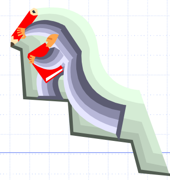
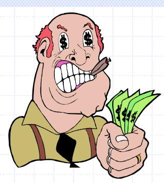
The Greedy Method

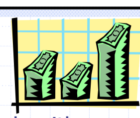


Outline and Reading



- The Greedy Method Technique (§5.1)
- Fractional Knapsack Problem (§5.1.1)
- Task Scheduling (§5.1.2)
- Minimum Spanning Trees (§7.3) [future lecture]

The Greedy Method Technique



- **The greedy method** is a general algorithm design paradigm, built on the following elements:
  - **configurations**: different choices, collections, or values to find
  - **objective function**: a score assigned to configurations, which we want to either maximize or minimize
- It works best when applied to problems with the **greedy-choice** property:
  - a globally-optimal solution can always be found by a series of local improvements from a starting configuration.

Making Change

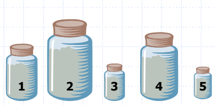
- Problem: A dollar amount to reach and a collection of coin amounts to use to get there.
- Configuration: A dollar amount yet to return to a customer plus the coins already returned
- Objective function: Minimize number of coins returned.
- Greedy solution: Always return the largest coin you can
- Example 1: Coins are valued \$.32, \$.08, \$.01
  - Has the greedy-choice property, since no amount over \$.32 can be made with a minimum number of coins by omitting a \$.32 coin (similarly for amounts over \$.08, but under \$.32).
- Example 2: Coins are valued \$.30, \$.20, \$.05, \$.01
  - Does not have greedy-choice property, since \$.40 is best made with two \$.20's, but the greedy solution will pick three coins (which ones?)

The Fractional Knapsack Problem

- Given: A set S of n items, with each item i having
  - b_i - a positive benefit
  - w_i - a positive weight
- Goal: Choose items with maximum total benefit but with weight at most W.
- If we are allowed to take fractional amounts, then this is the **fractional knapsack problem**.
  - In this case, we let x_i denote the amount we take of item i
  - Objective: maximize $\sum_{i \in S} b_i (x_i / w_i)$
  - Constraint: $\sum_{i \in S} x_i \le W$

Example

- Given: A set S of n items, with each item i having
  - b_i - a positive benefit
  - w_i - a positive weight
- Goal: Choose items with maximum total benefit but with weight at most W.



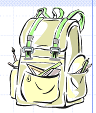
Items:	1	2	3	4	5
Weight:	4 ml	8 ml	2 ml	6 ml	1 ml
Benefit:	\$12	\$32	\$40	\$30	\$50
Value: (\$ per ml)	3	4	20	5	50

"knapsack"

10 ml

Solution:
- 1 ml of 5
- 2 ml of 3
- 6 ml of 4
- 1 ml of 2

The Fractional Knapsack Algorithm



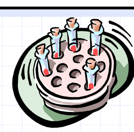
- Greedy choice: Keep taking item with highest **value** (benefit to weight ratio)
  - Since $\sum_{i \in S} b_i (x_i / w_i) = \sum_{i \in S} (b_i / w_i) x_i$
  - Run time: O(n log n). Why?
- Correctness: Suppose there is a better solution
  - there is an item i with higher value than a chosen item j, but $x_i < w_i$, $x_j > 0$ and $v_i < v_j$
  - If we substitute some i with j, we get a better solution
  - How much of i: $\min\{w_i - x_i, x_j\}$
  - Thus, there is no better solution than the greedy one

Algorithm *fractionalKnapsack*(*S*, *W*)
  Input: set S of items w/ benefit b_i and weight w_i; max. weight W
  Output: amount x_i of each item i to maximize benefit w/ weight at most W
  for *each item i in S*
    $x_i \leftarrow 0$
    $v_i \leftarrow b_i / w_i$ {value}
  $w \leftarrow 0$ {total weight}
  while $w < W$
    remove item i w/ highest v_i
    $x_i \leftarrow \min\{w_i, W - w\}$
    $w \leftarrow w + \min\{w_i, W - w\}$

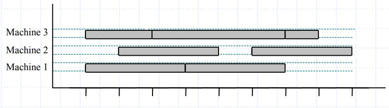
The Fractional Knapsack Algorithm- detailed soln

Algorithm *fractionalKnapsack*(*S*, *W*)
  Input: set S of n items w/ benefit b_i and weight w_i; max. total weight W
  Output: amount x_i of each item i to maximize benefit w/ weight at most W
  $PQ \leftarrow$ empty maximizing heap-based priority queue
  for *each item i in S*
    $x_i \leftarrow 0$
    $v_i \leftarrow b_i / w_i$ {value}
    PQ.insertItem(v_i,*i*) {insert item into PQ, sorted by value}
  $w \leftarrow 0$ {total weight so far}
  while $w < W$
    $k \leftarrow$ *PQ*.removeMax() {remove item *i* w/ highest v_i}
    addamt $\leftarrow \min\{w_k, W - w\}$ {amount of item k to take}
    $x_i \leftarrow$ *addamt*
    $w \leftarrow w$ + *addamt*

Task Scheduling



- Given: a set T of n tasks, each having:
  - A start time, s_i
  - A finish time, f_i (where $s_i < f_i$)
- Goal: Perform all the tasks using a minimum number of "machines."



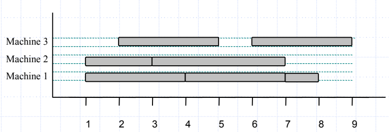
Task Scheduling Algorithm

- Greedy choice: consider tasks by their start time and use as few machines as possible with this order.
- Correctness:
  - When k^{th} machine is created to do task i (at time s_i), all k-1 other machines are busy with another task at time s_i;
  - There are k tasks that conflict with each other at time s_i
  - At least k machines necessary.
- Is it correct w/o ordering by start-time?

Algorithm *taskSchedule*(*T*)
  Input: set *T* of tasks w/ start time s_i and finish time f_i
  Output: non-conflicting schedule with minimum number of machines
  m ← *0*   {no. of machines}
  while *T is not empty*
    remove task i w/ smallest s_i
    if *there's a machine j for i* **then**
      schedule i on machine j
    else
      m ← *m* + *1*
      schedule i on machine m

Example

- Given: a set T of n tasks, each having:
  - A start time, s_i
  - A finish time, f_i (where $s_i < f_i$)
  - [1,4], [1,3], [2,5], [3,7], [4,7], [6,9], [7,8] (ordered by start)
- Goal: Perform all tasks on min. number of machines



Task Scheduling Algorithm

- Greedy choice: consider tasks by their start time and use as few machines as possible with this order.
- Make following operations fast:
  - removing task with smallest start time
  - checking scheduling conflicts
- Both steps above can be done in O(log n) time, where n is number of tasks. (How?)
- Thus, O(n log n).

Algorithm *taskSchedule*(*T*)
  Input: set *T* of tasks w/ start time s_i and finish time f_i
  Output: non-conflicting schedule with minimum number of machines
  schedule = list of (task, machine num) pairs
  m ← *0*   {no. of machines}
  while *T is not empty*
    remove task i w/ smallest s_i
    if *there's a machine j for i* **then**
      schedule i on machine j
    else
      m ← *m* + *1*
      schedule i on machine m
  return *schedule;*