

Physics 557 – Lecture 5

Group Theory: Since symmetries and the use of group theory is so much a part of recent progress in particle physics, we will take a small detour to introduce the basic structure (as seen by a physicist) of this most interesting field of study. See Chapter 5 in Rolnick and the group theory books on the list of texts. We will first define groups in the abstract and then proceed to think about their representations, typically in the form of matrices. In physics we are typically interested in “real” operators that act on “real” states but which form representations of the more abstract concept of groups.

Group (G): A set of (perhaps abstract) elements (things) – g_1, \dots, g_n , plus a definition of the multiplication operation, *i.e.*, a definition of the product of two of the elements

such that

1. $g_i \bullet g_j = g_k \in G$ - products of elements are also elements of group,
2. multiplication is *associative* – $(g_i \bullet g_j) \bullet g_k = g_i \bullet (g_j \bullet g_k)$,
3. the identity element exists as an element of the group, $1 \in G$,
 $1 \bullet g_j = g_j \bullet 1 = g_j$ (sometimes the left and right identities are distinct, but not generally in the context of physics),
4. the group includes a unique inverse for each element, $g_i \in G \Rightarrow g_i^{-1} \in G$ such that $g_i \bullet g_i^{-1} = g_i^{-1} \bullet g_i = 1$ (again the inverse is sometimes defined separately for left and right multiplication but this situation will not arise in this class).

Note: it is *not* necessary that the multiplication be commutative

- if $g_i \bullet g_j = g_j \bullet g_i$ (commutative) – an Abelian group
- if $g_i \bullet g_j \neq g_j \bullet g_i$ (non-commutative) – a non-Abelian group

If the number of elements is finite ($n < \infty$), then the group is called a finite or discrete group. There is a trivial group corresponding to $n = 1$ with $g = 1$ only. Clearly the group properties are all satisfied but in a trivial way. How about $n = 2$? Call the elements of the group 1 and \mathbf{P} . Evidently $\mathbf{P}^{-1} = \mathbf{P}$, $\mathbf{P} \bullet \mathbf{P} = 1$ in order to satisfy the

condition of being a group. There are, in fact, two related and physically interesting realizations of this group. One case is the reflection group where \mathbf{P} is a reflection in a plane (*i.e.*, one of the 3 possible planes in 3-D space). For example, reflection in the xy plane means $\mathbf{P} f(x,y,z) = f(x,y,-z)$ so that $\mathbf{P} \bullet \mathbf{P} f(x,y,z) = \mathbf{P} f(x,y,-z) = f(x,y,z)$ as required. Finite groups are commonly used in the study of solid state physics where discrete symmetries arise regularly. Another $n = 2$ group corresponds to reflection through the origin (in 3-D space), $\mathbf{P} (x,y,z) \rightarrow (-x,-y,-z)$. Again $\mathbf{P} \bullet \mathbf{P} = 1$, $\mathbf{P} \bullet \mathbf{P} f(x,y,z) = \mathbf{P} (-x,-y,-z) = f(x,y,z)$. This is the parity operation that we will discuss again shortly in the context of particle physics.

Typically the groups of interest in particle physics have an infinite number of elements but the individual elements are specified by (are functions of) a finite number (N) of parameters, $g = G(x_1, \dots, x_N)$. Of particular interest are those groups where the parameters vary continuously over some range. Thus the number of parameters is finite but the number of group elements is infinite. If the range of all of the parameters is bounded, the group is said to be compact, *e.g.*, the parameter space of the compact group $SO(3)$ is a sphere of radius π . Further, the groups we will employ have the added feature that that derivatives ($\partial g / \partial x_i$) with respect to all parameters exist. Groups with this property are called Lie Groups.

First we focus on the behavior near the origin of the parameter space. By definition

$$g(0, \dots, 0) \equiv 1,$$

the identity element. Near the origin of the parameter space the group elements correspond to infinitesimal transformations and the derivatives are especially important (as noted in the last lecture) and have a special name – the *generators* X_k .

$$\left. \frac{\partial g}{\partial x_k} \right|_{x_j=0, \text{ all } j} \equiv X_k.$$

The generators serve to define an N -dimensional algebra (vector space) where both addition (of elements of the algebra) and multiplication by constants are defined. The general element of this Lie algebra can be expressed as a linear combination of the generators

$$\vec{X} = \sum_{k=1}^N c_k X_k.$$

This is analogous to the familiar 3-dimensional vector space except that here the generators are the basis vectors (instead of $\hat{x}, \hat{y}, \hat{z}$). We can think of the generators as allowing a “Taylor series” expansion of the group elements near the origin. The group elements can be obtained from the elements of the algebra via exponentiation (recall the last lecture).

The algebra also supports the definition of an outer (or vector) product that produces another element of the algebra, *i.e.*, the algebra is closed under this operation. This product is just the familiar commutator

$$[X_k, X_l] \equiv X_k X_l - X_l X_k = C_{klm} X_m.$$

The tensor C_{ijk} is called the structure constant(s) of the algebra. It fully specifies the structure of the algebra and therefore of the group itself *near the origin* of the parameter space.

Recall that in the last lecture we considered the Lorentz group $SO(3,1)$ where we had

$$g(\vec{\theta}, \vec{y}) = \Lambda(\vec{\theta}, \vec{y}) = e^L, \quad L = -\vec{\theta} \cdot \vec{S} - \vec{y} \cdot \vec{K}.$$

Thus we recognize the matrices \vec{S} and \vec{K} as matrix representations of (minus) the generators of the group $SO(3,1)$. As we noted in the previous lecture, the different matrices do not commute nor do the elements of the group,

$$e^{-\theta_1 S_1} e^{-\theta_2 S_2} \neq e^{-\theta_2 S_2} e^{-\theta_1 S_1} \neq e^{-\theta_1 S_1 - \theta_2 S_2}.$$

This is a non-Abelian group. In general,

$$e^A e^B = e^B e^A \text{ iff } [A, B] = 0.$$

On the other hand, if

$$\left[A, [A, B] \right] = \left[B, [A, B] \right] = 0; [A, B] \neq 0,$$

then we have

$$e^A e^B = e^B e^A e^{[A, B]},$$

which is called the Baker-Hausdorf Lemma.

To make a full connection to the notation typically used in physics we should really add some “i”s above. The matrices S and K of the last lecture were defined so as to be real. Typically we want to relate the generators of the group to physically relevant operators in the quantum mechanical theory, which are Hermitian operators that preserve probability. Hence we expect them to be represented by Hermitian matrices (*i.e.*, $M^\dagger = (M^T)^* = M$). (This will be obvious for the unitary groups where the group elements are represented by unitary matrices $M^\dagger = M^{-1}$.) In this Hermitian notation we define 6 new matrices

$$J_k = iS_k \left[i = \sqrt{-1} \right]; \tilde{K}_k = iK_k \quad (k = 1, 2, 3),$$

where the J_k (but not the \tilde{K}_k) are Hermitian matrices.

$$\begin{aligned}
J_1 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}, J_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix}, J_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
\tilde{K}_1 &= \begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tilde{K}_2 = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tilde{K}_3 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}.
\end{aligned}$$

We recognize the J_k as the Hermitian representation of the angular momentum operators. Now the SO(3,1) group is represented by

$$\Lambda(\vec{\theta}, \vec{y}) = e^{i\vec{\theta} \cdot \vec{J} + i\vec{y} \cdot \vec{K}}.$$

Further we refine the definition of the generator to be

$$\left. \frac{1}{i} \frac{\partial g}{\partial x_k} \right|_{x_j=0, \text{ all } j} \equiv Y_k.$$

and

$$[Y_j, Y_k] = iC_{jkl} Y_l,$$

the more common form in the quantum mechanical context. Using the specific forms for the matrices above we quickly establish the elements of the structure constant of the group SO(3,1) (in the Hermitian form)

$$\begin{aligned} [J_j, J_k] &= i\epsilon_{jkl} J_l, \\ [J_j, \tilde{K}_k] &= i\epsilon_{jkl} \tilde{K}_l, \\ [\tilde{K}_j, \tilde{K}_k] &= -i\epsilon_{jkl} J_l. \end{aligned}$$

Here the symbol ϵ_{jkl} is the Levi-Civita symbol which is the (unique) fully anti-symmetric 3-tensor in 3 dimensions ($j,k,l = 1,2,3$),

$1 = \epsilon_{123} = \epsilon_{312} = \epsilon_{231} = -\epsilon_{213} = -\epsilon_{132} = -\epsilon_{321}$ with all other components vanishing.

(Note: $\epsilon_{ijk} \epsilon_{ijl} = 2\delta_{kl}$.)

One final piece of 4-D formalism – the matrices J_k and \tilde{K}_k can be thought of as the 6 non-zero components of an anti-symmetric tensor in 4-D rather than as 2 distinct 3-vectors (just like \vec{E} and \vec{B} in E&M are the components of the field tensor $F_{\mu\nu}$). To make this explicit define the tensors

$$\begin{aligned} J_{\mu\nu} (\mu, \nu = 0, 1, 2, 3): J_{kl} &= \epsilon_{klm} J_m (k, l, m = 1, 2, 3), \\ J_{k0} &= \tilde{K}_k = -J_{0k}, \end{aligned}$$

and

$$\begin{aligned} \omega^{\mu\nu} (\mu, \nu = 0, 1, 2, 3): \omega^{kl} &= \epsilon_{klm} \theta_m (k, l, m = 1, 2, 3), \\ \omega^{k0} &= y_k = -\omega^{0k}. \end{aligned}$$

Now the argument in the exponent for Λ becomes

$$i\vec{\theta} \cdot \vec{J} + i\vec{y} \cdot \vec{K} = \frac{i}{2} \omega^{\mu\nu} J_{\mu\nu},$$

which can be easily verified. Note the useful fact that the transformations of the group $SO(3,1)$ preserve not only the “length” of 4-vectors, *i.e.*, $r^\mu r_\mu$, but also leave invariant the two tensors: $g_{\mu\nu}$, the metric itself, and $\epsilon_{\mu\nu\alpha\beta}$, the 4-D analog of ϵ_{ijk} .

Now we shall go around the circle one more time. Let us look in quite general terms at the two groups that seem to arise most often in particle physics – the Orthogonal Group $SO(n)$ and the Unitary Group $SU(n)$. The former appear in the study of real n -D vector spaces, *e.g.*, space-time, and are defined by being transformations of the vector space that preserve the length of vectors or, more generally any scalar product (appropriately defined, if there is a nontrivial metric). (Thus, if two vectors have zero scalar product in one reference frame, *i.e.*, they are orthogonal, they will remain orthogonal in the rotated frame – hence the name of the group.) The Unitary Group appears in the study of complex n -D vector spaces, *e.g.*, quantum mechanics, and are defined by again preserving the length of (state) vectors, *i.e.*, probability. [Note that in both cases scalar products are those products that “use” all indices – nothing is left to “operate on”. Hence scalar products are left unchanged by the transformations.]

To see what properties of the groups these statements imply consider first a n -D real vector and its square, where the shapes in the following expressions are intended to imply how the indices are contracted.

$$r = \left[\begin{array}{c} \\ \\ \\ \end{array} \right], r^T = \left[\right],$$

$$r_2 \cdot r_1 \equiv r_2^T r_1 = \left[\right] \left[\begin{array}{c} \\ \\ \\ \end{array} \right]$$

Now consider the same vector in a transformed reference frame (or transform the vector)

$$r' = \Lambda r = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} \\ \\ \end{bmatrix}.$$

We demand that $r_2 \bullet r_1$ be preserved in the transformation for *any* r_1, r_2

$$\begin{aligned} r_2'^T r_1' &= (\Lambda r_2)^T \Lambda r_1 = r_2^T \Lambda^T \Lambda r_1 = r_2^T r_1 \\ \Rightarrow \Lambda^T \Lambda &= \mathbf{1}, \Lambda^{-1} = \Lambda^T. \end{aligned}$$

So the characteristic feature of the Orthogonal Group is that it is represented by real orthogonal matrices, *i.e.*, matrices whose inverses are their transposes. If the scalar product is defined with a non-trivial metric, the corresponding form is what we derived earlier for SO(3,1)

$$\begin{aligned} r_2'^T g r_1' &= (\Lambda r_2)^T g \Lambda r_1 = r_2^T \Lambda^T g \Lambda r_1 = r_2^T g r_1 \\ \Rightarrow \Lambda^T g \Lambda &= g, \Lambda^{-1} = g \Lambda^T g. \end{aligned}$$

Note that it follows from these equations and the properties of determinants that

$$\det[\Lambda^T \Lambda] = \det[\Lambda^T] \det[\Lambda] = \det[\Lambda]^2 = 1$$

or

$$\det[g \Lambda^T g \Lambda] = \det[g]^2 \det[\Lambda^T] \det[\Lambda] = \det[\Lambda]^2 = 1.$$

Typically we want only the “Special” (hence the “S” in the label of the group) or unimodular group (no reflections) and we require that the determinant of Λ be +1. Using (real) exponentiation to go from the algebra to the group, we write

$$\Lambda = e^{-\alpha S}$$

where α is a real parameter and S is a real $n \times n$ matrix. The orthogonal form means

$$\left(e^{-\alpha S}\right)^T = e^{-\alpha S^T} = \left(e^{-\alpha S}\right)^{-1} = e^{\alpha S} \Rightarrow S^T = -S.$$

Thus S is a real, anti-symmetric matrix. In the complex notation of quantum mechanics we replace S by $-iJ$ to find that $J = iS$ is Hermitian.

The constraint we imposed on the determinant of Λ translates into a constraint on the trace of S

$$\det \left[e^{-\alpha S} \right] = +1 \Rightarrow \text{Tr} [S] = 0,$$

which is trivially satisfied by an anti-symmetric matrix. For the more general case of a scalar product defined with a metric, S is still traceless and satisfies

$$gS^T g = -S,$$

i.e., S displays mixed symmetry defined by g . Next we can determine the number of independent components of S , *i.e.*, the dimensionality of the corresponding algebra (also called the order of the group). A real, anti-symmetric matrix has zeroes on the diagonal and all components below the diagonal are determined by those above. So we want $\frac{1}{2}$ the number of off-diagonal elements in an $n \times n$ matrix. Hence the algebra of $\text{SO}(n)$ has dimension

$$N[\text{SO}(n)] = \frac{n^2 - n}{2} = \frac{n(n-1)}{2}$$

$$N \begin{bmatrix} n=2 \\ n=3 \\ n=4 \end{bmatrix} = \begin{matrix} 1 \\ 3 \\ 6 \end{matrix}.$$

The same counting applies also to the Hermitian representations of the algebra of SO(n).

The corresponding exercise for the Unitary group now involves complex numbers and complex conjugation in the scalar product.

$$r = \begin{bmatrix} \\ \\ \end{bmatrix}, r^\dagger = \begin{bmatrix} * & * & * \end{bmatrix},$$

$$r_2 \cdot r_1 \equiv r_2^\dagger r_1 = \begin{bmatrix} * & * & * \end{bmatrix} \begin{bmatrix} \\ \\ \end{bmatrix}$$

Thus, if the unitary transformation is described by a matrix U , we have

$$r_2'^\dagger r_1' = (U r_2)^\dagger U r_1 = r_2^\dagger U^\dagger U r_1 = r_2^\dagger r_1$$

$$\Rightarrow U^\dagger U = \mathbf{1}, U^{-1} = U^\dagger,$$

i.e., U is a unitary matrix. In (complex) exponential notation

$$U = e^{i\beta T}, U^\dagger = e^{-i\beta T^\dagger} = U^{-1} = e^{-i\beta T}$$

$$\Rightarrow T^\dagger = T.$$

The generator is represented by a Hermitian matrix, as we expect from our earlier discussion. Again we have

$$\det[U^\dagger U] = \det[U]^* \det[U] = 1$$

and we focus on the Special version of the group, SU(n),

$$\det[U] \equiv 1 \Rightarrow \text{Tr}[T] = 0.$$

Hence the algebra is defined by traceless, Hermitian matrices in the appropriate number of dimensions. In n-D a nxn complex matrix has $2n^2$ components. Being Hermitian reduces this by a factor of 2 and the constraint of zero trace removes another degree of freedom. Thus the order of the special unitarity group in n-D is

$$N[\text{SU}(n)] = \frac{2n^2}{2} - 1 = n^2 - 1$$

$$N \begin{bmatrix} n = 1 \text{ (really U(1))} \\ n = 2 \\ n = 3 \end{bmatrix} = \begin{matrix} 1 \\ 3 \\ 8 \end{matrix}.$$

Note that the algebras of U(1) and SO(2) have the same (trivial) dimension. You might expect that they are related and they are! They are identical or isomorphic as groups

$$\text{U}(1) \cong \text{SO}(2).$$

This becomes more obvious if we note that rotations in a plane, SO(2), can be performed in any order, *i.e.*, SO(2) is an Abelian group like U(1). You might have thought that SO(2) had 2-D representations, unlike U(1), but, in fact, these representations can, by an appropriate choice of basis vectors, be *reduced* to the canonical 1-D representations $e^{in\theta}$, which are the irreducible representations of U(1) (see, *e.g.*, Chapter 5.2.1 in Rolnick). [Another way to see this is to note that 2-D problems, *i.e.*, SO(2), can always be mapped onto the complex plane, *i.e.*, U(1).]

The groups SO(3) and SU(2) are also related. Again the algebras are identical. The *algebras* are determined by the structure constant in the form (using Hermitian representations for both)

$$\left[Y_j, Y_k \right] = i C_{jkl} Y_l.$$

In the special case $j, k, l = 1, 2, 3$, *i.e.*, $N=3$ for both $SO(3)$ and $SU(2)$, there is a unique choice for the anti-symmetric tensor C_{jkl} . It must be equal to ϵ_{jkl} since this is the only $3 \times 3 \times 3$ fully antisymmetric tensor – another application of the “what else can it be?” theorem! This implies that the *groups* must be identical near the origin of the 3-D parameter space. On the other hand the groups are not isomorphic (identical) when we consider the full parameter space. Instead $SU(2)$ is, in some sense, larger. For every element of $SO(3)$ there are two elements in $SU(2)$ (see, *e.g.*, Chapter 3 in Rolnick and HW II). This relationship is called a homomorphism with a 2 to 1, $SU(2)$ to $SO(3)$, mapping.

Another way to think about this is that the parameter space of $SO(3)$ is like a sphere of radius π . Next consider how the parameter space maps onto the group space. Each point in the sphere specifies a direction from the origin, which is the axis of the rotation, and a distance from the origin, which is the magnitude of the rotation. When we get to the surface at π , we must identify antipodal points, a rotation through π in one direction is equivalent to a rotation of π about the exactly opposite direction. This means we can define a path in both the parameter space and the group space by starting at the origin, going out to π in one direction, hopping to (exactly) the other side of the sphere, and coming back to the origin. This is a closed path in the group space that cannot be shrunk to zero in the parameter space! Thus the space is not simply connected. On the other hand for $SU(2)$ we define a similar picture but the sphere extends to 2π and now, no matter what direction we left the origin along, we reach the transformation -1 at 2π (recall that, when we rotate a spin $\frac{1}{2}$ state by 2π , we don't get back to the original state but to minus the original state). Thus the entire surface at 2π is identified as a single point (no issues about only antipodal points in this case). All closed paths can be shrunk to zero and the space is simply connected. $SU(2)$ is called the covering group for $SO(3)$.

Aside: The structure of the matrices and structure constants for $SU(3)$ are available on the class [web page](#).

Let's think just a bit more about the representations of groups. First we need the concept of reducible and irreducible. If a representation (*i.e.*, the matrices) of the group elements can be reduced to block diagonal form by some choice of basis vectors

$$\begin{bmatrix} \begin{bmatrix} \end{bmatrix} & 0 & 0 \\ 0 & \begin{bmatrix} \end{bmatrix} & 0 \\ 0 & 0 & \begin{bmatrix} \end{bmatrix} \end{bmatrix},$$

then that representation is reducible. If it cannot, it is irreducible. We are familiar with this concept from the addition of angular momentum. If we add angular momentum 1 to angular momentum 1 (*e.g.*, 2 vector particles collide with zero orbital angular momentum), the possibilities are angular momentum 0, 1 or 2 ($1+3+5 = 9 = 3 \times 3$ possible states),

$$\underline{\underline{3}} \times \underline{\underline{3}} = \underline{\underline{1}} + \underline{\underline{3}} + \underline{\underline{5}}.$$

As we noted earlier, the apparently 2-D representations of SO(2) are reducible to 1-D representations. In fact, the irreducible representations of Abelian groups are all 1-D. The smallest dimension representation that faithfully represents the group, *i.e.*, displays all of its structure, is called the defining or fundamental representation. All groups have 1-D (scalar) representations but they are not faithful for non-Abelian groups. [Recall that 1-D representations are just numbers, which must commute unlike matrices.] For SO(3) the fundamental representation is the vector representation, $\underline{\underline{3}}$. For SU(2) the fundamental representation is the spinor representation, $\underline{\underline{2}}$. The half-integer spin representations ($J = 1/2, 3/2, \dots$), are often referred to as the spinor representations of SO(3) but they are strictly the representations of SU(2). The fundamental representation of the algebra of SU(2) looks like that above for SO(3) but with the Hermitian 3x3 matrices, J , replaced by the Pauli matrices, $\sigma/2$, where

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

[As discussed in the homework the basis states in this representation are the eigenstates of the spin operator rather than the 3-vector representation used for SO(3) above.]

We can also interpret the algebra itself as providing a representation of the group – called the adjoint representation. The generators themselves are the basis vectors. The transformation of a basis vector by a generator is defined as the commutator of the generator with the “basis vector” (*i.e.*, the other generator). Hence the structure constants, the C_{jkl} , define a matrix representation of the algebra and the group (by exponentiation). For SO(3) the adjoint representation and the fundamental representation are identical. Recall that the fundamental 3-vector (Hermitian) representation for SO(3) looks like (just read them off from the SO(3,1) forms presented earlier)

$$J_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, J_y = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}, J_z = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

These matrices can be written in the form

$$\left[J_j \right]_{kl} = i \varepsilon_{jlk},$$

where it is important to note the order of the last two indices. Thus for SO(3) the adjoint representation provided by the structure constants (properly defined) is identical to the fundamental representation.

Similarly the generators of SU(2) correspond not only to the adjoint representation of SU(2) but also form a fundamental (and adjoint) representation of SO(3), *i.e.*, the Pauli matrices transform like a 3-vector under rotation.

We will close this introduction to group theory with a few brief observations and one look way ahead. If we combine the group of space-time rotations, SO(3,1) with generators $J_{\mu\nu}$, with the space-time translations generated by the total 4-momentum P_μ we obtain the Poincaré group. This group describes a complete set of “external” space-time or Lorentz symmetries (corresponding, via Noether, to the usual global

conservation laws for energy, 3-momentum and angular momentum). As the full importance of the “internal” symmetries (weak isospin, QCD, *etc.*, which operate in a space “internal” to space-time) became apparent in the last 40 years, an important issue was whether we can “grand unify” the various internal symmetries with the external space-time symmetries. This was answered first in 1967 (Coleman and Mandula) with a “No-Go” theorem stating that it is impossible to mix them in a nontrivial way. If T_a represents the generators of the internal symmetry, then we must have that

$$\left[P_\mu, T_a \right] = \left[J_{\mu\nu}, T_a \right] = 0.$$

The way out of this symmetry dead-end is via an enlargement of the class of allowed symmetries, along a path long known to the mathematicians. We now want to allow 2 classes of generators to define the algebra. The first variety of generators, which we will label “even”, corresponds to the generators discussed above and obeys commutation relations. The second (and new) variety of generators, which we will label “odd”, obeys *anticommutation* relations. The general structure of the enlarged algebra (called a “graded Lie algebra”) is given by

$$\begin{aligned} [\text{even}, \text{even}] &= \text{even}, \\ \{\text{odd}, \text{odd}\} &= \text{even}, \\ [\text{even}, \text{odd}] &= \text{odd}. \end{aligned}$$

This structure is easily understood if we think of the odd generators as carrying “fermion number”. Amongst themselves they obey anticommutation relations as expected for fermions. Further, any operator with an odd number of odd generators will be fermion-like, *i.e.*, again be an odd generator, while even numbers of odd generators result in boson-like even generators. The operation of the odd generators on state vectors will result in a change in fermion number and the mixing of fermionic and bosonic states. Thus larger symmetries, with external and internal components (as in string theory), lead “naturally” to the appearance of supersymmetry, corresponding to a degeneracy between bosons and fermions whose quantum numbers, other than spin, are identical! We will discuss this at greater length in future lectures.