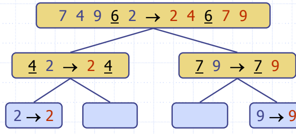
Quick-Sort

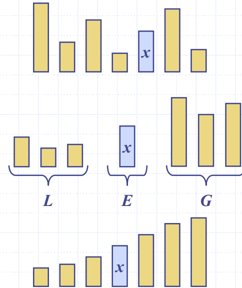


Outline and Reading

- Quick-sort (§4.3)
  - Algorithm
  - Partition step
  - Quick-sort tree
  - Execution example
- Analysis of quick-sort (4.3.1)

Quick-Sort

- Quick-sort is a randomized sorting algorithm based on the divide-and-conquer paradigm:
  - Divide: pick a random element x (called pivot) and partition S into
    - L elements less than x
    - E elements equal x
    - G elements greater than x
  - Recur: sort L and G
  - Conquer: join L, E and G



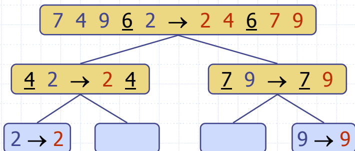
Partition

- We partition an input sequence as follows:
  - We remove, in turn, each element y from S and
  - We insert y into L, E or G, depending on the result of the comparison with the pivot x
- Each insertion and removal is at the beginning or at the end of a sequence, and hence takes $O(1)$ time
- Thus, the partition step of quick-sort takes $O(n)$ time

```
Algorithm partition(S, p)
  Input sequence S, position p of pivot
  Output subsequences L, E, G of the
     elements of S less than, equal to,
     or greater than the pivot, resp.
  L, E, G ← empty sequences
  x ← S.remove(p)
  while ¬S.isEmpty()
     y ← S.remove(S.first())
     if y < x
        L.insertLast(y)
     else if y = x
        E.insertLast(y)
     else { y > x }
        G.insertLast(y)
  return L, E, G
```

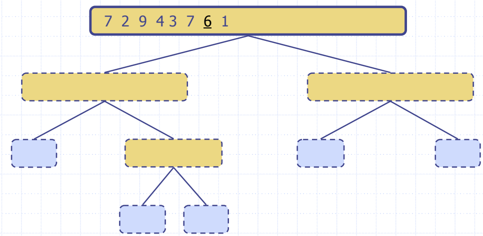

Quick-Sort Tree

- An execution of quick-sort is depicted by a binary tree
  - Each node represents a recursive call of quick-sort and stores
    - Unsorted sequence before the execution and its pivot
    - Sorted sequence at the end of the execution
  - The root is the initial call
  - The leaves are calls on subsequences of size 0 or 1



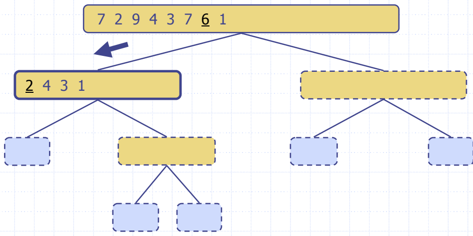
Execution Example

- Pivot selection



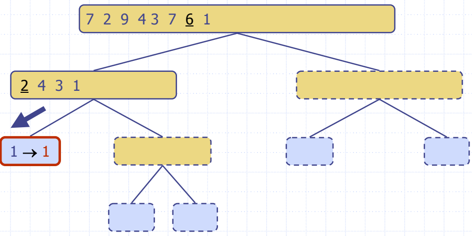
Execution Example (cont.)

- Partition, recursive call, pivot selection



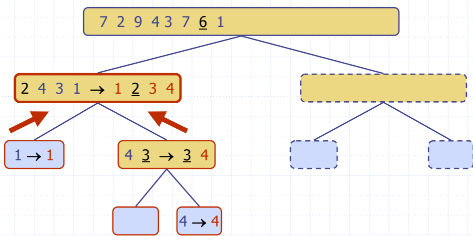
Execution Example (cont.)

- Partition, recursive call, base case



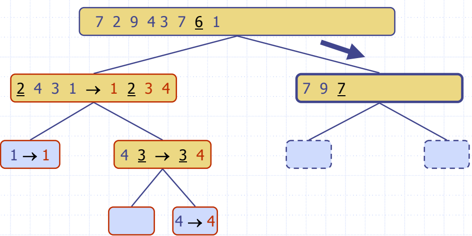
Execution Example (cont.)

- Recursive call, …, base case, join



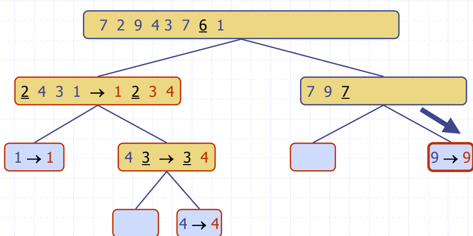
Execution Example (cont.)

- Recursive call, pivot selection



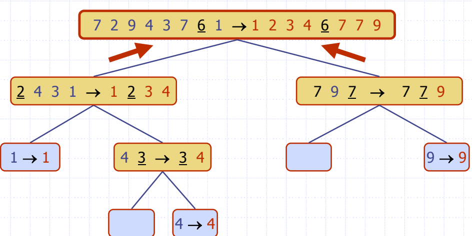
Execution Example (cont.)

- Partition, …, recursive call, base case



Execution Example (cont.)

- Join, join

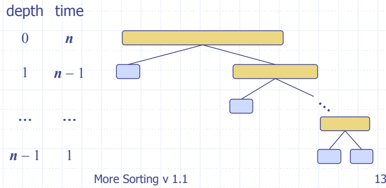


Worst-case Running Time

- The worst case for quick-sort occurs when the pivot is the unique minimum or maximum element
- One of L and G has size $n - 1$ and the other has size 0
- The running time is proportional to the sum

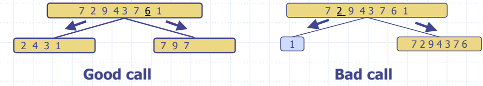
$$n + (n - 1) + \ldots + 2 + 1$$

- Thus, the worst-case running time of quick-sort is $O(n^2)$



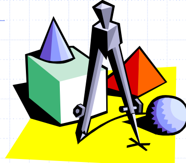
Expected Running Time

- Consider a recursive call of quick-sort on a sequence of size s
  - **Good call:** the sizes of L and G are each less than $3s/4$
  - **Bad call:** one of L and G has size greater than $3s/4$



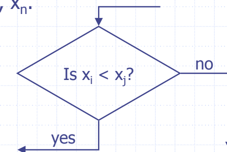
- A call is good with probability $1/2$
  - 1/2 of the possible pivots cause good calls:

Sorting Lower Bound



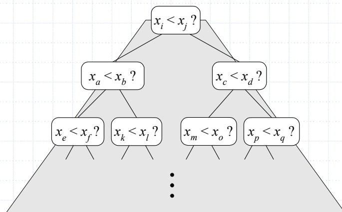
Comparison-Based Sorting (§ 4.4)

- Many sorting algorithms are comparison based.
  - They sort by making comparisons between pairs of objects
  - Examples: bubble-sort, selection-sort, insertion-sort, heap-sort, merge-sort, quick-sort, ...
- Let us therefore derive a lower bound on the running time of any algorithm that uses comparisons to sort n elements, $x_1, x_2, \ldots, x_n$.



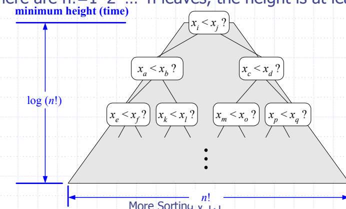
Counting Comparisons

- Let us just count comparisons then.
- Each possible run of the algorithm corresponds to a root-to-leaf path in a **decision tree**

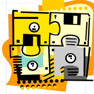


Decision Tree Height

- The height of this decision tree is a lower bound on the running time
- Every possible input permutation must lead to a separate leaf output.
  - If not, some input ...4...5... would have same output ordering as ...5...4..., which would be wrong.
- Since there are n!=1*2*...*n leaves, the height is at least log (n!)



The Lower Bound

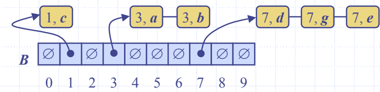


- Any comparison-based sorting algorithms takes at least log (n!) time
- Therefore, any such algorithm takes time at least

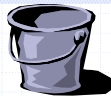
$$\log(n!) \geq \log\left(\frac{n}{2}\right)^{\frac{n}{2}} = (n/2)\log(n/2).$$

- That is, any comparison-based sorting algorithm must run in Ω(n log n) time.

Bucket-Sort and Radix-Sort



Bucket-Sort (§ 4.5.1)

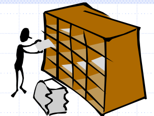


- Let be S be a sequence of n (key, element) items with keys in the range $[0, N-1]$
- Bucket-sort uses the keys as indices into an auxiliary array B of sequences (buckets); N total buckets.

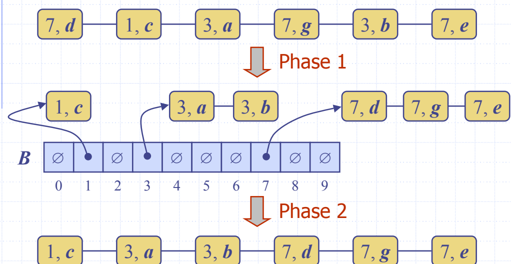
  Phase 1: Empty sequence S by moving each item (k, o) into its bucket $B[k]$

  Phase 2: For $i = 0, \ldots, N-1$, move the items of bucket $B[i]$ to the end of sequence S

Example



- Key range $[0, 9]$



Bucket-Sort (§ 4.5.1)

Phase 1: Move items into buckets

Phase 2: Move buckets into sequence, in order.

- Analysis:
  - Phase 1 takes $O(n)$ time
  - Phase 2 takes $O(n + N)$ time

  Bucket-sort takes $O(n + N)$ time
- Correctness:
  - What are loop invariants for Phase 1 and 2?

```
Algorithm bucketSort(S, N)
  Input sequence S of (key, element)
     items with keys in the range
     [0, N – 1]
  Output sequence S sorted by
     increasing keys
  B ← array of N empty sequences
  while ¬S.isEmpty()
     f ← S.first()
     (k, o) ← S.remove(f)
     B[k].insertLast((k, o))
  for i ← 0 to N – 1
     while ¬B[i].isEmpty()
        f ← B[i].first()
        (k, o) ← B[i].remove(f)
        S.insertLast((k, o))
```


Bucket-Sort Properties

- Keys have a fixed range of values.
- Keys are NOT compared.
- bucketSort is a stable sort.
- **Stable** Sort Property:
  - Any two items with the same key will be in the same relative order after sorting.

Lexicographic Order

- A d-tuple is a sequence of d keys $(k_1, k_2, \ldots, k_d)$, where key k_i is said to be the i-th dimension of the tuple
- Example:
  - The Cartesian coordinates of a point in space are a 3-tuple
- The lexicographic order of two d-tuples is recursively defined as follows

$$(x_1, x_2, \ldots, x_d) < (y_1, y_2, \ldots, y_d)$$

$$x_1 < y_1 \lor x_1 = y_1 \land (x_2, \ldots, x_d) < (y_2, \ldots, y_d)$$

I.e., the tuples are compared by the first dimension, then by the second dimension, etc.

Radix-Sort (§ 4.5.2)

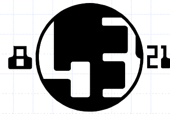
- Radix-sort uses bucket-sort to sort each dimension in a stable manner.
- Radix-sort is applicable to tuples where the keys in each dimension i are integers in the range $[0, N-1]$
- Radix-sort runs in time $O(d(n+N))$

Algorithm *radixSort(S, N)*
Input sequence S of d-tuples such that $(0, \ldots, 0) \leq (x_1, \ldots, x_d)$ and $(x_1, \ldots, x_d) \leq (N-1, \ldots, N-1)$ for each tuple $(x_1, \ldots, x_d)$ in S
Output sequence S sorted in lexicographic order
for $i \leftarrow d$ **downto** 1
  bucketSort(S, N, i)
  (bucketSorts S on dimension i)

Example:

(7,4,6) (5,1,5) (2,4,6) (2, 1, 4) (3, 2, 4)
(2, 1, 4) (3, 2, 4) (5,1,5) (7,4,6) (2,4,6)
(2, 1, 4) (5,1,5) (3, 2, 4) (7,4,6) (2,4,6)
(2, 1, 4) (2,4,6) (3, 2, 4) (5,1,5) (7,4,6)

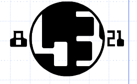
Radix-Sort for Base 10 Numbers



- Consider a sequence of n d-digit integers
  $x = x_{d-1} \ldots x_1 x_0$
- We represent each element as a d-tuple of integers in the range $[0, 9]$ and apply bucket-sort with $N = 10$
- This application of the radix-sort algorithm runs in $O(dn)$ time
- For example, we can sort a sequence of 10-digit integers in linear time

Algorithm *base10RadixSort(S)*
Input sequence S of d-digit integers
Output sequence S sorted
replace each element x of S with the item $(0, x)$
for $i \leftarrow 0$ **to** $d-1$
  replace the key k of each item (k, x) of S with digit x_i of x
  bucketSort(S, 10)

Example



- Sorting a sequence of 4-digit integers