Appendix to Lecture 4 - Fetter & Walecka – Exercise 2.6: Studying motion in a rotating frame to second order in the rotation velocity $\omega$.

We want to try keeping terms to second order in the rotation frequency $\omega$ in the solution of the falling body problem. We can proceed as in the lecture notes (see Eq. 4.24 or Eq. 11.8 in F&W). We have (use $g$ and $\omega$ instead of $g_{eff}$ and $\omega_E$)

$$m\ddot{\vec{r}}'(t)\Big|_{\text{acceler}} \simeq -mg\hat{z}' - 2m\vec{\omega}\times\dot{\vec{r}}'(t)_{\text{acceler}},$$

and try the Ansatz

$$\dot{\vec{r}}' \equiv -g\,t\,\hat{z}' + \dot{\vec{\delta}}(t).$$

On substitution this definition yields

$$\ddot{\vec{\delta}} \simeq -2\vec{\omega}\times\left(-g\,t\,\hat{z}' + \dot{\vec{\delta}}(t)\right).$$

We obtain the result to first order in $\omega$ by ignoring the second term in the parenthesis compared to the first (recall $\hat{z} = \cos\theta\hat{z}' - \sin\theta\hat{x}'$),

$$\ddot{\vec{\delta}} \simeq -2\vec{\omega}\times\left(-gt\hat{z}'\right) \Rightarrow \vec{\delta} \simeq \omega g\frac{t^3}{3}\left(\hat{z}\times\hat{z}'\right) = \omega g\sin\theta\frac{t^3}{3}\hat{y}'.$$

This in terms implies that $\dot{\vec{\delta}} \simeq \omega\, g\sin\theta t^2\hat{y}'$ to leading order in $\omega$ and we can iterate in the equation above to get the answer to order $\omega^2$. Thus we have

$$\begin{aligned}\ddot{\vec{\delta}} &\simeq -2\vec{\omega}\times\left(-gt\hat{z}' + \dot{\vec{\delta}}(t)\right) \simeq -2\vec{\omega}\times\left(-gt\hat{z}' + \omega\, g\sin\theta t^2\hat{y}'\right)\\ &\simeq 2\omega\, g\sin\theta t\hat{y}' - 2\omega^2 g\sin\theta t^2\left(\hat{z}\times\hat{y}'\right)\\ &\simeq 2\omega\, g\sin\theta t\hat{y}' + 2\omega^2 g\sin\theta t^2\left(\sin\theta\hat{z}' + \cos\theta\hat{x}'\right).\end{aligned}$$

So at this order there are two new terms in the acceleration. The $\hat{z}'$ term changes the (local) vertical motion to

$$z'(t) = -g\frac{t^2}{2} + \omega^2 g \sin^2\theta \frac{t^4}{6}.$$

Masses fall more slowly and the time to fall a distance $h$ changes from (the freshman physics result) $\tau_1 = \sqrt{2h/g}$ to

$$\begin{aligned}\tau_2 &\simeq \sqrt{\frac{2}{g}\left(h + \omega^2 g \sin^2\theta \frac{\tau_1^4}{6}\right)} \simeq \sqrt{\frac{2h}{g}}\left(1 + \frac{\omega^2 g \sin^2\theta}{12h}\frac{4h^2}{g^2}\right) \\ &\simeq \tau_1 + \frac{1}{6}\omega^2 \sin^2\theta \left(\sqrt{\frac{2h}{g}}\right)^3 = \tau_1 + \frac{\tau_1}{6}(\omega\tau_1)^2 \sin^2\theta,\end{aligned}$$

where, to obtain the correction, we can use the unperturbed time $\tau_1$. The $\hat{x}'$ term in the acceleration leads to a drift towards the equator (in either hemisphere due to the $\cos\theta$ factor) in the amount

$$\begin{aligned}x'(\tau_2) &\simeq x'(\tau_1) \simeq \omega^2 g \sin\theta\cos\theta \frac{\tau_1^4}{6} \simeq \frac{2h^2}{3g}\omega^2 \sin\theta\cos\theta \\ &\simeq \frac{h^2}{3g}\omega^2 \sin 2\theta = \frac{h}{6}(\omega\tau_1)^2 \sin 2\theta.\end{aligned}$$

This analysis reproduces the required expressions.

However, we might ask about the role of the second order terms in the effective gravity, (see Eq. 11.6 expressed in the coordinates of the rotating frame)

$$\vec{g}_{eff}(\theta) = -\left(\frac{G}{M_E R_E} - \omega_E^2 R_E \sin^2\theta\right)\hat{z}' + \omega_E^2 R_E \sin\theta\cos\theta\hat{x}'.$$

The quantity in the parenthesis can be taken as a definition of what we mean by $g$ in the equations above, *i.e.*, already including an order $\omega^2$ correction in the above expression for $\tau_1$. Note that this correction is of magnitude $\Delta\tau \simeq \tau_1\omega^2 R_E \sin^2\theta/2g = \left(\tau_1^3\omega^2\sin^2\theta/4\right)\left(R_E/h\right)$, *i.e.*, larger than the one above. The second term in the expression for $\vec{g}_{eff}$ also leads to a drift towards the equator of magnitude $\Delta x' \simeq \tau_1^2 \omega^2 R_E \sin\theta \cos\theta / 2 = \left(h^2 \omega^2 \sin\theta \cos\theta / g\right)\left(R_E / h\right)$. Again this correction is larger than the Coriolis term above. In fact, the earth responds to the non-radial, centrifugal force by becoming oblate as discussed in F&W 2.7 in the HW.