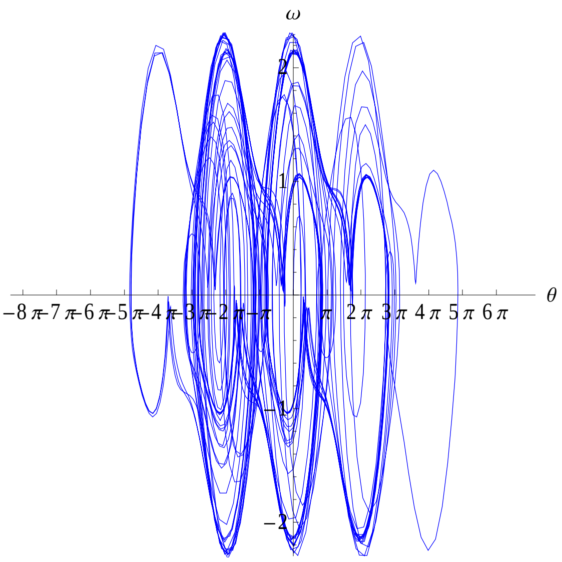


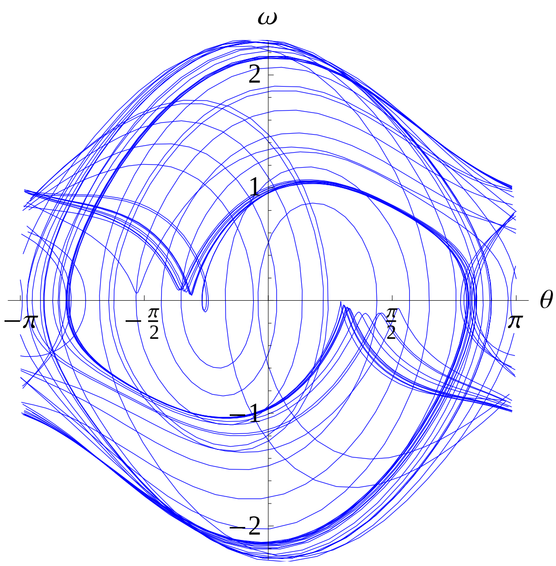
$\omega_s = \omega_D$



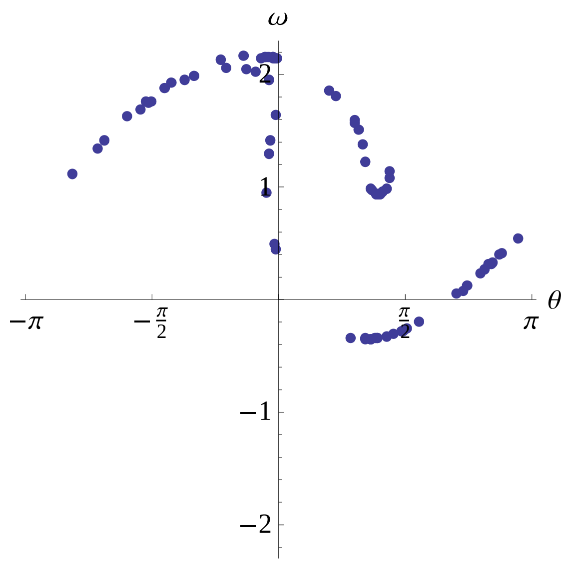
$\omega_s = \omega_D / 3$

As in B&G we next consider the "nearby" case $g = 1.15$. Now we are meant to see chaos. The first point is that the trajectory in phase space in much more complicated. We see first the full trajectory for $t < 1000$ (left) and then display the trajectory mapped onto the angular interval around the origin as in B&G and appropriate for a real pendulum (right).

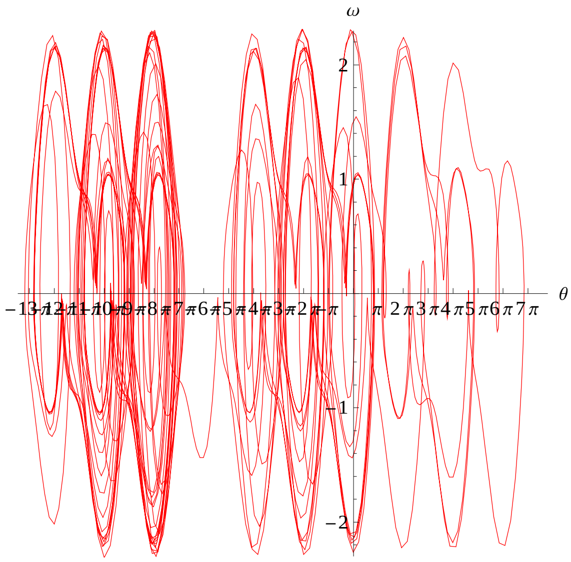




It is clear that there is no indication of periodic or attractor behavior. The characteristic feature of chaos is that the phase portrait will "fill up" with non-repeating motion, while the points in the Poincaré section will systematically fall along a typically complicated curve as indicated in the figure to the right (see also Fig. 3.4c in B&G).



Another indication of the underlying chaotic behavior is the rapid change in the trajectory for even a small change in the initial conditions or the parameters characterizing the pendulum. Here we simply consider the resulting trajectory if we set $g = 1.151$ as in the figure below and compare it to the figure above - note that it is *very* different!



As indicated in B&G, further increases in g can lead us to revisit periodic behavior, then observe more period doubling, and finally see chaotic behavior (trajectories filling the phase portrait and filling a curve in the Poincaré section, see the remaining parts of Fig. 3.4). The reader is encouraged to use the associated Mma notebook to perform this exploration.

Interestingly the points of bifurcation (period doubling) are a characteristic route from periodicity to chaos and exhibit a certain universal feature. The asymptotic ratio of contiguous spacing of the points of bifurcations is given by the Feigenbaum number. Define g_k as the value at the k$^{\text{th}}$ bifurcation, where g is some parameter of the system in question. Then we have, for *all* systems with this behavior,

$$\lim_{k\to\infty} \frac{g_k - g_{k-1}}{g_{k+1} - g_k} = 4.6692016\ldots, \qquad (16.4)$$

i.e., the points get sequentially closer together by a factor of about 4.67.

Other observed routes to chaos include:

- Quasiperiodic behavior – periodic limit cycles in at least 2 coordinates whose frequencies are incommensurate.

- Intermittency - regions of periodic behavior in parameter space interspersed with chaos leading finally to continuing chaos.

- Conservative chaos arising, *e.g.*, from chaotic (but conservative) orbital motion. Another interesting example is the compound pendulum demonstrated in a simulation from the Wikipedia web site and included on our web page; also see a compound pendulum in real life on YouTube. http://www.youtube.com/watch?v=z3W5aw-VKKA

Another remarkable feature of chaotic motion is that simple structures in phase space for non-chaotic motion, like 0-D (fixed points) and 1-D (limit cycles) attractors (and the separatrix defining basins of attraction), become more complicated in chaotic motion, *i.e.*, more fuzzy. The underlying concept here is fractal behavior (see Chapter 5 in B&G), *i.e.*, these structures become fractals. They exhibit non-integer dimensions larger than their "natural" values. The corresponding chaotic attractors are called "strange attractors" with dimension >0 and >1, respectively. This is a pretty confusing concept so let's consider some simple examples. The first issue is to define what we mean by dimension. Here we consider the so-called capacity dimension (labeled d_C in B&G). Consider first a ("normal") line made up out of the usual set of contiguous segments, each of length ε. We are accustomed to saying that, if the line is of total length L, then the relationship between the total length, the length of each segment and the number of segments, $N(\varepsilon)$, is

$$L = N(\varepsilon)\times\varepsilon \Rightarrow N(\varepsilon) = \frac{L}{\varepsilon}. \tag{16.5}$$

More generally, we would say that a d-dimensional object of "volume" L^d (in the usual engineering sense) can be built out of d-dimensional elements of "volume" ε^d, where the number of elements required is given by

$$N(\varepsilon) = \frac{L^d}{\varepsilon^d}. \tag{16.6}$$

Thus we can define the dimension generally by taking logarithms,

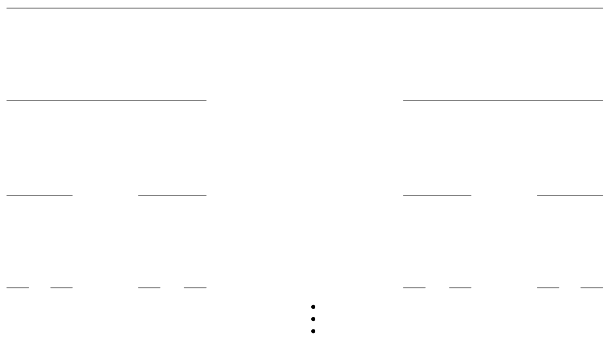
$$d = \frac{\ln\left(N(\varepsilon)\right)}{\ln L - \ln\varepsilon}, \tag{16.7}$$

and consider the limit $\varepsilon \to 0$ to obtain the capacity dimension via

$$d_C = \lim_{\varepsilon\to 0} \frac{\ln\left(N\left(\varepsilon\right)\right)}{\ln L - \ln\varepsilon} = \lim_{\varepsilon\to 0} \frac{\ln\left(N\left(\varepsilon\right)\right)}{\ln\left(1/\varepsilon\right)}. \tag{16.8}$$

The simplest initially 1-D case with nontrivial behavior is the Cantor set obtained by sequentially dividing each line segment into 3 equal parts and throwing out the middle one. The sequence looks like

$N(\varepsilon)$	ε
1	1
2	1/3
4	1/9
8	1/27
⋮	⋮



We easily see that for the *n*th element in the set we have $N\left(\varepsilon\right) = 2^{n-1}$ and $\varepsilon = \left(1/3\right)^{n-1}$. Thus the (capacity) dimension for the Cantor set is given by

$$d_{\text{Cantor}} = \lim_{n\to\infty} \frac{\left(n-1\right)\ln 2}{\left(n-1\right)\ln 3} = \frac{\ln 2}{\ln 3} = 0.6309\ldots. \tag{16.9}$$

Not surprisingly, since the actual amount of line is shrinking at each step, the fractal dimension is less than the naïve dimension. For the Koch curve considered in the HW (and on page 112 in B&G), the lines gets longer (it spreads out in 2-D) and has dimension > 1. The latter case of increasing dimension is what happens in chaotic motion as things fuzz out. Note also that for both the Cantor set and the Koch curve the structures observed at each step are of similar form, *i.e.*, the basic geometry looks the same at any level of magnification. This feature of self-similarity is common but not universal for fractals.

Clearly chaotic systems, and nonlinear systems in general, exhibit a remarkable richness of possible behaviors. You will see more of this as you pursue your career in physics.