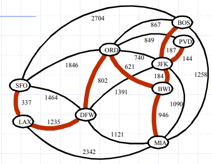
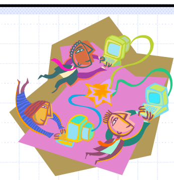
Minimum Spanning Trees



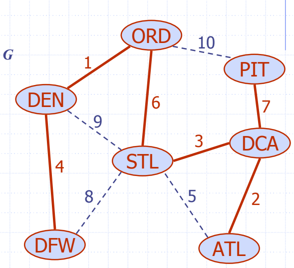
Outline and Reading



- Minimum Spanning Trees (§7.3)
  - Definitions
  - A crucial fact
- The Prim-Jarnik Algorithm (§7.3.2)
- Kruskal's Algorithm (§7.3.1)
- Baruvka's Algorithm (§7.3.3)

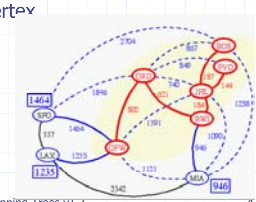
Minimum Spanning Tree

- Spanning subgraph
  - Subgraph of a graph G containing all the vertices of G
- Spanning tree
  - Spanning subgraph that is itself a (free) tree
- Minimum spanning tree (MST)
  - Spanning tree of a weighted graph with minimum total edge weight
- Applications
  - Communications networks
  - Transportation networks

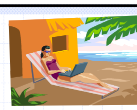


Prim-Jarnik's Algorithm

- Like Dijkstra's algorithm only simpler
- Grow the MST from arbitrary vertex s
- Greedily add vertices into cloud based on distance to any vertex in cloud
- At v, need to store $d(v)$ = minimum weight edge connecting v to a cloud vertex
- At each step:
  - We add to the cloud the vertex u outside the cloud with the smallest distance label
  - We update the labels of the vertices adjacent to u (edge relaxation)

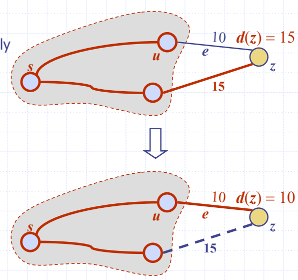


Prim's Edge Relaxation

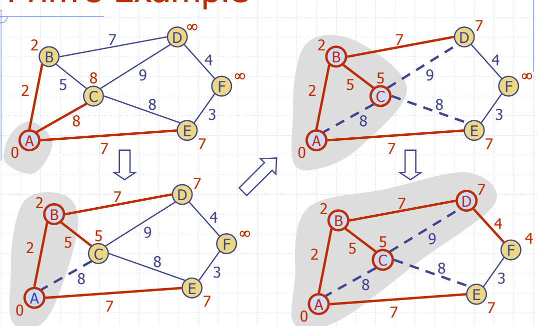


- Consider an edge $e = (u,z)$ such that
  - u is the vertex most recently added to the cloud
  - z is not in the cloud
- The relaxation of edge e updates distance $d(z)$ as follows:

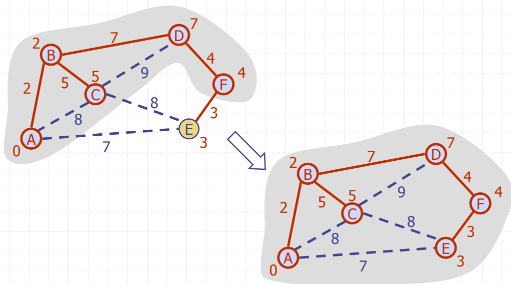
$d(z) \leftarrow \min\{d(z), e\}$



Prim's Example



Example (contd.)



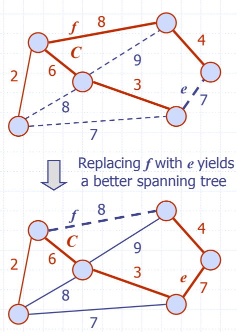
Cycle Property

Cycle Property:

- Let T be a minimum spanning tree of a weighted graph G
- Let e be an edge of G that is not in T and C let be the cycle formed by e with T
- For every edge f of C, $weight(f) \le weight(e)$

Proof:

- By contradiction
- If $weight(f) > weight(e)$ we can get a spanning tree of smaller weight by replacing e with f



Correctness of Prim's

Let T_k be tree produced by Prim's after kth iteration. Let G_k be the the subgraph of G induced by T_k. Then T_k is a MST of G_k.

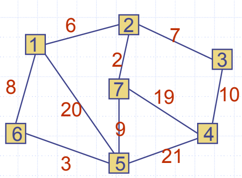
Prim-Jarnik's Algorithm (cont.)

- A priority queue stores the vertices outside the cloud
  - Key: distance
  - Element: vertex
- Locator-based methods
  - *insert*(*k*,*e*) returns a locator
  - *replaceKey*(*l*,*k*) changes the key of an item
- We store three labels with each vertex:
  - Distance
  - Tree edge in MST
  - Locator in priority queue

```
Algorithm PrimJarnikMST(G)
  Q ← new heap-based priority queue
  s ← a vertex of G
  for all v ∈ G.vertices()
    if v = s
      setDistance(v, 0)
    else
      setDistance(v, ∞)
    setTreeEdge(v, ∅)
    l ← Q.insert(getDistance(v), v)
    setLocator(v,l)
  while ¬Q.isEmpty()
    u ← Q.removeMin()
    for all e ∈ G.incidentEdges(u)
      z ← G.opposite(u,e)
      r ← weight(e)
      if r < getDistance(z)
        setDistance(z,r)
        setTreeEdge(z,e)
        Q.replaceKey(getLocator(z),r)
```


Example graph

Start at 1, run Prim's



Analysis

- Graph operations
  - Method incidentEdges is called once for each vertex
- Label operations
  - We set/get the distance, tree and locator labels of vertex z $O(\deg(z))$ times
  - Setting/getting a label takes $O(1)$ time
- Priority queue operations
  - Each vertex inserted and removed once taking $O(\log n)$ time each time for 2n times.
  - The key of a vertex w in the priority queue is modified at most $\deg(w)$ times, where each key change takes $O(\log n)$ time
- Prim-Jarnik's algorithm runs in $O((n + m) \log n)$ time provided the graph is represented by the adjacency list structure
- The running time is $O(m \log n)$ since the graph is connected
- What is running time for unsorted-sequence based priority queue?

Kruskal's MST algorithm

- Another greedy strategy for finding MST
- Gradually turn forest into tree as edges are added
- Add cheapest edge possible
  - Don't add edge if it forms cycle
- Overview:

```
kruskalMST (Graph G)
    Initalize F (forest) to empty.
    Place all edges in PQ according to cost
    For each edge (u,v) in PQ (in sorted order)
        if (u,v) does not make a cycle in F
            add (u,v) to F
    return F;
```

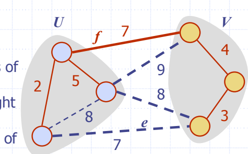

Partition Property

Partition Property:

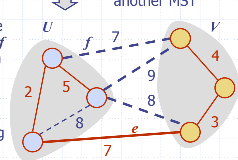
- Consider a partition of the vertices of G into subsets U and V
- Let e be an edge of minimum weight across the partition
- There is a minimum spanning tree of G containing edge e

Proof:

- Let T be an MST of G
- If T does not contain e, consider the cycle C formed by e with T and let f be an edge of C across the partition
- By the cycle property, $weight(f) \leq weight(e)$
- Thus, $weight(f) = weight(e)$
- We obtain another MST by replacing f with e



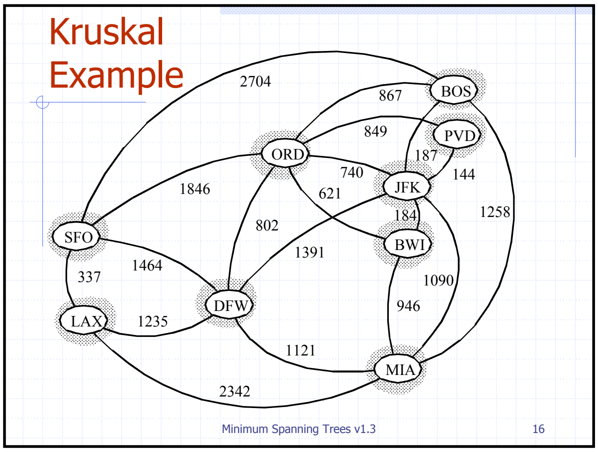
Replacing f with e yields another MST



Kruskal's Algorithm

- Each vertex starts in its own cloud (a partition)
- Clouds merge together as edges are added
- A priority queue stores edges in weight order
  - Key: weight
  - Element: edge
- Only edges between clouds will not form cycles
  - add cheapest edge between clouds
- At end of algorithm:
  - All vertices in one cloud
  - Edges added form MST

Algorithm *KruskalMST(G)*
  for each vertex V in G **do**
    define a ***Cloud(v)*** of $\leftarrow \{v\}$
  let Q be a priority queue.
  Insert all edges into Q using their weights as the key
  $T \leftarrow \varnothing$
  while T has fewer than n-1 edges **do**
    edge e = ***T.removeMin()***
    Let u, v be the endpoints of e
    if ***Cloud(v)*** $\neq$ ***Cloud(u)*** **then**
      Add edge e to T
      Merge ***Cloud(v)*** and ***Cloud(u)***
  return T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Minimum Spanning Trees v1.3
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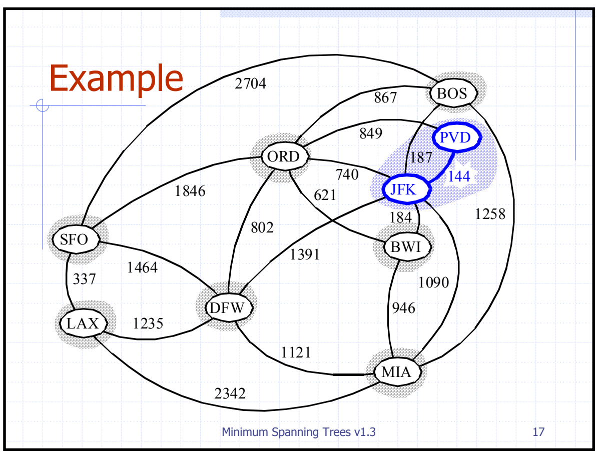
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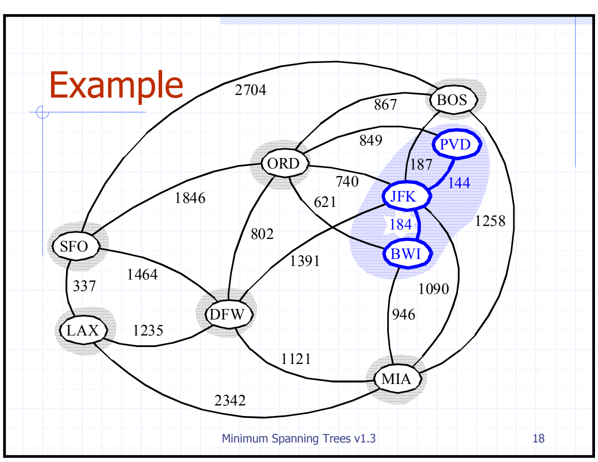
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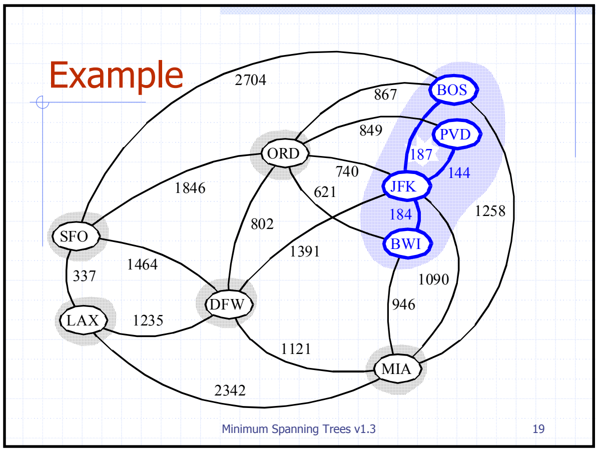
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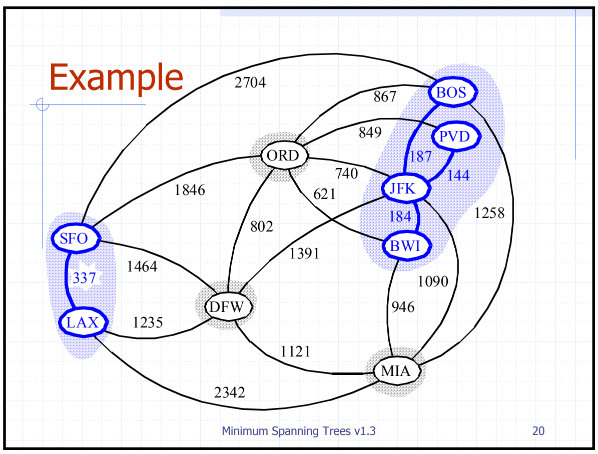
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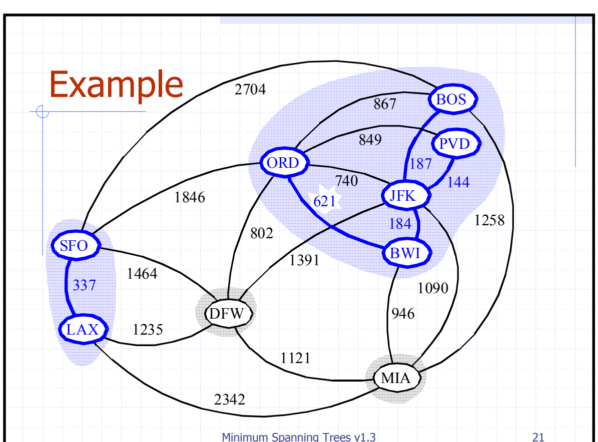
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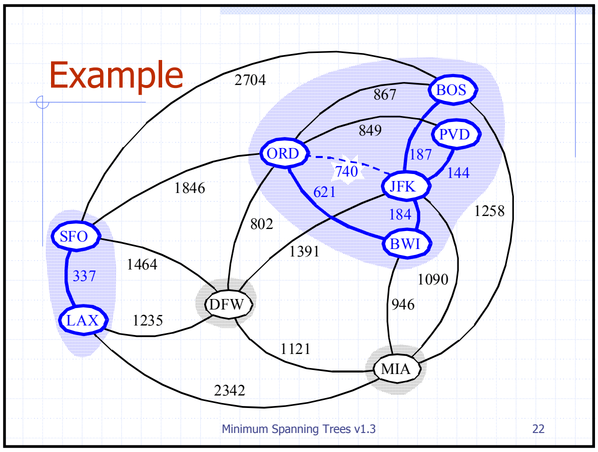
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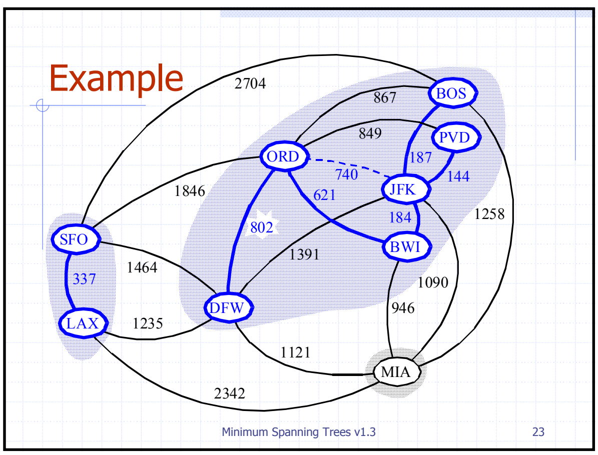
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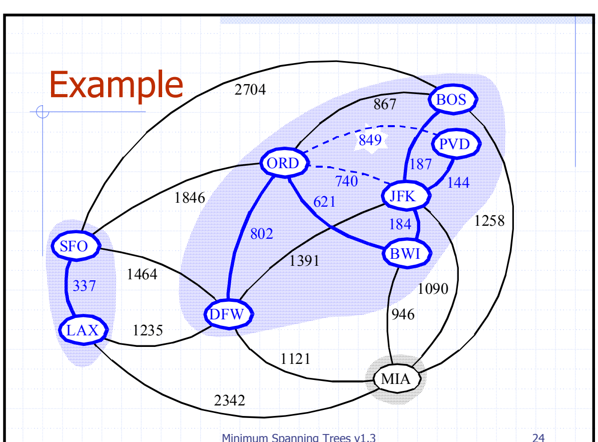
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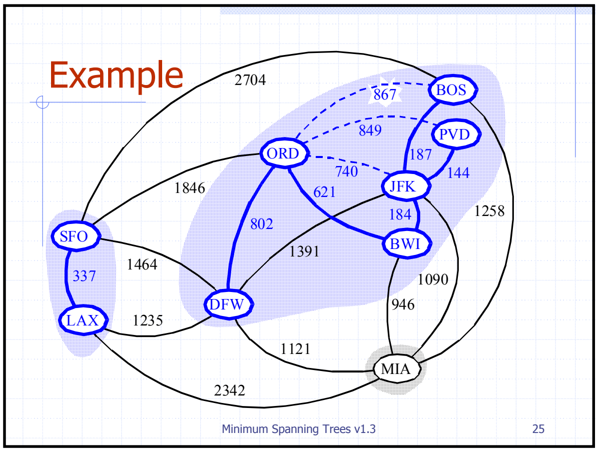
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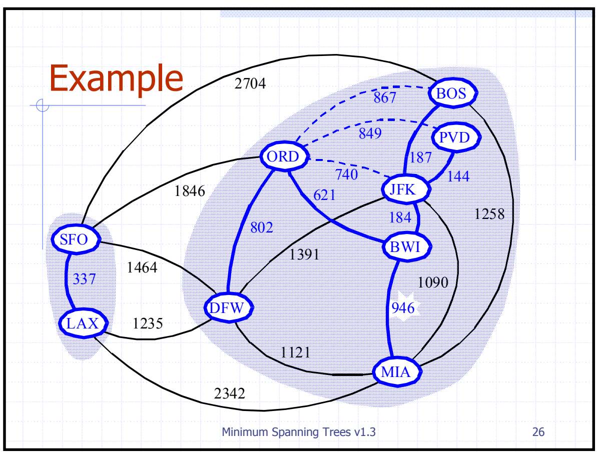
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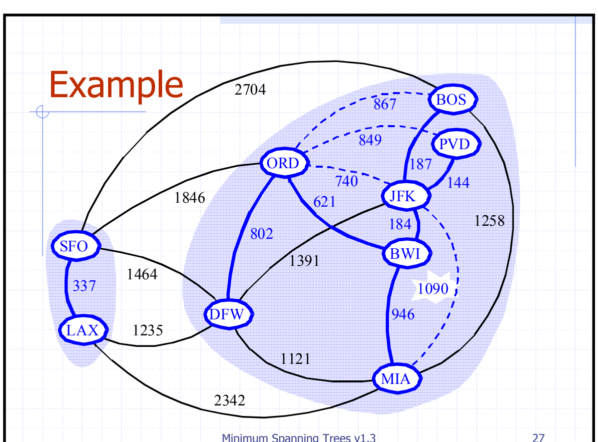
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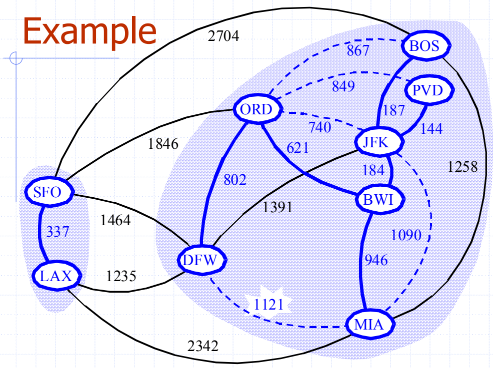
Example



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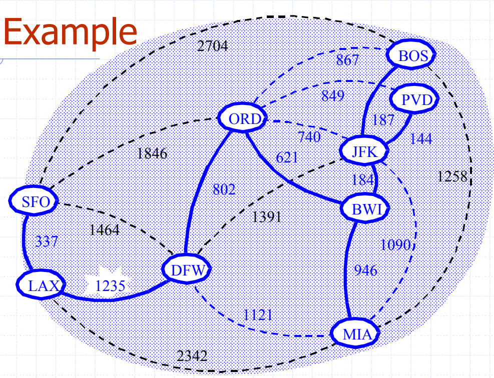
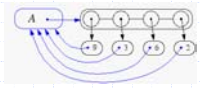


Data Structure for Kruskal Algortihm

- The algorithm maintains a forest of trees
- An edge is accepted it if connects distinct trees
- We need a data structure that maintains a **partition**, i.e., a collection of disjoint sets, with the operations:
  - -**find**(u): return the set storing u
  - -**union**(u,v): replace the sets storing u and v with their union

Representation of a Partition



- Each set is stored in a sequence
- Each element has a reference back to the set
  - operation find(u) takes O(1) time, and returns the set of which u is a member.
  - in operation union(u,v), we move the elements of the smaller set to the sequence of the larger set and update their references
  - the time for operation union(u,v) is $\min(n_u,n_v)$, where n_u and n_v are the sizes of the sets storing u and v
- Whenever an element is processed, it goes into a set of size at least double, hence each element is processed at most log n times

Partition-Based Implementation

- A partition-based version of Kruskal's Algorithm performs cloud merges as unions and tests as finds.

```
Algorithm Kruskal(G):
  Input: A weighted graph G.
  Output: An MST T for G.
Let P be a partition of the vertices of G, where each vertex forms a separate set.
Let Q be a priority queue storing the edges of G, sorted by their weights
Let T be an initially-empty tree
while Q is not empty do
  (u,v) ← Q.removeMinElement()
  if P.find(u) != P.find(v) then
      Add (u,v) to T
      P.union(u,v)
return T
```

Running time: O(m log n)

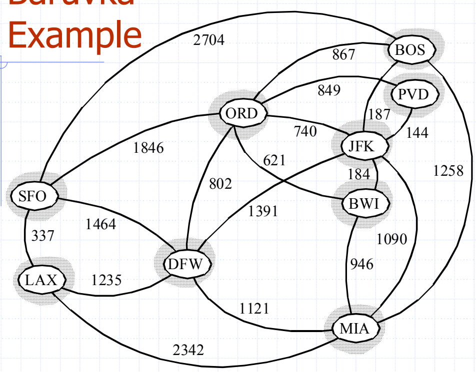
Baruvka's Algorithm

- Like Kruskal's Algorithm, Baruvka's algorithm grows many "clouds" at once.

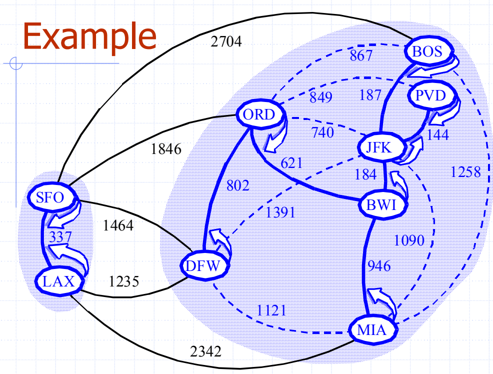
```
Algorithm BaruvkaMST(G)
    T ← V {just the vertices of G}
  while T has fewer than n-1 edges do
    for each connected component C in T do
      Let edge e be the smallest-weight edge from C to another component in T.
      if e is not already in T then
        Add edge e to T
  return T
```

- Each iteration of the while-loop halves the number of connected components in T.
  - The running time is O(m log n).

Baruvka Example



Example



Example