PROBABILITY and MATHEMATICAL STATISTICS I.

CLASS NOTES FOR MATH/STAT 394-5, AMATH/STAT 506, and STAT 512

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PROBABILITY and MATHEMATICAL STATISTICS II.

CLASS NOTES FOR STAT 513 AND AMATH/STAT 506

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