Lecture 7 Appendix: Examples of Flows in Phase Space

At the end of Lecture 7 we suggested that the imagery of flow patterns in the phase space of the canonical variables (q, p) is a powerful tool for (at least qualitatively) analyzing the behavior of mechanical (and other) systems, even in situations where we cannot solve the equations of motion analytically. The formalism allows us to make more complete (and sometimes nonstandard) definitions of familiar concepts in mechanics, such as conservative, solvable, integrable, chaotic, *etc*.

As at the end of Lecture 7 we consider a system described by a Hamiltonian that is a function of *f* generalized variables, the conjugate momenta and time, $H\left(q_1,\ldots q_f, p_1,\ldots, p_f, t\right)$. The equations of motion have the canonical form

$$\dot{q}_k = \frac{\partial H}{\partial p_k},\ \dot{p}_k = -\frac{\partial H}{\partial q_k}. \tag{A.7.1}$$

The solutions of these equations define a trajectory in the corresponding 2*f*-dimensional phase space parameterized by the time *t*, $q_k(t), p_k(t)$. The family of such solutions corresponding to various initial values, $q_k(t_0), p_k(t_0)$, serve to define a "flow pattern" in phase space, which we want to analyze much as we would analyze fluid flow. As suggested at the end of Lecture 7, there are many similarities between the Hamilton problem of interest here and the flow of actual fluids, which offers the possibility of an intuitive understanding of the more general mechanics problem. The flow problem is characterized by a velocity field $\vec{V}$ (with *n* components) and a stream function ψ. For us the components of the velocity field are the right-hand-sides of the 2*f* equations for $\dot{q}_k$ and $\dot{p}_k$, and the role of the stream function is played by the Hamiltonian. The phase space is typically formally treated as if it was Cartesian (even when the mechanics variables are not actually Cartesian) but without explicit reference to a metric (*e.g.*, it is not a Euclidean space). There is, in fact, no direct meaning for the concept of distance in a space where the coordinates are both locations *and* momenta. For the general flow problem we start with an autonomous system of coupled ordinary equations

$$\dot{x}_k = V_k\left(x_1,\ldots,x_n\right), \tag{A.7.2}$$

where the label autonomous means there is no explicit time dependence on the right-hand-side (*i.e.*, the Hamiltonian has no explicit time dependence). The set of problems defined by Eq. (A.7.2) is, in fact, more general than Hamiltonian mechanics. A solution to such a problem corresponds to finding *n* ordinary functions, $x_k\left(x_{1,0},\ldots,x_{n,0},t\right)$, of the initial conditions and time, where (in mechanics) we typically demand that these functions be unique, differentiable, finite at all finite times and constructible (at least approximately) by recursion methods (successive approximations). A problem with such solutions is labeled as solvable and defines an acceptable flow pattern. We expect such systems to display deterministic behavior. We will return eventually to the issue of how random, chaotic behavior can arise.

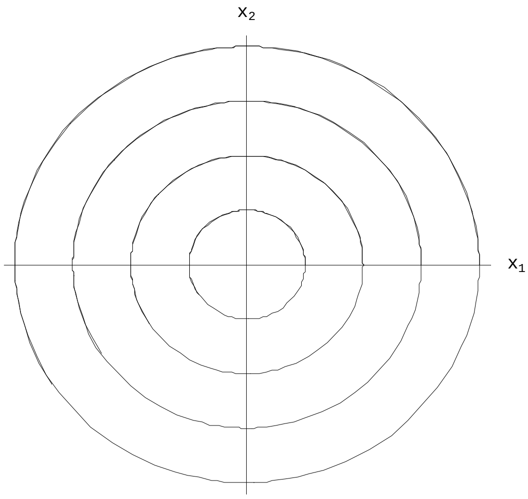
The qualitative behavior of such systems can be summarized in terms of plots of the flow pattern, which exhibit vectors at a grid of points in phase space that are locally tangent to the velocity field. The magnitudes of these vectors are not generally important except at equilibrium points, where the right-hand-sides of all *n* equations in Eq. (A.7.2) vanish. In this discussion we will focus on examples of planar flow, *f* = 1, *i.e.*, a particle moving in one physical dimension ($x_1 = q_1, x_2 = p_1$), leaving higher dimensional examples until later. Three simple examples of conservative systems are the following.

1) A free particle, $\vec{V}=\left(x_2,0\right)$, with streamlines parallel to the x_1 axis (lines of constant momentum x_2) as indicated in the figure

$$\begin{aligned}&\dot{x}_1=V_1=x_2=\frac{\partial H}{\partial x_2},\dot{x}_2=V_2=0=-\frac{\partial H}{\partial x_1}\\&\Rightarrow x_2=\text{constant}, H=\frac{1}{2}x_2^2=\text{constant.}\end{aligned}\tag{A.7.3}$$

The specific values are determined by the initial conditions. Note the essential (non-chaotic) feature that trajectories that start nearby each other stay nearby (stability of trajectories).

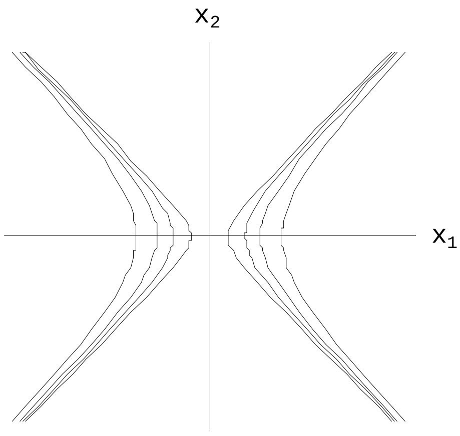
2) A linear restoring force (the harmonic oscillator with unit mass and spring constant), $\vec{V} = (x_2, -x_1)$, with streamlines forming circles around the origin ($G(x,\dot{x}) = G(x_1,x_2) = x_1^2 + x_2^2$ = constant, as indicated in the figure for various values of the constant), which is a stable equilibrium point,



$$\begin{aligned}
&\dot{x}_1 = V_1 = x_2 = \frac{\partial H}{\partial x_2} \Rightarrow H = \frac{1}{2}x_2^2 + f(x_1), \\
&\dot{x}_2 = V_2 = -x_1 = -\frac{\partial H}{\partial x_1} \Rightarrow H = \frac{1}{2}x_1^2 + \tilde{f}(x_2) \\
&\Rightarrow H = \frac{1}{2}x_2^2 + \frac{1}{2}x_1^2\left(+\text{constant}\right), \\
&\Rightarrow \dot{H} = x_2\dot{x}_2 + x_1\dot{x}_1 = -x_2x_1 + x_1x_2 = 0.
\end{aligned} \tag{A.7.4}$$

The trajectories repeat themselves each time around yielding periodic orbits ($x_1 = a_0\cos(t-t_0), x_2 = -a_0\sin(t-t_0)$) and the orbits are stable, *i.e.*, nearby starting points yield nearby orbits. The Hamiltonian is conserved.

3) A linear repulsive force, $\vec{V} = (x_2, x_1)$, with hyperbolic streamlines ($G(x,\dot{x}) = G(x_1,x_2) = x_2^2 - x_1^2$ = constant, as indicated in the figure for various values of the constant) centered at the origin



$$\begin{aligned}
&\dot{x}_1 = V_1 = x_2 = \frac{\partial H}{\partial x_2} \Rightarrow H = \frac{1}{2}x_2^2 + f\left(x_1\right), \\
&\dot{x}_2 = V_2 = x_1 = -\frac{\partial H}{\partial x_1} \Rightarrow H = -\frac{1}{2}x_1^2 + \tilde{f}\left(x_2\right) \\
&\Rightarrow H = \frac{1}{2}x_2^2 - \frac{1}{2}x_1^2\left(+\text{constant}\right) \\
&\Rightarrow \dot{H} = x_2\dot{x}_2 - x_1\dot{x}_1 = x_2 x_1 - x_1 x_2 = 0.
\end{aligned} \tag{A.7.5}$$

Again the conserved quantity is just the Hamiltonian (the total energy) and again the trajectories are stable in the sense that nearby starting points yield nearby trajectories. In contrast to the previous example, there is no stable equilibrium point here. Note also that, although the trajectories in this last example go off to infinity,

$$\begin{aligned}
&x\left(t\right) = x_1\left(t\right) = c_1 e^{t} + c_2 e^{-t}, \\
&p\left(t\right) = x_2\left(t\right) = c_1 e^{t} - c_2 e^{-t}, \\
&c_1 + c_2 = x\left(0\right), c_1 - c_2 = p\left(0\right),
\end{aligned} \tag{A.7.6}$$

it takes infinite time to get there.

Those students who want to get ahead may want to read the following notes, which we will not discuss in class until later. Connecting again to the last lecture (and the work of Lie) we can think of the solutions of Eq. (A.7.2),

$$x_k\left(t\right) = \Psi_k\left(x_1\left(t_0\right),\ldots,x_n\left(t_0\right),t-t_0\right), \tag{A.7.7}$$

as a 1-parameter coordinate transformation from the initial point in phase space to the point at time *t*. The corresponding inverse transformation is just

$$x_k\left(t_0\right) = \Psi_k\left(x_1\left(t\right),\ldots,x_n\left(t\right),t_0-t\right). \tag{A.7.8}$$

The infinitesimal transformation (near the initial time) is given by

$$dx_k = V_k\left(x_1\left(t_0\right),\ldots,x_n\left(t_0\right)\right)dt. \tag{A.7.9}$$

We can also express this coordinate transformation symbolically as

$$x_k\left(t\right)=U\left(t-t_0\right)x_k\left(t_0\right), \tag{A.7.10}$$

where U is the time evolution operator. This operator represents the simultaneous solution of the n differential equations. In the case that the velocity field, *i.e.*, the Hamiltonian, is analytic, we can use Taylor series expansion techniques to express

$$x_k\left(t+\Delta t\right)=x_k\left(t\right)+\Delta t\dot{x}_k+\frac{\Delta t^2}{2}\ddot{x}_k+\cdots, \tag{A.7.11}$$

and also, with no *explicit* time dependence (*i.e.*, the time dependence arises from the flow), write

$$\frac{d}{dt}=\dot{x}_k\frac{\partial}{\partial x_k}=\vec{V}\cdot\vec{\nabla}\equiv L. \tag{A.7.12}$$

Note that this L is NOT the Lagrangian (we are in Hamiltonian mode here). An example is

$$L_{\mathrm{HO}}=x_2\frac{\partial}{\partial x_1}-x_1\frac{\partial}{\partial x_2} \tag{A.7.13}$$

for the harmonic oscillator of example 2) above. In this case we can identify

$$\begin{aligned}x_k\left(t+\Delta t\right)&=x_k\left(t\right)+\Delta tLx_k\left(t\right)+\frac{\Delta t^2L^2}{2}x_k\left(t\right)+\cdots\\&=e^{\Delta tL}x_k\left(t\right)\\\Rightarrow U\left(t-t_0\right)&=e^{\Delta tL}.\end{aligned} \tag{A.7.14}$$

At least formally, this exponential operator expresses the integration of the original equations. (Such exponential expressions are typical of the transition from infinitesimal transformations, and generators, to finite transformations or members of the corresponding group. Recall also that, in the context of quantum mechanics and quantum field theory, it is the Hamiltonian operator that is the generator of infinitesimal translations in time.) Is important to note that the radius of convergence of Eq. (A.7.14) is not *a priori* known, *i.e.*, is not known until problem is fully solved. These considerations can be extended, at least formally, to driven systems with a velocity field (and Hamiltonian) that has explicit time dependence, as we will shortly discuss.

Next we want to consider how the volume of our "fluid" changes as it flows along its trajectories. Consider a small volume element at the initial time,

$$\delta\Omega(t_0) = \delta x_1(t_0)\ldots\delta x_n(t_0). \qquad \text{(A.7.15)}$$

Each point within this volume corresponds to the initial conditions for a trajectory of the system. We will not consider the difficult problem of changes in shape but rather simply consider how the size of this volume element changes along the trajectories. For an infinitesimal element, the transformation reduces to the Jacobi determinant of the overall coordinate transformation,

$$\begin{aligned} \delta\Omega(t) &= J(t,t_0)\,\delta\Omega(t_0), \\ J(t,t_0) &= \frac{\partial\left(x_1(t),\ldots,x_n(t)\right)}{\partial\left(x_1(t_0),\ldots,x_n(t_0)\right)}. \end{aligned} \qquad \text{(A.7.16)}$$

ASIDE: If this result is unfamiliar, review some differential geometry. As an explicit example consider n = 2,

$$\begin{aligned} &d^2\Omega(t) = dx_1(t)\,dx_2(t),\; d^2\Omega(t_0) = dx_1(t_0)\,dx_2(t_0), \\ &dx_1(t)\,dx_2(t) \equiv \left|d\vec{x}_1(t)\times d\vec{x}_2(t)\right| \\ &\qquad = \left|\frac{\partial x_1(t)}{\partial x_1(t_0)}\frac{\partial x_2(t)}{\partial x_2(t_0)} - \frac{\partial x_2(t)}{\partial x_1(t_0)}\frac{\partial x_1(t)}{\partial x_2(t_0)}\right|\left|d\vec{x}_1(t_0)\times d\vec{x}_2(t_0)\right| \\ &\qquad = \left|\det\begin{pmatrix} \dfrac{\partial x_1(t)}{\partial x_1(t_0)} & \dfrac{\partial x_1(t)}{\partial x_2(t_0)} \\ \dfrac{\partial x_2(t)}{\partial x_1(t_0)} & \dfrac{\partial x_2(t)}{\partial x_2(t_0)} \end{pmatrix}\right| dx_1(t_0)\,dx_2(t_0) \\ &\qquad = J\,dx_1(t_0)\,dx_2(t_0). \end{aligned}$$

We can learn about this Jacobi determinant by considering its time dependence,

$$\begin{aligned}\frac{dJ}{dt} &= \frac{\partial\left(\dot{x}_1(t),\ldots,x_n(t)\right)}{\partial\left(x_1(t_0),\ldots,x_n(t_0)\right)} + \frac{\partial\left(x_1(t),\dot{x}_2\ldots,x_n(t)\right)}{\partial\left(x_1(t_0),\ldots,x_n(t_0)\right)} \\ &+\ldots+\frac{\partial\left(x_1(t),\ldots,\dot{x}_n(t)\right)}{\partial\left(x_1(t_0),\ldots,x_n(t_0)\right)}.\end{aligned} \tag{A.7.17}$$

By the chain rule for such determinants we can express a specific term as

$$\begin{aligned}&\frac{\partial\left(x_1(t),\ldots,\dot{x}_k(t)\ldots,x_n(t)\right)}{\partial\left(x_1(t_0),\ldots,x_n(t_0)\right)} \\ &\quad= \frac{\partial\left(x_1(t),\ldots,\dot{x}_k(t)\ldots,x_n(t)\right)}{\partial\left(x_1(t),\ldots,x_n(t)\right)} J(t,t_0) \\ &\quad= \frac{\partial\dot{x}_k(t)}{\partial x_k(t)} J(t,t_0).\end{aligned} \tag{A.7.18}$$

In this last expression the subscript *k* is not summed over. However, when we return to Eq. (A.7.17), we do want the sum,

$$\begin{aligned}\frac{dJ}{dt} &= \sum_{k=1}^{n}\frac{\partial\dot{x}_k(t)}{\partial x_k(t)} J(t,t_0), \\ \dot{J} &= \left(\vec{\nabla}\cdot\vec{V}\right)J \equiv \sum_{k=1}^{n}\frac{\partial V_k(t)}{\partial x_k(t)} J(t,t_0).\end{aligned} \tag{A.7.19}$$

Note that again we are using Cartesian notation even though the underlying coordinates may not be Cartesian and we do not necessarily have a metric.

Eq. (A.7.19) allows us to define conservative dynamical systems in a new fashion. We will define conservative systems to correspond to incompressible flow in phase space, *i.e.*, the velocity field is divergence free, $\vec{\nabla}\cdot\vec{V}=0$. In this case the Jacobi determinant is a constant ($\dot{J}=0$), as is the volume element, along the trajectory. Conservation of the flowing “mass” density (really probability) yields

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \left(\vec{V} \rho \right) = 0$$
$$\Rightarrow \frac{\partial \rho}{\partial t} + \vec{V} \cdot \vec{\nabla} \left(\rho \right) = \frac{d\rho}{dt} = 0. \qquad \text{(A.7.20)}$$

Thus the mass density or the density of systems in phase space is constant along the trajectory. This is essentially Liouville's theorem from statistical mechanics stating that the density of states does not change in conservative systems. Note that conservative flow in phase space corresponds to a flow pattern with no sources or sinks, as we would intuitively expect.

Returning to the mechanics problem of 2*f* canonical variables describing a mechanical system with Hamiltonian (stream function),

$$H\left(q_1, \ldots, q_f, p_1, \ldots, p_k \right)$$
$$\dot{q}_k = \frac{\partial H}{\partial p_k}, \dot{p}_k = -\frac{\partial H}{\partial q_k} \left[k = 1, \ldots, f \right], \qquad \text{(A.7.21)}$$

we immediately have incompressible flow,

$$\vec{\nabla} \cdot \vec{V} = \sum_{k=1}^{f} \left(\frac{\partial \dot{q}_k}{\partial q_k} + \frac{\partial \dot{p}_k}{\partial p_k} \right) = \sum_{k=1}^{f} \left(\frac{\partial^2 H}{\partial p_k \partial q_k} - \frac{\partial^2 H}{\partial q_k \partial p_k} \right) = 0. \qquad \text{(A.7.22)}$$

Thus the canonical Hamiltonian "flow" is incompressible. Our new definition of conservative is consistent with the old one (recall the examples above). This can be true even if the Hamiltonian has explicit time dependence and even though the energy may not be conserved (in which case we may want to think of a volume-preserving flow in 2*f* + 1 dimensions, $q_{2f+1} = t$). We will consider an example below.

This incompressible flow is to be contrasted with truly dissipative systems. As an example consider the 1-dimensional damped, but driven harmonic oscillator, described by the 2nd order differential equation

$$\ddot{y} + \beta \dot{y} + \omega^2 y = F\left(t \right). \qquad \text{(A.7.23)}$$

We identify $x_1 = y$ and $x_2 = \dot{y}$ and ignore the driving force initially. The corresponding velocity field is

$$\vec{V} = \left(x_2, -\beta x_2 - \omega^2 x_1\right), \tag{A.7.24}$$

and

$$\begin{aligned} \vec{\nabla} \cdot \vec{V} &= \frac{\partial \dot{x}_1}{\partial x_1} + \frac{\partial \dot{x}_2}{\partial x_2} \\ &= \frac{\partial \dot{y}}{\partial y} + \frac{\partial \ddot{y}}{\partial \dot{y}} = -\beta < 0. \end{aligned} \tag{A.7.25}$$

Thus the flow is *not* incompressible, but rather the comoving volume shrinks exponentially,

$$\delta\Omega(t) = J(t)\,\delta\Omega(0) = e^{-\beta t}\delta\Omega(0), \tag{A.7.26}$$

as the energy drains out of the system. Independent of the initial conditions all trajectories lead to the zero-dimension (*i.e.*, a point) *attractor* at $y = \dot{y} = 0$ (a sink) as *t* goes to infinity. If instead there is a driving force, we must consider a time dependent potential (and the phase space can be considered as 3-dimensional with $x_3 = t$). We know that the full solution includes the particular solution (integral) to match the driving force and a damped component to match the initial conditions. Thus, independent of the initial conditions, all trajectories flow to the 1-dimensional (a line in phase space) attractor provided by the particular solution as time goes to infinity. As we will see higher dimensional attractors (or repellers) are also possible.

This description is to be contrasted with the canonical Hamiltonian description of the same (nondriven) system studied in the HW. With the identifications $x \to y$, $m = 1$ and $\gamma \to \beta$ the equations of motion are identical. However, in the HW we use the canonical momentum $p = e^{\beta t}\dot{y} = x_2$ and the Hamiltonian

$$
\begin{aligned}
H &= \frac{p^2}{2} e^{-\beta t} + \frac{1}{2}\omega^2 e^{\beta t} x^2 \\
&= \frac{x_2^2}{2} e^{-\beta t} + \frac{1}{2}\omega^2 e^{\beta t} x_1^2 \\
\Rightarrow V_1 &= \frac{\partial H}{\partial x_2} = x_2 e^{-\beta t}, V_2 = -\frac{\partial H}{\partial x_1} = x_1 \omega^2 e^{\beta t} \\
\Rightarrow \vec{\nabla} \cdot \vec{V} &= 0.
\end{aligned}
\tag{A.7.27}
$$

Thus, as suggested above, the flow in canonical phase space (q,p) is volume preserving. While y and $\dot{y}$ both "shrink" like $e^{-\beta t/2}$, $q = y$ shrinks but p grows like $e^{+\beta t/2}$ so $\delta\Omega_1 = \delta y \delta \dot{y} \sim e^{-\beta t}$ but $\delta\Omega_2 = \delta q \delta p \sim 1$. The definitions of the phase spaces are different and the same physics looks different in the two cases.

An equilibrium point of the flow corresponds to a point in phase space, $\bar{X} = \left(\bar{x}_1, \ldots, \bar{x}_n\right)$, where the right-hand-side(s) of Eq. (A.7.2) vanish,

$$
V_k\left(\bar{x}_1, \ldots, \bar{x}_n\right) = 0. \tag{A.7.28}
$$

Then the question arises, is this equilibrium point stable or unstable? Do trajectories that come nearby the equilibrium point stay nearby as time increases, or do they move away? The former situation describes stability, while the later means instability. For most such situations we can decide this question by expanding the velocity field around the equilibrium point (usually) to linear order,

$$
\begin{aligned}
&V_k\left(x_1, \ldots, x_n\right) = V_k\left(\bar{x}_1, \ldots, \bar{x}_n\right) + \delta x_j \left.\frac{\partial V_k}{\partial x_j}\right|_{X=\bar{X}} + \ldots, \\
&\delta x_j = x_j - \bar{x}_j.
\end{aligned}
\tag{A.7.29}
$$

In this way we arrive at a linear (vector) variational problem

$$\delta \dot{X}_j = V_j(X) - V_j(\bar{X}) \simeq \left.\frac{\partial V_j}{\partial x_k}\right|_{X=\bar{X}} \delta X_k \equiv A_{jk}\delta X_k,$$
$$\delta X = \begin{pmatrix} \delta x_1 \\ \vdots \\ \delta x_n \end{pmatrix}, \qquad \text{(A.7.30)}$$

which formally has the solution

$$\delta X(t) = e^{At}\delta X(0). \qquad \text{(A.7.31)}$$

We can make this analysis more explicit if we can diagonalize the matrix A and find the eigenvalues and eigenvectors,

$$\begin{aligned} A\hat{e}_k &= \lambda_k \hat{e}_k, \\ A^n\hat{e}_k &= \lambda_k^n \hat{e}_k. \end{aligned} \qquad \text{(A.7.32)}$$

Assuming that we have chosen the eigenvectors to be linearly independent, which we must do explicitly if there are degenerate eigenvalues, they can be used as basis vectors (and the exponential in Eq. (A.7.31) is easy to evaluate),

$$\begin{aligned} \delta X(0) &= \sum_{j=1}^{n} c_j \hat{e}_j, \\ \delta X(t) &= \sum_{j=1}^{n} c_j e^{\lambda_j t}\hat{e}_j. \end{aligned} \qquad \text{(A.7.33)}$$

As an example consider the case of a 2-D phase space (1-D configuration space) where we can solve the eigenvalue problem. In the (general) conservative case Liouville tells us that

$$\vec{\nabla}\cdot\vec{V} = \sum_{j=1}^{n} \frac{\partial \delta \dot{x}_j}{\partial \delta x_j} = \mathrm{Tr}[A] = 0. \qquad \text{(A.7.34)}$$

Thus in the 2-D case we have

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix},$$
$$\det\left|A - \lambda\mathbf{1}\right| = 0 \Rightarrow \lambda = \pm\sqrt{bc + a^2}. \tag{A.7.35}$$

For $a^2 + bc < 0$ there are two distinct imaginary eigenvalues corresponding to elliptic streamlines and an attractive (linear) force, and the equilibrium point is called elliptic (stable). For the opposite case $a^2 + bc > 0$ we have hyperbolic stream lines, a repulsive force and a hyperbolic equilibrium point (unstable). (And for the boundary case of $a^2 + bc = 0$ there is no force.)

Finally consider the driven but damped oscillator in 1-D. As we have seen, in this case $\vec{\nabla} \cdot \vec{V} \neq 0$ and the trace of A will generally not vanish. If the real part of the trace still vanishes, then the linear analysis is insufficient to understand the full behavior and we must keep higher order terms. The possible cases with $n = 2$ are:

1) Two real eigenvalues with opposite signs, which yields hyperbolic behavior with the eigenvectors along the asymptotes of the hyperbolae.

2) Two real eigenvalues with the same sign, which yields flow out from a source (positive values) or into a sink (negative values). The flow will tend to be along the eigenvector of the smaller (real) eigenvalue, unless it is arranged to be exactly along the other eigenvector (or exactly radial).

3) Two complex eigenvalues with identical real parts, such that a) if the real parts vanish, the solution is elliptical (periodic orbits, no damping); b) the real parts are positive and the flow spirals out from the source at the equilibrium point; c) the real parts of the eigenvalues are negative and the flow spirals in to the sink at the equilibrium point.

In summary, conservative systems can exhibit neither sinks nor sources, but can have either elliptic (with localized, perhaps periodic trajectories) or hyperbolic (with scattering trajectories) equilibrium points. Systems with dissipation can have both sinks (attractors) or sources (repellers), which themselves can be points (0-D), lines (1-D limit cycle) or of higher dimension.

In the systems discussed so far the corresponding flow patterns in phase space (so-called phase portraits) are "stable" in that nearby initial conditions yield nearby trajectories (at least for finite time) and small changes in the dynamics (*e.g.*, changes in the spring constant) yield small changes in the flow patterns. This reasonable and familiar behavior is to be contrasted with systems that exhibit chaotic behavior (*e.g.*, strange attractors), *i.e.*, unstable behavior in the sense that large changes in trajectories result from small changes in initial conditions or in the system parameters. As an amusing and illustrative introduction to this behavior consider a "step-wise" dynamic system acting on numbers:

Rule: Given a decimal number, truncate everything to the left of the decimal point and multiply by 10. Repeat.

In more mathematical language this rule says $x_n = \left[10x_{n-1}\right] \bmod 1$ and is called the decimal Bernoulli shift. Here "mod 1" means keep only the (decimal) fractional part of the number. Thus the range of values is given by $0 \le x_n < 1$.

If we start with 3.14159, one step gives 0.4159, then 0.159, then 0.59, and so on. If we change the number in the 6th decimal place, which is a very nearby initial condition, we find a different result after 6 steps. If we really start with π to arbitrary accuracy, we never get a repeating sequence. Consider now the expansion of 1/3 = 0.3333, which after one step moves to a stable equilibrium point. If we consider the expansion of 1/7 = 0.142859142859…, we see periodic behavior of period six. Such periodic behavior (always) arises when we start with a rational number. On the other hand, the nearby starting point 10/69 = 0.1449275362318840579710…, yields period 22. This strong dependence on the initial conditions is characteristic of unstable periodic behavior. As illustrated above non-periodic behavior arises from a starting point that is an irrational number and, in fact, we saw that it is unstable non-periodic behavior. As another example consider the sequence of integers that arise in the expansion of $\sqrt{3} = 1.73205\ldots$ and compare it to the expansion of $\sqrt{2.99} = 1.72916\ldots$. No matter how close the starting point, eventually the sequences diverge from each other. You are encouraged to employ Mma to study such behavior (*e.g.*, see the notebook Bern.nb on the web page).