The Z-test

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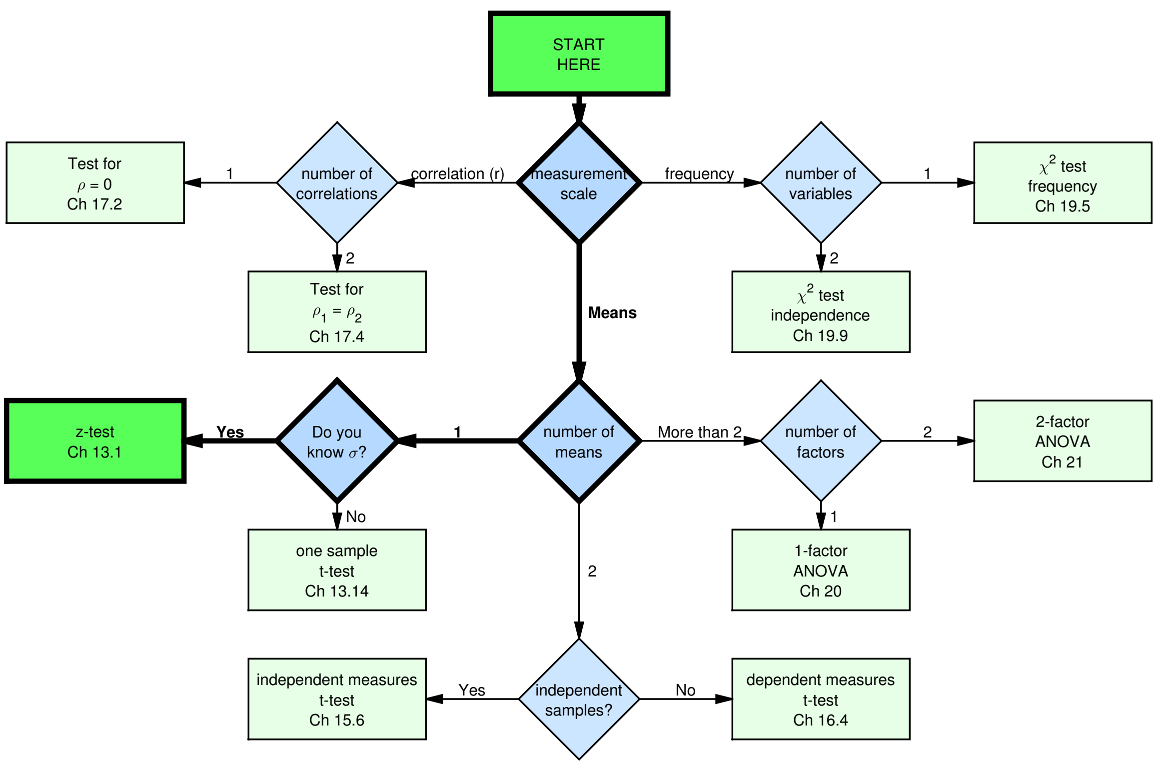
The z-test is a hypothesis test to determine if a **single observed mean** is significantly different (or greater or less than) the mean under the null hypothesis, μ_{hyp} **when you know the standard deviation of the population**. Here's where the z-test sits on our flow chart.



The assumption is that if the null hypothesis is true, then our observed mean, $\bar{x}$ is drawn from a normal distribution with mean μ_{hyp} and standard deviation equal to the standard error of the mean:

$$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{n}}$$

Where n is the sample size and σ_x is the population standard deviation.

To conduct the test we convert our observed mean, $\bar{x}$, to a z-score (standard deviation units):

$$z = \frac{(\bar{x} - \mu_{hyp})}{\sigma_{\bar{x}}} = \frac{(\bar{x} - \mu_{hyp})}{\frac{\sigma_x}{\sqrt{n}}}$$

We can then look up the probability of our observed mean under the null hypothesis in the z-table.

Example 1: (one tailed z-test)

The population of all verbal GRE scores are known to have a standard deviation of 8.5. The UW Psychology department hopes to receive applicants with a verbal GRE scores over 210. This year, the mean verbal GRE scores for the 42 applicants was 212.79. Using a value of $\alpha = 0.05$ is this new mean significantly greater than the desired mean of 210?

For this example, the mean under the null hypothesis is $\mu_{hyp} = 210$, the population standard deviation is $\sigma_x = 8.5$, and the observed mean is $\bar{x} = 212.79$.

The standard error of the mean is therefore:

$$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{n}} = \frac{8.5}{\sqrt{42}} = 1.31$$

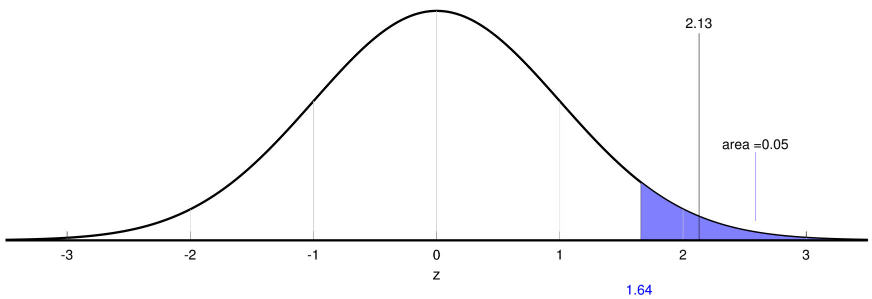
To find the probability of finding a mean above 212.79 we convert our observed mean, $\bar{x}$, to a z-score:

$$z = \frac{(\bar{x} - \mu_{hyp})}{\sigma_{\bar{x}}} = \frac{(212.79 - 210)}{1.31} = 2.13$$

This will be a one tailed test because we're only rejecting H_0 if our observed mean is significantly larger than 210. To make our decision we need to find the critical value of z, which is the z for which the area above is 0.05. Looking at our z-table for $\alpha = 0.05$:

z	Area between mean and z	Area beyond z
⋮	⋮	⋮
1.62	0.4474	0.0526
1.63	0.4484	0.0516
1.64	**0.4495**	**0.0505**
1.65	0.4505	0.0495
1.66	0.4515	0.0485
⋮	⋮	⋮

You can see that the critical value of z is 1.64:

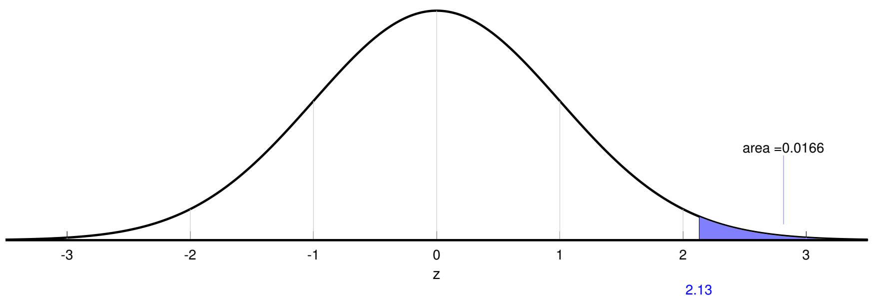


Our observed value of z is 2.13 which is greater than the critical value of 1.64. We therefore reject H_0.

Equivalently, we can calculate the p-value for our observed mean and compare it to alpha. For this one-tailed test, the p-value is the area under the normal distribution above our observed value of z. From the z-table:

z	Area between mean and z	Area beyond z
⋮	⋮	⋮
2.11	0.4826	0.0174
2.12	0.4830	0.0170
2.13	**0.4834**	**0.0166**
2.14	0.4838	0.0162
2.15	0.4842	0.0158
⋮	⋮	⋮

You can see that our p-value is p = 0.0166.



Our p-value is less than alpha (0.05). If the null hypothesis is true, then the probability of obtaining our observed mean or greater is less than 0.05. We therefore reject H_0 and state that (in APA format):

The verbal GRE scores of applicants (M = 212.79) is significantly greater than 210, z=2.13, p = 0.0166.

We could also use the 'pnorm' function R to calculate this p-value. Remember, we need to divide the population standard deviation ($\sigma = 8.5$) by the square root of n ($\sqrt{42}$):

```
1-pnorm(212.79,210,8.5/sqrt(42))
[1] 0.01670141
```

Example 2: (two tailed z-test)

Suppose you start up a company that has developed a drug that is supposed to increase IQ. You know that the standard deviation of IQ in the general population is 15. You test your drug on 36 patients and obtain a mean IQ of 97.65. Using an alpha value of 0.05, is this IQ significantly different than the population mean of 100?

To solve this, we first calculate the standard error of the mean:

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{15}{\sqrt{36}} = 2.5$$

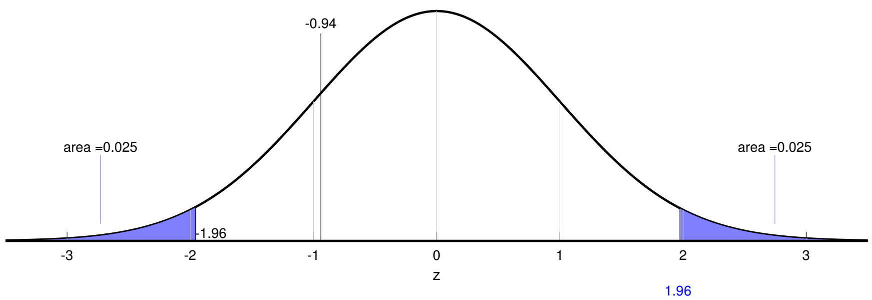
and then convert our observed mean to a z-score:

$$z = \frac{(\bar{x} - \mu_{hyp})}{\sigma_{\bar{x}}} = \frac{(97.65 - 100)}{2.5} = -0.94$$

We then compare our observed value of z to the critical values of z for alpha = 0.05. We are looking for a significant **difference**, so this will be a two-tailed test. We reject the null hypothesis if our observed mean is either significantly larger or smaller than 100. Our critical values of z are therefore the two values that span the middle 95% of the area under the standard normal distribution. This means that the areas in each of the two tails is $\frac{0.05}{2}$ = 0.025:

z	Area between mean and z	Area beyond z
⋮	⋮	⋮
1.94	0.4738	0.0262
1.95	0.4744	0.0256
1.96	**0.4750**	**0.0250**
1.97	0.4756	0.0244
1.98	0.4761	0.0239
⋮	⋮	⋮

Which corresponds to a critical value of $z = 1.96$.

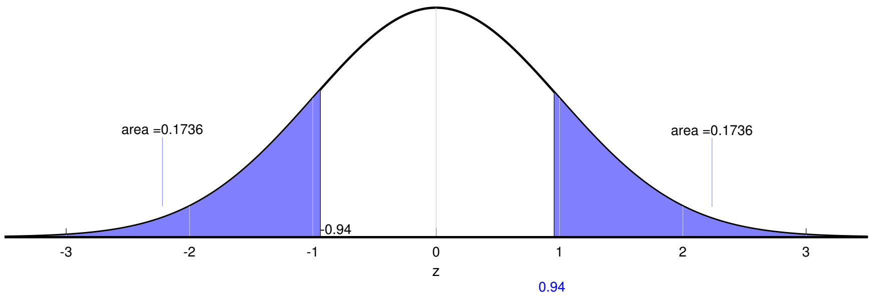


The rejection region contains values of z less than -1.96 and greater than 1.96. Our observed value of z falls outside the rejection region, so we fail to reject H_0 and conclude that our drug did not have a significant effect on IQ.

To calculate the p-value we need to find the area under the standard normal distribution beyond our observed value of z and **double it**. This is because for a two-tailed test we want the probability of obtaining our observed value or more extreme in either direction. This makes sense if you think about what happens if the observed value of z falls exactly on the critical value (1.96 in this example). The area beyond the observed value of z in both the positive direction and the negative direction will add up to alpha (0.05).

z	Area between mean and z	Area beyond z
-0.96	-0.3315	0.8315
-0.95	-0.3289	0.8289
-0.94	**-0.3264**	**0.8264**
-0.93	-0.3238	0.8238
-0.92	-0.3212	0.8212
⋮	⋮	⋮

For this example, the area above z = 0.94 plus the area below z = -0.94 is 0.1736 + 0.1736 = 0.3472



Since our p-value of 0.3472 is greater than 0.05, we fail to reject H_0 and state that:

The IQ of superb drug patients (M = 97.65) is not significantly different than 100, z=-0.94, p = 0.3472.

To do this in R, we need to be sure to double our p-value since this is a two-tailed test. Here's how to calculate the p-value in one step. Note the '2*(1-pnorm...', which doubles the p-value:

```
2*(1-pnorm(97.65,100,15/sqrt(36)))
[1] 1.652782
```

Questions

Here are 15 practice z-test questions followed by their answers, including how to use R to find p-values.

1) Suppose the arousal of hot cats has a population that is normally distributed with a standard deviation of 6. Tomorrow you sample 49 hot cats from this population and obtain a mean arousal of 46.44 and a standard deviation of 5.6968.
Using an alpha value of $\alpha = 0.01$, is this observed mean significantly less than an expected arousal of 47?

2) Suppose the jewelry of exams has a population that is normally distributed with a standard deviation of 5. You are walking down the street and sample 9 exams from this population and obtain a mean jewelry of 28.95 and a standard deviation of 6.3802.
Using an alpha value of $\alpha = 0.01$, is this observed mean significantly different than an expected jewelry of 27?

3) Suppose the courage of psychologists has a population that is normally distributed with a standard deviation of 10. You decide to sample 57 psychologists from this population and obtain a mean courage of 34.81 and a standard deviation of 9.0579.
Using an alpha value of $\alpha = 0.05$, is this observed mean significantly greater than an expected courage of 34?

4) Suppose the speed of quaint weather events has a population that is normally distributed with a standard deviation of 8. One day you sample 44 quaint weather events from this population and obtain a mean speed of 5.4 and a standard deviation of 7.3023.
Using an alpha value of $\alpha = 0.05$, is this observed mean significantly greater than an expected speed of 3?

5) Suppose the price of fingers has a population that is normally distributed with a standard deviation of 5. For your first year project you sample 36 fingers from this population and obtain a mean price of 66.09 and a standard deviation of 4.6836.
Using an alpha value of $\alpha = 0.01$, is this observed mean significantly different than an expected price of 67?

6) Suppose the courage of hateful republicans has a population that is normally distributed with a standard deviation of 6. Let's pretend that you sample 58 hateful republicans from this population and obtain a mean courage of 23.59 and a standard deviation of 6.61.
Using an alpha value of $\alpha = 0.05$, is this observed mean significantly less than an expected courage of 26?

7) Suppose the anxiety of movies has a population that is normally distributed with a standard deviation of 1. Let's sample 40 movies from this population and obtain a mean anxiety of 94.1 and a standard deviation of 1.2039.
Using an alpha value of $\alpha = 0.05$, is this observed mean significantly greater than an expected anxiety of 94?

8) Suppose the life expectancy of Seattleites has a population that is normally distributed with a standard deviation of 1. You go out and sample 45 Seattleites from this population and obtain a mean life expectancy of 88.51 and a standard deviation of 1.0815.
Using an alpha value of $\alpha = 0.05$, is this observed mean significantly different than an expected life expectancy of 89?

9) Suppose the width of bus riders has a population that is normally distributed with a standard deviation of 10. Suppose that before graduation, your first job was to sample 98 bus riders from this population and obtain a mean width of 49.98 and a standard deviation of 10.3386.
Using an alpha value of $\alpha = 0.01$, is this observed mean significantly different than an expected width of 52?

10) Suppose the width of makeshift personalities has a population that is normally distributed with a standard deviation of 7. You want to sample 22 makeshift personalities from this population and obtain a mean width of 87.19 and a standard deviation of 7.257.
Using an alpha value of $\alpha = 0.01$, is this observed mean significantly less than an expected width of 89?

11) Suppose the amount of beer has a population that is normally distributed with a standard deviation of 9. You are walking down the street and sample 67 beer from this population and obtain a mean amount of -0.54 and a standard deviation of 9.9197.
Using an alpha value of $\alpha = 0.05$, is this observed mean significantly less than an expected amount of 0?

12) Suppose the cash of photoreceptors has a population that is normally distributed with a standard deviation of 5. For your first year project you sample 19 photoreceptors from this population and obtain a mean cash of 7.16 and a standard deviation of 5.261.
Using an alpha value of $\alpha = 0.05$, is this observed mean significantly different than an expected cash of 6?

13) Suppose the life expectancy of two candy bars has a population that is normally distributed with a standard deviation of 3. Tomorrow you sample 41 two candy bars from this population and obtain a mean life expectancy of 46.45 and a standard deviation of 3.5814.

Using an alpha value of $\alpha = 0.01$, is this observed mean significantly different than an expected life expectancy of 48?

14) Suppose the news of frail galaxies has a population that is normally distributed with a standard deviation of 7. For some reason you sample 78 frail galaxies from this population and obtain a mean news of 23.3 and a standard deviation of 7.7705.
Using an alpha value of $\alpha = 0.05$, is this observed mean significantly different than an expected news of 22?

15) Suppose the hospitality of wary otter pops has a population that is normally distributed with a standard deviation of 7. Your friend gets you to sample 58 wary otter pops from this population and obtain a mean hospitality of 65.29 and a standard deviation of 6.0893.
Using an alpha value of $\alpha = 0.01$, is this observed mean significantly different than an expected hospitality of 66?

Answers

1) One tailed z-test

$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{n}} = \frac{6}{\sqrt{49}} = 0.8571$

$z_{obs} = \frac{\bar{x}-\mu_{hyp}}{\sigma_{\bar{x}}} = \frac{(46.44-47)}{0.8571} = -0.65$

z_{crit} for $\alpha = 0.01$ (One tailed) is -2.33

We fail to reject H_0.

The arousal of hot cats (M = 46.44) is not significantly less than 47, z=-0.65, p = 0.2578.

```
# Using R:
pnorm(46.44,47,6/sqrt(49))
[1] 0.2567707
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2) Two tailed z-test

$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{n}} = \frac{5}{\sqrt{9}} = 1.6667$

$z_{obs} = \frac{\bar{x}-\mu_{hyp}}{\sigma_{\bar{x}}} = \frac{(28.95-27)}{1.6667} = 1.17$

z_{crit} for $\alpha = 0.01$ (Two tailed) is ± 2.58

We fail to reject H_0.

The jewelry of exams (M = 28.95) is not significantly different than 27, z=1.17, p = 0.242.

```
# Using R: since our observed mean is greater than the mean for H0:
2*(1-pnorm(28.95,27,5/sqrt(9)))
[1] 0.242001
```

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3) One tailed z-test

$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{n}} = \frac{10}{\sqrt{57}} = 1.3245$

$z_{obs} = \frac{\bar{x}-\mu_{hyp}}{\sigma_{\bar{x}}} = \frac{(34.81-34)}{1.3245} = 0.61$

z_{crit} for $\alpha = 0.05$ (One tailed) is 1.64

We fail to reject H_0.

The courage of psychologists (M = 34.81) is not significantly greater than 34, z=0.61, p = 0.2709.

```
# Using R:
1-pnorm(34.81,34,10/sqrt(57))
[1] 0.2704222
```

---

4) One tailed z-test

$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{n}} = \frac{8}{\sqrt{44}} = 1.206$

$z_{obs} = \frac{\bar{x} - \mu_{hyp}}{\sigma_{\bar{x}}} = \frac{(5.4-3)}{1.206} = 1.99$

z_{crit} for $\alpha = 0.05$ (One tailed) is 1.64

We reject H_0.

The speed of quaint weather events (M = 5.4) is significantly greater than 3, z=1.99, p = 0.0233.

```
# Using R:
1-pnorm(5.4,3,8/sqrt(44))
[1] 0.02329685
```

---

5) Two tailed z-test

$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{n}} = \frac{5}{\sqrt{36}} = 0.8333$

$z_{obs} = \frac{\bar{x} - \mu_{hyp}}{\sigma_{\bar{x}}} = \frac{(66.09-67)}{0.8333} = -1.09$

z_{crit} for $\alpha = 0.01$ (Two tailed) is ± 2.58

We fail to reject H_0.

The price of fingers (M = 66.09) is not significantly different than 67, z=-1.09, p = 0.2757.

```
# Using R: since our observed mean is less than the mean for H0:
2*pnorm(66.09,67,5/sqrt(36))
[1] 0.2748331
```

---

6) One tailed z-test

$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{n}} = \frac{6}{\sqrt{58}} = 0.7878$

$z_{obs} = \frac{\bar{x} - \mu_{hyp}}{\sigma_{\bar{x}}} = \frac{(23.59-26)}{0.7878} = -3.06$

z_{crit} for $\alpha = 0.05$ (One tailed) is -1.64

We reject H_0.

The courage of hateful republicans (M = 23.59) is significantly less than 26, z=-3.06, p = 0.0011.

```
# Using R:
pnorm(23.59,26,6/sqrt(58))
[1] 0.001110378
```

---

7) One tailed z-test

$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{n}} = \frac{1}{\sqrt{40}} = 0.1581$

$z_{obs} = \frac{\bar{x}-\mu_{hyp}}{\sigma_{\bar{x}}} = \frac{(94.1-94)}{0.1581} = 0.63$

z_{crit} for $\alpha = 0.05$ (One tailed) is 1.64

We fail to reject H_0.

The anxiety of movies (M = 94.1) is not significantly greater than 94, z=0.63, p = 0.2643.

```
# Using R:
1-pnorm(94.1,94,1/sqrt(40))
[1] 0.2635446
```

---

8) Two tailed z-test

$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{n}} = \frac{1}{\sqrt{45}} = 0.1491$

$z_{obs} = \frac{\bar{x}-\mu_{hyp}}{\sigma_{\bar{x}}} = \frac{(88.51-89)}{0.1491} = -3.29$

z_{crit} for $\alpha = 0.05$ (Two tailed) is ± 1.96

We reject H_0.

The life expectancy of Seattleites (M = 88.51) is significantly different than 89, z=-3.29, p = 0.001.

```
# Using R: since our observed mean is less than the mean for H0:
2*pnorm(88.51,89,1/sqrt(45))
[1] 0.001012537
```

---

9) Two tailed z-test

$\sigma_{\bar{x}} = \frac{\sigma x}{\sqrt{n}} = \frac{10}{\sqrt{98}} = 1.0102$

$z_{obs} = \frac{\bar{x} - \mu_{hyp}}{\sigma_{\bar{x}}} = \frac{(49.98 - 52)}{1.0102} = -2.00$

z_{crit} for $\alpha = 0.01$ (Two tailed) is ± 2.58

We fail to reject H_0.

The width of bus riders (M = 49.98) is not significantly different than 52, z=-2, p = 0.0455.

```
# Using R: since our observed mean is less than the mean for H0:
2*pnorm(49.98,52,10/sqrt(98))
[1] 0.04553289
```

---

10) One tailed z-test

$\sigma_{\bar{x}} = \frac{\sigma x}{\sqrt{n}} = \frac{7}{\sqrt{22}} = 1.4924$

$z_{obs} = \frac{\bar{x} - \mu_{hyp}}{\sigma_{\bar{x}}} = \frac{(87.19 - 89)}{1.4924} = -1.21$

z_{crit} for $\alpha = 0.01$ (One tailed) is -2.33

We fail to reject H_0.

The width of makeshift personalities (M = 87.19) is not significantly less than 89, z=-1.21, p = 0.1131.

```
# Using R:
pnorm(87.19,89,7/sqrt(22))
[1] 0.1126017
```

---

11) One tailed z-test

$\sigma_{\bar{x}} = \frac{\sigma x}{\sqrt{n}} = \frac{9}{\sqrt{67}} = 1.0995$

$z_{obs} = \frac{\bar{x} - \mu_{hyp}}{\sigma_{\bar{x}}} = \frac{(-0.54 - 0)}{1.0995} = -0.49$

z_{crit} for $\alpha = 0.05$ (One tailed) is -1.64

We fail to reject H_0.

The amount of beer (M = -0.54) is not significantly less than 0, z=-0.49, p = 0.3121.

```
# Using R:
pnorm(-0.54,0,9/sqrt(67))
[1] 0.3116704
```

---

12) Two tailed z-test

$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{n}} = \frac{5}{\sqrt{19}} = 1.1471$

$z_{obs} = \frac{\bar{x}-\mu_{hyp}}{\sigma_{\bar{x}}} = \frac{(7.16-6)}{1.1471} = 1.01$

z_{crit} for $\alpha = 0.05$ (Two tailed) is ± 1.96

We fail to reject H_0.

The cash of photoreceptors (M = 7.16) is not significantly different than 6, z=1.01, p = 0.3125.

```
# Using R: since our observed mean is greater than the mean for H0:
2*(1-pnorm(7.16,6,5/sqrt(19)))
[1] 0.3118898
```

---

13) Two tailed z-test

$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{n}} = \frac{3}{\sqrt{41}} = 0.4685$

$z_{obs} = \frac{\bar{x}-\mu_{hyp}}{\sigma_{\bar{x}}} = \frac{(46.45-48)}{0.4685} = -3.31$

z_{crit} for $\alpha = 0.01$ (Two tailed) is ± 2.58

We reject H_0.

The life expectancy of two candy bars (M = 46.45) is significantly different than 48, z=-3.31, p = 0.0009.

```
# Using R: since our observed mean is less than the mean for H0:
2*pnorm(46.45,48,3/sqrt(41))
[1] 0.0009387062
```

---

14) Two tailed z-test

$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{n}} = \frac{7}{\sqrt{78}} = 0.7926$

$z_{obs} = \frac{\bar{x}-\mu_{hyp}}{\sigma_{\bar{x}}} = \frac{(23.3-22)}{0.7926} = 1.64$

z_{crit} for $\alpha = 0.05$ (Two tailed) is ± 1.96

We fail to reject H_0.

The news of frail galaxies (M = 23.3) is not significantly different than 22, z=1.64, p = 0.101.

```
# Using R: since our observed mean is greater than the mean for H0:
2*(1-pnorm(23.3,22,7/sqrt(78)))
[1] 0.1009669
```

---

15) Two tailed z-test

$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{n}} = \frac{7}{\sqrt{58}} = 0.9191$

$z_{obs} = \frac{\bar{x}-\mu_{hyp}}{\sigma_{\bar{x}}} = \frac{(65.29-66)}{0.9191} = -0.77$

z_{crit} for $\alpha = 0.01$ (Two tailed) is ± 2.58

We fail to reject H_0.

The hospitality of wary otter pops (M = 65.29) is not significantly different than 66, z=-0.77, p = 0.4413.

```
# Using R: since our observed mean is less than the mean for H0:
2*pnorm(65.29,66,7/sqrt(58))
[1] 0.4398438
```