Lecture 9: Oscillations in Large N and Continuous Systems (The remainder of Chapter 4 in F&W)

Now let's think about what happens when we have N coupled variables, say N identical point masses and N+1 identical springs (longitudinal motion as in F&W Fig 24.1), or N identical point masses equally spaced along a massless string (transverse motion as in F&W Fig. 24.2). Both systems can be treated as normal mode problems, but now we expect to find N modes. The first (longitudinal motion) system is described by the Lagrangian (with η representing the longitudinal displacement from equilibrium)

$$L = T - U = \frac{m}{2}\sum_{j=1}^{N}\dot{\eta}_j^2 - \frac{k}{2}\sum_{j=0}^{N}\left(\eta_{j+1} - \eta_j\right)^2, \tag{9.1}$$

with boundary conditions $\eta_0 = \eta_{N+1} = 0$. The second situation, transverse displacements, is described by identical mathematics in the limit that the transverse displacements are small. If the spacing between masses is a, the tension in the string is τ and the transverse displacement is represented by μ, the restoring force on a given mass (in the small angle limit, *i.e.*, the transverse restoring force is given by $\tau \sin\theta \simeq \tau\, \mu/a$) is $F_j = \tau\left[-\left(\mu_j - \mu_{j-1}\right)/a + \left(\mu_{j+1} - \mu_j\right)/a\right]$. The corresponding Lagrangian is

$$L = T - U = \frac{m}{2}\sum_{j=1}^{N}\dot{\mu}_j^2 - \frac{\tau}{2a}\sum_{j=0}^{N}\left(\mu_{j+1} - \mu_j\right)^2, \tag{9.2}$$

with boundary conditions $\mu_0 = \mu_{N+1} = 0$. Thus, by the rule of Feynman that the same equations have the same solutions, we need only solve one of these two problems and then substitute $\mu_k \Leftrightarrow \eta_k$ and $k \Leftrightarrow \tau/a$. So let's focus on the latter situation and its equation of motion

$$m\ddot{\mu}_j + \frac{\tau}{a}\left(\mu_j - \mu_{j-1}\right) - \frac{\tau}{a}\left(\mu_{j+1} - \mu_j\right) = m\ddot{\mu}_j + 2\frac{\tau}{a}\mu_j - \frac{\tau}{a}\left(\mu_{j+1} + \mu_{j-1}\right) = 0. \tag{9.3}$$

As in previous discussions we have coupled second order linear differential equations with constant coefficients and we are encouraged to try an exponential form as an Ansatz,

$$\mu_j(t) \to \mu(x_j,t) = \mu(aj,t) = Ae^{i(kaj-\omega t)} \tag{9.4}$$

where ω is an angular frequency and *k* is the wave number ($k = 2\pi/\lambda$ where λ is the wavelength). Note that, looking ahead, we have written the solution as a function of the continuous coordinate *x*, which we sample only at the *discrete* values $x_j = aj$ (*j* an integer). Substituting this Ansatz into the equation of motion and canceling all common factors yields

$$\begin{aligned} &-m\omega^2 + \frac{2\tau}{a} - \frac{\tau}{a}\left(e^{ika} + e^{-ika}\right) = 0 \\ &\Rightarrow \omega^2 = \frac{2\tau}{ma}\left(1-\cos ka\right) = \frac{4\tau}{ma}\sin^2\frac{ka}{2}. \end{aligned} \tag{9.5}$$

This sort of relationship between ω and *k* is called a dispersion relation. It corresponds to the more familiar relationship between energy and momentum for a free particle, $E^2 = p^2 + m^2$. To the extent that ω is not linear in *k*, waves of different wavelengths will travel with different velocities and “disperse”.

To turn this into an eigenvalue problem we need to consider the boundary conditions for the problem. Before considering the “fixed end” case discussed above, we first consider what are called periodic boundary conditions (corresponding physically to a closed loop of string). Thus we set the transverse displacement at one of the string equal to the displacement at the other end, $\mu(0) = \mu(Na)$, or more generally $\mu(ja) = \mu(ja + Na)$ (translational invariance when translated by a distance *Na*, the circumference of the loop). In terms of the form of the solution introduced above this means

$$\begin{aligned} &Ae^{i(kja-\omega t)} = Ae^{i(kja+kNa-\omega t)} \Rightarrow 1 = e^{ikNa} \Rightarrow kNa = 2\pi n \\ &\Rightarrow k_n = \frac{2\pi n}{Na}\begin{cases} n = 0, \pm 1, \ldots, \pm\dfrac{N-1}{2} & N \text{ odd} \\ n = 0, \pm 1, \ldots, \pm\dfrac{N-2}{2}, \dfrac{N}{2} & N \text{ even} \end{cases}, \\ &\Rightarrow \omega_n = \sqrt{\frac{4\tau}{ma}}\sin\left(\frac{\pi|n|}{N}\right). \end{aligned} \tag{9.6}$$

For either even or odd N, there are (as expected) N possible eigenvalues of k and ω. (Other values of the parameter n, $> N$ or < 1, simply reproduce the same N solutions.) Note that the solutions tend to come in pairs with opposite signs for k corresponding to motion in both directions in x. We can think about this motion as the motion of points in the wave form with constant phase, ϕ_0, as in $e^{i(k_n x - \omega_n t)} = e^{i\phi_0}$ $\left(\mathrm{Re} \Rightarrow \cos\phi_0 = \cos\left(k_n x - \omega_n t\right)\right)$. So we can define a "phase velocity" via

$$c_n = \omega_n / k_n = \sqrt{\frac{4\tau}{ma}} \sin\left(\frac{\pi|n|}{N}\right)\left(\frac{Na}{2\pi n}\right) = \sqrt{\frac{\tau a}{m}} \frac{\sin\left(\pi|n|/N\right)}{\left(\pi n/N\right)}. \tag{9.7}$$

This expression first tells us that the direction of motion changes with the sign of n (we have eigenmodes moving in both directions). We also see that the characteristic magnitude of the phase velocity is $\sqrt{\tau a/m}$. This is the actual magnitude for $n \to 0$, when the second factor approaches unity. However, for $|n| \to N/2$ we find $c_{n=N/2} \to 2\sqrt{\tau a/m}/\pi < \sqrt{\tau a/m}$. (This n dependence of the phase velocity leads to dispersion.) The corresponding wavelength of the oscillations $\left(\lambda_n = 2\pi/|k_n| = Na/|n|\right)$ varies in the range $\lambda_{n=0} \to \infty$ to $\lambda_{n=N/2} = 2a$, *i.e.*, the spacing between the masses can be at most ½ of a cycle. This reminds us of the essential feature of a chain of discrete masses – it cannot support arbitrarily short wavelengths. Note as expected that the n = 1 Eigen mode has wavelength $\lambda_1 = Na$, the translational invariance length.

Now we return to consider the case of fixed ends, $\mu(0) = \mu(Na + a) = 0$. To satisfy this condition we make use of the opposite moving solutions noted in the previous paragraph and write (recall that linear equations always allow linear superposition)

$$\mu(x,t) = Ae^{-i\omega t}\left(e^{ikx} - e^{-ikx}\right). \tag{9.8}$$

This expression will still satisfy the equation of motion if k and ω satisfy the dispersion relation in Eq. (9.5) and by construction this form satisfies the boundary condition at x = 0 due to the minus sign. Now the eigenvalue problem arises from the other boundary condition,

$$\begin{aligned}
&\mu\left(Na+a,t\right)=Ae^{-i\omega t}\left(e^{ika\left(N+1\right)}-e^{-ika\left(N+1\right)}\right)=0\\
&\Rightarrow \sin ka\left(N+1\right)=0\Rightarrow ka\left(N+1\right)=n\pi\\
&\Rightarrow k_n=\frac{n\pi}{a\left(N+1\right)}\left\{n=1,2,\ldots,N\right.\\
&\Rightarrow \omega_n=\sqrt{\frac{4\tau}{ma}}\sin\left(\frac{\pi n}{2\left(N+1\right)}\right).
\end{aligned}\tag{9.9}$$

Note the differences from the periodic boundary condition case of Eq. (9.6), especially the factor of 2 in the numerator and the absence of solutions of both signs. This latter point just reminds us that, with fixed ends, the eigenmodes are standing waves rather than propagating waves, *i.e.*, there are nodes or zeros in the wave shape that occur at *fixed* positions in *x*. Thus the n$^{\text{th}}$ normal mode looks like

$$\begin{aligned}
&\mu\left(x_j,t\right)=2iA_n\left[\sin\frac{n\pi x_j}{a\left(N+1\right)}\right]e^{-i\omega_n t}\\
&\Rightarrow \text{Real Part}=2A_n\left[\sin\frac{n\pi j}{\left(N+1\right)}\right]\sin\omega_n t.
\end{aligned}\tag{9.10}$$

This expression illustrates that the *x* and *t* dependences have factored into separate functions. Instead of moving waves as in the periodic boundary condition example (where the *x* and *t* dependence stay coupled when we take the real part), we have standing waves. Also in contrast to the periodic case the n = 1 eigenmode here has wavelength $\lambda_1=2a\left(N+1\right)=2l$ or twice the length of the string (*i.e.*, ½ a cycle corresponds to the distance between nodes, which must be fixed at the endpoints).

Finally we want to consider what happens in the continuum limit, *i.e.*, $a\to 0$, $N\to\infty$ with length, $a\left(N+1\right)=l$, fixed. This is the limit that we use to study an actual matter system, even though matter is really composed of a discrete (but very large) set or atoms. We can basically take the corresponding limit of the above expressions where we replace m/a by a (linear) mass density σ, which could be a function of *x* (in which case the tension will also typically be a function of *x*). Following F&W we also change the label of the displacement from equilibrium to remind us of the continuum limit, $\mu\left(x,t\right)\to u\left(x,t\right)$. For periodic boundary conditions (now a loop of continuous string) the dispersion relation for any fixed mode n as $N \to \infty$ yields

Periodic Boundary Conditions

$$\begin{aligned}
&k_n \to \frac{2\pi n}{l}\left\{n = 0, \pm 1, \pm 2, \ldots,\right. \\
&\omega_n \to \sqrt{\frac{4\tau}{ma}}\frac{\pi|n|}{N} \to \sqrt{\frac{\tau}{\sigma}}\frac{2\pi|n|}{l}, \\
&c_n = \frac{\omega_n}{k_n} \to \sqrt{\frac{\tau}{\sigma}}, \\
&\lambda_n = \frac{2\pi}{|k_n|} = \frac{l}{|n|}.
\end{aligned} \tag{9.11}$$

Thus all (finite n) modes have the same phase velocity (*i.e.*, no dispersion), and now we see arbitrarily short wavelength modes present. The n = 1 case still corresponds to a full cycle in the distance of periodicity, l.

For the fixed end case there remains a factor of 2 difference from Eq. (9.11) in various places,

Fixed Ends Boundary Conditions

$$\begin{aligned}
&k_n \to \frac{\pi n}{l}\left\{n = 1, 2, \ldots,\right. \\
&\omega_n \to \sqrt{\frac{4\tau}{ma}}\frac{\pi|n|}{2(N+1)} \to \sqrt{\frac{\tau}{\sigma}}\frac{\pi|n|}{l}, \\
&c_n = \frac{\omega_n}{k_n} \to \sqrt{\frac{\tau}{\sigma}}, \\
&\lambda_n = \frac{2\pi}{k_n} = \frac{2l}{n}.
\end{aligned} \tag{9.12}$$

The case n = 1 still corresponds to just ½ cycle along the length of the string.

Returning to the equation of motion, Eq. (9.3), and taking the continuum limit, we can make the replacements $\left(\mu_j - \mu_{j-1}\right)/a \rightarrow \partial u/\partial x$ and (being careful about possible x dependence in the tension) $\left[\tau_j\left(\mu_j - \mu_{j-1}\right)/a - \tau_{j+1}\left(\mu_{j+1} - \mu_j\right)/a\right] \rightarrow a\,\partial\left(\tau(x)\partial u/\partial x\right)/\partial x$ to obtain the string equation

$$\sigma(x)\frac{\partial^2}{\partial t^2}u(x,t) - \frac{\partial}{\partial x}\left[\tau(x)\frac{\partial}{\partial x}u(x,t)\right] = 0. \tag{9.13}$$

In the simple, but typical, case that we can treat the mass density and tension as constant along the string we have

$$\begin{aligned} &\frac{1}{c^2}\frac{\partial^2}{\partial t^2}u(x,t) - \frac{\partial^2}{\partial x^2}u(x,t) = \frac{1}{c^2}\ddot{u}(x,t) - u''(x,t) = 0, \\ &c = \sqrt{\frac{\tau}{\sigma}}, \end{aligned} \tag{9.14}$$

the familiar wave equation in 1-D. To obtain the corresponding Lagrangian from Eq. (9.2) we replace the mass by an element of mass, $m \rightarrow dm \rightarrow \sigma dx$ (defining the kinetic energy per unit length), the product $\tau a \rightarrow \tau dx$ (defining the potential energy stored per unit length) and then sum (integrate) over x to obtain

$$L = \int_0^l dx\left[\frac{\sigma}{2}\left(\frac{\partial u}{\partial t}\right)^2 - \frac{\tau}{2}\left(\frac{\partial u}{\partial x}\right)^2\right]. \tag{9.15}$$

The integrand in this expression is called the Lagrangian density, $\mathcal{L}$, and the action is now an integral over space-time,

$$A = \int dt\,dx\,\mathcal{L}. \tag{9.16}$$

The Lagrangian density plays a central role in the application of these ideas to field theory.

The general solution to the excitation of a system (string) described by such a Lagrangian can, as expected, be written as a linear combination of the normal modes. For the fixed end boundary conditions we can write the normal coordinates as (following the notation of F&W)

$$\begin{aligned} \varsigma_n(t) &= C_n \cos(\omega_n t + \phi_n), \\ \ddot{\varsigma}_n &= -\omega_n^2 \varsigma_n, \; \omega_n = \frac{n\pi}{l} c = \frac{n\pi}{l}\sqrt{\frac{\tau}{\sigma}}, \end{aligned} \tag{9.17}$$

where the constants C_n, ϕ_n are fixed by the initial conditions (the temporal boundary conditions). The corresponding orthonormal eigenfunctions, matched to the spatial boundary conditions, are

$$\begin{aligned} &\rho_n(x) = \sqrt{\frac{2}{l\sigma}} \sin(k_n x), \\ &\rho_n'' - k_n^2 \rho_n = 0, \; k_n = \frac{\omega_n}{c} = \frac{n\pi}{l}, \; \lambda_n = \frac{2l}{n}, \\ &\int_0^l dx \sigma \rho_m(x) \rho_n(x) = \delta_{mn}. \end{aligned} \tag{9.18}$$

The last line expresses the usual orthonormality of the eigenfunctions with respect to the diagonalized metric $m = \sigma$. Thus the general solution has the form (note again the separation of variables term-by-term)

$$u(x,t) = \sum_{n=1}^{\infty} \rho_n(x) \varsigma_n(t). \tag{9.19}$$

Expressed in terms of the normal modes we can perform the *x* integral in the Lagrangian (see Eq. (9.15), either using the explicit form of ρ_n or by performing an integrations by parts) and perform one of the sums using the orthogonality of the modes to find the diagonal form

$$L = \frac{1}{2} \sum_{n=1}^{\infty} \left(\dot{\zeta}_n^2 - \omega_n^2 \zeta^2 \right). \tag{9.20}$$

In terms of the normal modes the finite length, continuum problem corresponds to an infinite, but still discrete, set of decoupled harmonic oscillators.

As a final comment we note that Hamilton's principle of least action applied to the continuum form of the action in Eq. (9.16) yields the continuum form of Lagrange's Equation,

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial u/\partial t\right)}\right)+\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial u/\partial x\right)}\right)-\frac{\partial \mathcal{L}}{\partial u}=0. \tag{9.21}$$

Here we have applied the calculus of variations to variations with respect to 3 functions, $u\left(x,t\right), \partial u/\partial x = u', \partial u/\partial t = \dot{u}$, all of which are functions of the 2 independent variables *x* and *t*. Applying this result to the string Lagrangian density in Eq. (9.15), $\mathcal{L} = \left(\sigma/2\right)\dot{u}^2 - \left(\tau/2\right)u'^2$, yields the wave equation, Eq. (9.14).