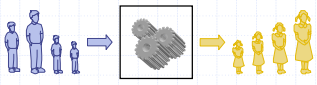
Analysis of Algorithms



Input Algorithm Output

An **algorithm** is a step-by-step procedure for solving a problem in a finite amount of time.

Math you need to know

- Summations (Sec. 1.3.1)
- Logarithms and Exponents (Sec. 1.3.2)
  - **properties of logarithms:**
    $\log_b(xy) = \log_b x + \log_b y$
    $\log_b (x/y) = \log_b x - \log_b y$
    $\log_b x^a = a\log_b x$
    $\log_b a = \log_x a/\log_x b$
  - **properties of exponentials**:
    $a^{(b+c)} = a^b a^c$
    $a^{bc} = (a^b)^c$
    $a^b / a^c = a^{(b-c)}$
    $b = a^{\log_a b}$
    $b^c = a^{c*\log_a b}$
- Proof techniques (Sec. 1.3.3)
- Basic probability (Sec. 1.3.4)

Math you need to know

- **Proofs are**
  - a sequence of statements
  - Each statement is true, based on
    - Definitions
    - Hypotheses
    - Well-known math principles
    - Previous statements
  - Statements lead towards conclusion

Induction proof

- Method of proving statements for (infinitely) large values of n, (n is the induction variable).
- Math way of using a loop in a proof.

Example induction proof

- Prove: for all int x, for all int y, for all int n,
  If n is positive, then $x^n - y^n$ is divisible by x-y.
- Let S_n denote "for all x and y, $x^n - y^n$ is divisible by x-y"

Example induction proof

- Prove: for all int x, for all int y, for all int n,
  If n is positive, then $x^n - y^n$ is divisible by x-y.
- Let S_n denote "for all x and y, $x^n - y^n$ is divisible by x-y"
- Proof with induction:
  - Base case: show S_1
  - Inductive Hypothesis (IH): for all $k \geq 1$, if S_k is true, than S_{k+1} is true.
    OR
    Inductive Hypothesis (IH): for all $k \geq 2$, if S_{k-1} is true, than S_k is true.

Example induction proof

- Prove: for all int x, for all int y, for all int n,
  If n is positive, then $x^n - y^n$ is divisible by x-y.
- Let S_n denote "for all x and y, $x^n - y^n$ is divisible by x-y"
- Proof with induction:

Pseudocode (§1.1)

- Mixture of English, math expressions, and computer code
- Less detailed than a program
- Preferred notation for describing algorithms
- Hides program design issues
- Can write at different levels of detail.

Very High-level pseudocode:

```
Algorithm arrayMax(A, n)
  Input array A of n integers
  Output maximum element of A

  currentMax ← A[0]
  Step through each element in A,
    updating currentMax when a
    bigger element is found
  return currentMax
```


Pseudocode (§1.1)

- Mixture of English, math expressions, and computer code
- Less detailed than a program
- Preferred notation for describing algorithms
- Hides program design issues
- Can write at different levels of detail.

Detailed pseudocode

```
Algorithm arrayMax(A, n)
  Input array A of n integers
  Output maximum element of A

  currentMax ← A[0]
  for i ← 1 to n − 1 do
    if A[i] > currentMax then
      currentMax ← A[i]
  return currentMax
```

Pseudocode Details

- Control flow
  - **if** … **then** … [**else** …]
  - **while** … **do** …
  - **repeat** … **until** …
  - **for** … **do** …
  - Indentation replaces braces
- Method declaration

  Algorithm *method* (*arg* [, *arg*…])

  Input …

  Output …
- Method call

  var.method (*arg* [, *arg*…])
- Return value

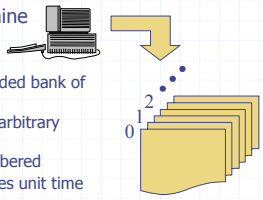
  return *expression*
- Expressions
  - ← Assignment (like = in Java)
  - = Equality testing (like == in Java)
  - n^2 Superscripts and other mathematical formatting allowed

Primitive Operations

- Basic computations performed by an algorithm
- Identifiable in pseudocode
- Largely independent from the programming language
- Examples:
  - Evaluating an expression
  - Assigning a value to a variable
  - Indexing into an array
  - Calling a method
  - Returning from a method

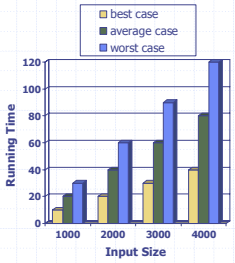
Estimating performance

- Count Primitive Operations
- = time needed by RAM model
- Random Access Machine (RAM) Model has:
  - A **CPU**
  - An potentially unbounded bank of **memory** cells
  - Each cell can hold an arbitrary number or character
  - Memory cells are numbered
  - Accessing any cell takes unit time



Running Time (§1.1)

- The running time grows with the input size.
- Running time varies with different input
- Worst-case: look at input causing most operations
- Best-case: look at input causing least number of operations
- Average case: between best and worst-case.



Counting Primitive Operations (§1.1)

- Worst-case primitive operations count, as a function of the input size

Algorithm *arrayMax*(*A*, *n*)	# operations
currentMax ← *A*[0]	2
for *i* ← 1 **to** *n* − 1 **do**	$1 + n$
if *A*[*i*] > *currentMax* **then**	$2(n - 1)$
currentMax ← *A*[*i*]	$2(n - 1)$
{ increment counter *i* }	$2(n - 1)$
return *currentMax*	1
Total	$7n - 2$

Counting Primitive Operations (§1.1)

- Best-case primitive operations count, as a function of the input size

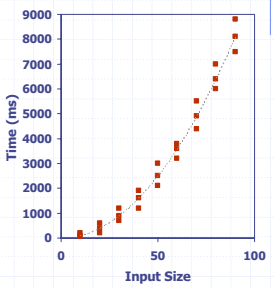
Algorithm *arrayMax*(*A*, *n*)	# operations
currentMax ← *A*[0]	2
for *i* ← 1 **to** *n* − 1 **do**	$1 + n$
if *A*[*i*] > *currentMax* **then**	$2(n - 1)$
currentMax ← *A*[*i*]	0
{ increment counter *i* }	$2(n - 1)$
return *currentMax*	1
Total	$5n$

Defining Worst [W(n)], Best [B(N)], and Average [A(n)]

- Let I_n = set of all inputs of size n.
- Let t(i) = # of primitive ops by alg on input i.
- W(n) = maximum t(i) taken over all i in I_n
- B(n) = minimum t(i) taken over all i in I_n
- A(n) = $\sum_{i \in I_n} p(i)t(i)$, p(i) = prob. of i occurring.
- We focus on the worst case
  - Easier to analyze
  - Usually want to know how bad can algorithm be
  - average-case requires knowing probability; often difficult to determine

Experimental Studies (§ 1.6)

- Implement your algorithm
- Run your implementation with inputs of varying size and composition
- Measure running time of your implementation (e. g., with System.currentTimeMillis())
- Plot the results



Limitations of Experiments

- Implement may be time-consuming and/or difficult
- Results may not be indicative of the running time on other inputs not included in the experiment.
- In order to compare two algorithms, the same hardware and software environments must be used
- Infeasible to test for correct-ness on all possible inputs.

Theoretical Analysis

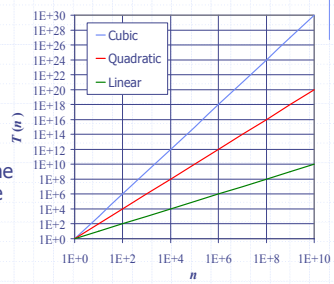
- Uses a high-level description of the algorithm instead of an implementation
- Characterizes running time as a function of the input size, n.
- Takes into account all possible inputs
- Allows us to evaluate the speed of an algorithm independent of the hardware/software environment
- Can prove correctness

Growth Rate of Running Time

- Changing the hardware/ software environment
  - Affects ***running time*** by a constant factor;
  - Does not alter its growth rate
- Example: linear growth rate of ***arrayMax*** is an intrinsic property of algorithm.

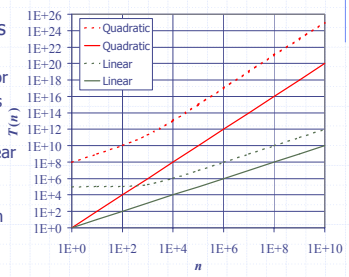
Growth Rates

- Growth rates of functions:
  - Linear $\approx n$
  - Quadratic $\approx n^2$
  - Cubic $\approx n^3$
- In a log-log chart, the slope of the line corresponds to the growth rate of the function (for polynomials)



Constant Factors

- The growth rate is not affected by
  - constant factors or
  - lower-order terms
- Examples
  - $10^2n + 10^5$ is a linear function
  - $10^5n^2 + 10^8n$ is a quadratic function



Big-Oh and Growth Rate

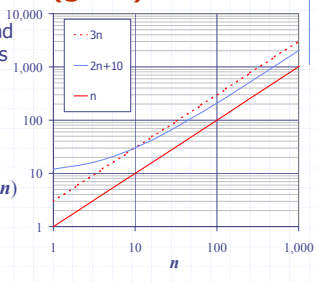
- The big-Oh notation gives an upper bound on the growth rate of a function
- The statement "$f(n)$ is $O(g(n))$" means that the growth rate of $f(n)$ is no more than the growth rate of $g(n)$
- We can use the big-Oh notation to rank functions according to their growth rate

	$f(n)$ is $O(g(n))$	$g(n)$ is $O(f(n))$
$g(n)$ grows more	Yes	No
$f(n)$ grows more	No	Yes
Same growth	Yes	Yes

Big-Oh Notation (§1.2)

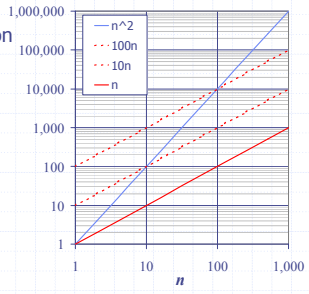
- Given functions $f(n)$ and $g(n)$, we say that $f(n)$ is $O(g(n))$ if there are positive constants c and n_0 such that

  $f(n) \leq cg(n)$ for $n \geq n_0$
- Example: $2n + 10$ is $O(n)$
  - $2n + 10 \leq cn$
  - $(c - 2)\, n \geq 10$
  - $n \geq 10/(c - 2)$
  - Pick $c = 3$ and $n_0 = 10$



Big-Oh Example

- Example: the function n^2 is not $O(n)$
  - $n^2 \leq cn$
  - $n \leq c$
  - The above inequality cannot be satisfied since c must be a constant



More Big-Oh Examples

- 7n-2

  7n-2 is O(n)
  need $c > 0$ and $n_0 \geq 1$ such that $7n-2 \leq c \bullet n$ for $n \geq n_0$
  this is true for $c = 7$ and $n_0 = 1$

- $3n^3 + 20n^2 + 5$

  $3n^3 + 20n^2 + 5$ is $O(n^3)$
  need $c > 0$ and $n_0 \geq 1$ such that $3n^3 + 20n^2 + 5 \leq c \bullet n^3$ for $n \geq n_0$
  this is true for $c = 4$ and $n_0 = 21$

- $3 \log n + \log \log n$

  $3 \log n + \log \log n$ is $O(\log n)$
  need $c > 0$ and $n_0 \geq 1$ such that $3 \log n + \log \log n \leq c \bullet \log n$ for $n \geq n_0$
  this is true for $c = 4$ and $n_0 = 2$

Big-Oh Rules

- If is $f(n)$ a polynomial of degree d, then $f(n)$ is $O(n^d)$, i.e.,
  1. Drop lower-order terms
  2. Drop constant factors
- Use the smallest possible class of functions
  - Say "$2n$ is $O(n)$" instead of "$2n$ is $O(n^2)$"
- Use the simplest expression of the class
  - Say "$3n + 5$ is $O(n)$" instead of "$3n + 5$ is $O(3n)$"

Asymptotic Algorithm Analysis

- asymptotic analysis = determining an algorithms running time in big-Oh notation
- asymptotic analysis steps:
  - We find the worst-case number of primitive operations executed as a function of the input size
  - We express this function with big-Oh notation
- Example:
  - We determine that algorithm *arrayMax* executes at most $7n - 2$ primitive operations
  - We say that algorithm *arrayMax* "runs in $O(n)$ time" or "runs in order n time"
- Since constant factors and lower-order terms are eventually dropped, we can disregard them when counting primitive operations!

Intuition for Asymptotic Notation

Big-Oh
- f(n) is O(g(n)) if f(n) is asymptotically **less than or equal** to g(n)

big-Omega
- f(n) is Ω(g(n)) if f(n) is asymptotically **greater than or equal** to g(n)

big-Theta
- f(n) is Θ(g(n)) if f(n) is asymptotically **equal** to g(n)

little-oh
- f(n) is o(g(n)) if f(n) is asymptotically **strictly less** than g(n)

little-omega
- f(n) is ω(g(n)) if is asymptotically **strictly greater** than g(n)

Relatives of Big-Oh

- **big-Omega**
  - f(n) is Ω(g(n)) if there is a constant c > 0 and an integer constant $n_0 \geq 1$ such that $f(n) \geq c \bullet g(n)$ for $n \geq n_0$
- **big-Theta**
  - f(n) is Θ(g(n)) if there are constants c' > 0 and c'' > 0 and an integer constant $n_0 \geq 1$ such that $c' \bullet g(n) \leq f(n) \leq c'' \bullet g(n)$ for $n \geq n_0$
- **little-oh**
  - f(n) is o(g(n)) if, for any constant c > 0, there is an integer constant $n_0 \geq 0$ such that $f(n) \leq c \bullet g(n)$ for $n \geq n_0$
- **little-omega**
  - f(n) is ω(g(n)) if, for any constant c > 0, there is an integer constant $n_0 \geq 0$ such that $f(n) \geq c \bullet g(n)$ for $n \geq n_0$

Example Uses of the Relatives of Big-Oh

- **$5n^2$ is $\Omega(n^2)$**

  f(n) is $\Omega(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $f(n) \geq c \bullet g(n)$ for $n \geq n_0$

  let $c = 5$ and $n_0 = 1$
- **$5n^2$ is $\Omega(n)$**

  f(n) is $\Omega(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $f(n) \geq c \bullet g(n)$ for $n \geq n_0$

  let $c = 1$ and $n_0 = 1$
- **$5n^2$ is $\omega(n)$**

  f(n) is $\omega(g(n))$ if, for any constant $c > 0$, there is an integer constant $n_0 \geq 0$ such that $f(n) \geq c \bullet g(n)$ for $n \geq n_0$

  need $5n_0^2 \geq c \bullet n_0 \rightarrow$ given c, the n_0 that satifies this is $n_0 \geq c/5 \geq 0$

More math tools & proofs

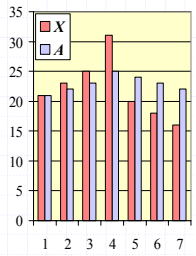
- Correctness of computing average
  - loop invariants and induction
- Recurrence equations
- Strong induction
- Cost of recursive algorithms with recurrence equations.

Computing Prefix Averages

- asymptotic analysis examples: two algorithms for prefix averages
- The *i*-th prefix average of an array X is average of the first $(i + 1)$ elements of X:

$$A[i] = (X[0] + X[1] + \ldots + X[i])/(i+1)$$

- Computing the array A of prefix averages of another array X has applications to financial analysis



Prefix Averages (Quadratic)

- The following algorithm computes prefix averages in quadratic time by applying the definition

Algorithm *prefixAverages1(X, n)*
Input array X of n integers
Output array A of prefix averages of X #operations

1. $A \leftarrow$ new array of n integers
2. **for** $i \leftarrow 0$ **to** $n - 1$ **do**
3. $s \leftarrow X[0]$
4. **for** $j \leftarrow 1$ **to** i **do**
5. $s \leftarrow s + X[j]$
6. $A[i] \leftarrow s / (i + 1)$
7. **return** A

Prefix Averages (Quadratic)

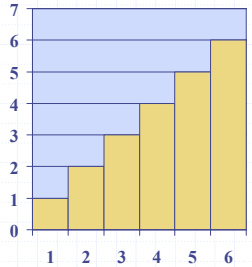
- The following algorithm computes prefix averages in quadratic time by applying the definition

Algorithm *prefixAverages1(X, n)*
Input array X of n integers
Output array A of prefix averages of X #operations

1.	$A \leftarrow$ new array of n integers	n
2.	**for** $i \leftarrow 0$ **to** $n - 1$ **do**	n
3.	$s \leftarrow X[0]$	$2n$
4.	**for** $j \leftarrow 1$ **to** i **do**	$1 + 2 + \ldots + (n - 1)$
5.	$s \leftarrow s + X[j]$	$3(1 + 2 + \ldots + (n - 1))$
6.	$A[i] \leftarrow s / (i + 1)$	$4n$
7.	**return** A	1

Arithmetic Progression

- The running time of *prefixAverages1* is $O(1 + 2 + \ldots + n)$
- The sum of the first n integers is $n(n + 1) / 2$
  - There is a simple visual proof of this fact
- Thus, algorithm *prefixAverages1* runs in $O(n^2)$ time



Prefix Averages (Linear, non-recursive)

- The following algorithm computes prefix averages in linear time by keeping a running sum

```
Algorithm prefixAverages2(X, n)
    Input array X of n integers
    Output array A of prefix averages of X        #operations
    A ← new array of n integers                   n
    s ← 0                                         1
    for i ← 0 to n − 1 do                         n
        s ← s + X[i]                              n
        A[i] ← s / (i + 1)                        n
    return A                                      1
```

- Algorithm *prefixAverages2* runs in $O(n)$ time

Prefix Averages (Linear)

- The following algorithm computes prefix averages in linear time by computing prefix sums (and averages)

```
Algorithm recPrefixSumAndAverage(X, A, n)
    Input array X of n ≥ 1 integer.
          Empty array A; A is same size as X.
    Output array A[0]…A[n-1] changed to hold prefix averages of X.
          returns sum of X[0], X[1],…,X[n-1]
1.  if n=1
2.        A[0] ← X[0]
3.        return A[0]
4.  tot ← recPrefixSumAndAverage(X,A,n-1)
5.  tot ← tot + X[n-1]
6.  A[n-1] ← tot / n
7.  return tot;
```


Prefix Averages (Linear)

- The following algorithm computes prefix averages in linear time by computing prefix sums (and averages)

```
Algorithm recPrefixSumAndAverage(X, A, n)          T(n) operations
    Input array X of n ≥ 1 integer.
          Empty array A; A is same size as X.
    Output array A[0]…A[n-1] changed to hold prefix averages of X.
          returns sum of X[0], X[1],…,X[n-1]       #operations
    if n=1                                         1
        A[0] ← X[0]                                3
        return A[0]                                2
    tot ← recPrefixSumAndAverage(X,A,n-1)          3+T(n-1)
    tot ← tot + X[n-1]                             4
    A[n-1] ← tot / n                               4
    return tot;                                    1
```

Prefix Averages, Linear

- Recurrence equation
  - T(1) = 6
  - T(n) = 13 + T(n-1) for n>1.
- Solution of recurrence is
  - T(n) = 13(n-1) + 6
- T(n) is O(n).