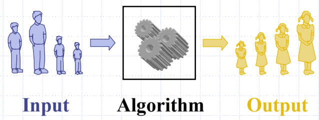
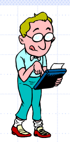
Analysis of Algorithms



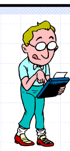
An **algorithm** is a step-by-step procedure for solving a problem in a finite amount of time.

Math you need to know



- Summations (Sec. 1.3.1)
- Logarithms and Exponents (Sec. 1.3.2)
  - **properties of logarithms:**
    - $\log_b(xy) = \log_b x + \log_b y$
    - $\log_b (x/y) = \log_b x - \log_b y$
    - $\log_b x^a = a\log_b x$
    - $\log_b a = \log_x a/\log_x b$
  - **properties of exponentials**:
    - $a^{(b+c)} = a^b a^c$
    - $a^{bc} = (a^b)^c$
    - $a^b / a^c = a^{(b-c)}$
    - $b = a^{\log_a b}$
    - $b^c = a^{c*\log_a b}$
- Proof techniques (Sec. 1.3.3)
- Basic probability (Sec. 1.3.4)

Math you need to know



- **Proofs are**
  - a sequence of statements
  - Each statement is true, based on
    - Definitions
    - Hypotheses
    - Well-known math principles
    - Previous statements
  - Statements lead towards conclusion

Induction proof

- Method of proving statements for (infinitely) large values of n, (n is the induction variable).
- Math way of using a loop in a proof.

Example induction proof

- Prove: for all int x, for all int y, for all int n,
  If n is positive, then $x^n - y^n$ is divisible by x-y.
- Let S_n denote "for all x and y, $x^n - y^n$ is divisible by x-y"

Example induction proof

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  If n is positive, then $x^n - y^n$ is divisible by x-y.
- Let S_n denote "for all x and y, $x^n - y^n$ is divisible by x-y"
- Proof with induction:
  - Base case: show S_1
  - Inductive Step: for all k ≥ 1, if S_k is true, than S_{k+1} is true.
    OR
    Inductive Step: for all k ≥ 2, if S_{k-1} is true, than S_k is true.
  - S_k sometimes called inductive hypothesis.

Example induction proof

- Prove: for all int x, for all int y, for all int n,
  If n is positive, then $x^n - y^n$ is divisible by x-y.
- Let S_n denote "for all x and y, $x^n - y^n$ is divisible by x-y"
- Proof with induction:

Pseudocode (§1.1)

- Mixture of English, math expressions, and computer code
- Less detailed than a program
- Preferred notation for describing algorithms
- Hides program design issues
- Can write at different levels of detail.

Very High-level pseudocode:

Algorithm *arrayMax*(*A*, *n*)
  Input array *A* of *n* integers
  Output maximum element of *A*

  currentMax ← *A*[0]
  Step through each element in A, updating *currentMax* when a bigger element is found
  return *currentMax*

Pseudocode (§1.1)

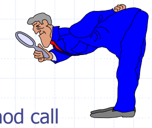
- Mixture of English, math expressions, and computer code
- Less detailed than a program
- Preferred notation for describing algorithms
- Hides program design issues
- Can write at different levels of detail.

Detailed pseudocode

```
Algorithm arrayMax(A, n)
  Input array A of n integers
  Output maximum element of A

  currentMax ← A[0]
  for i ← 1 to n − 1 do
    if A[i] > currentMax then
      currentMax ← A[i]
  return currentMax
```

Pseudocode Details



- Control flow
  - **if** … **then** … [**else** …]
  - **while** … **do** …
  - **repeat** … **until** …
  - **for** … **do** …
  - Indentation replaces braces
- Method declaration

  Algorithm *method* (*arg* [, *arg*…])

  Input …

  Output …
- Method call

  var.method (*arg* [, *arg*…])
- Return value

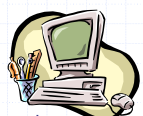
  return *expression*
- Expressions

  ← Assignment (like = in Java)

  = Equality testing (like == in Java)

  n^2 Superscripts and other mathematical formatting allowed

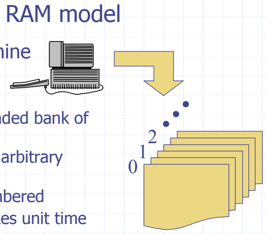
Primitive Operations



- Basic computations performed by an algorithm
- Identifiable in pseudocode
- Largely independent from the programming language
- Examples:
  - Evaluating an expression
  - Assigning a value to a variable
  - Indexing into an array
  - Calling a method
  - Returning from a method

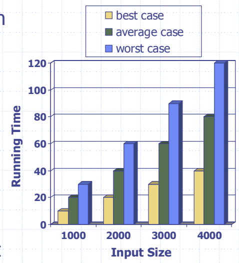
Estimating performance

- Count Primitive Operations
- = time needed by RAM model
- Random Access Machine (RAM) Model has:
  - A **CPU**
  - An potentially unbounded bank of **memory** cells
  - Each cell can hold an arbitrary number or character
  - Memory cells are numbered
  - Accessing any cell takes unit time



Running Time (§1.1)

- The running time grows with the input size.
- Running time varies with different input
- Worst-case: look at input causing most operations
- Best-case: look at input causing least number of operations
- Average case: between best and worst-case.



Counting Primitive Operations (§1.1)

- Worst-case primitive operations count, as a function of the input size

Algorithm *arrayMax*(A, n)	# operations
currentMax ← $A[0]$	2
for i ← 1 **to** $n - 1$ **do**	$1 + n$
if $A[i] >$ *currentMax* **then**	$2(n - 1)$
currentMax ← $A[i]$	$2(n - 1)$
{ increment counter i }	$2(n - 1)$
return *currentMax*	1
Total	$7n - 2$

Counting Primitive Operations (§1.1)

- Best-case primitive operations count, as a function of the input size

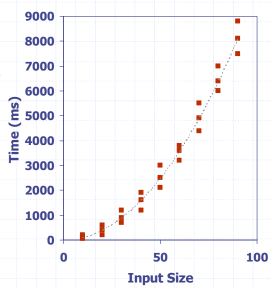
Algorithm *arrayMax*(A, n)	# operations
currentMax ← $A[0]$	2
for i ← 1 **to** $n - 1$ **do**	$1 + n$
if $A[i] >$ *currentMax* **then**	$2(n - 1)$
currentMax ← $A[i]$	0
{ increment counter i }	$2(n - 1)$
return *currentMax*	1
Total	$5n$

Defining Worst [W(n)], Best [B(N)], and Average [A(n)]

- Let I_n = set of all inputs of size n.
- Let t(i) = # of primitive ops by alg on input i.
- W(n) = maximum t(i) taken over all i in I_n
- B(n) = minimum t(i) taken over all i in I_n
- A(n) = $\sum_{i \in I_n} p(i)t(i)$, p(i) = prob. of i occurring.
- We focus on the worst case
  - Easier to analyze
  - Usually want to know how bad can algorithm be
  - average-case requires knowing probability; often difficult to determine

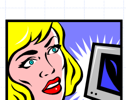
Experimental Studies (§ 1.6)

- Implement your algorithm
- Run your implementation with inputs of varying size and composition
- Measure running time of your implementation (e. g., with System.currentTimeMillis())
- Plot the results

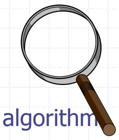


Limitations of Experiments

- Implement may be time-consuming and/or difficult
- Results may not be indicative of the running time on other inputs not included in the experiment.
- In order to compare two algorithms, the same hardware and software environments must be used
- Infeasible to test for correctness on all possible inputs.



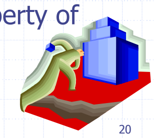
Theoretical Analysis



- Uses a high-level description of the algorithm instead of an implementation
- Characterizes running time as a function of the input size, *n*.
- Takes into account all possible inputs
- Allows us to evaluate the speed of an algorithm independent of the hardware/software environment
- Can prove correctness

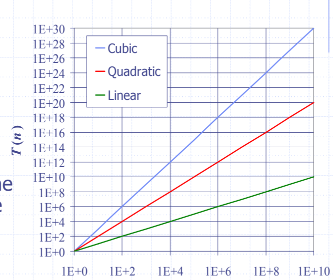
Growth Rate of Running Time

- Changing the hardware/ software environment
  - Affects ***running time*** by a constant factor;
  - Does not alter its growth rate
- Example: linear growth rate of ***arrayMax*** is an intrinsic property of algorithm.



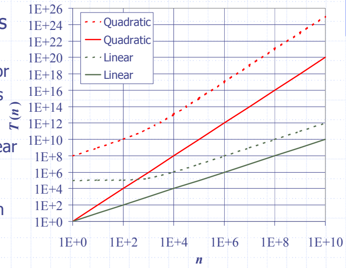
Growth Rates

- Growth rates of functions:
  - Linear $\approx n$
  - Quadratic $\approx n^2$
  - Cubic $\approx n^3$
- In a log-log chart, the slope of the line corresponds to the growth rate of the function (for polynomials)



Constant Factors

- The growth rate is not affected by
  - constant factors or
  - lower-order terms
- Examples
  - $10^2n + 10^5$ is a linear function
  - $10^5n^2 + 10^8n$ is a quadratic function



Big-Oh and Growth Rate

- The big-Oh notation gives an upper bound on the growth rate of a function
- The statement "$f(n)$ is $O(g(n))$" means that the growth rate of $f(n)$ is no more than the growth rate of $g(n)$
- We can use the big-Oh notation to rank functions according to their growth rate

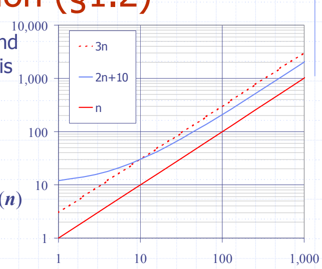
	$f(n)$ is $O(g(n))$	$g(n)$ is $O(f(n))$
$g(n)$ grows more	Yes	No
$f(n)$ grows more	No	Yes
Same growth	Yes	Yes

Big-Oh Notation (§1.2)

- Given functions $f(n)$ and $g(n)$, we say that $f(n)$ is $O(g(n))$ if there are positive constants c and n_0 such that

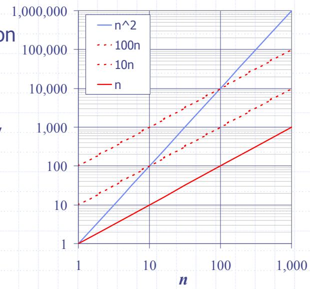
$$f(n) \le cg(n) \quad \text{for } n \ge n_0$$

- Example: $2n + 10$ is $O(n)$
  - $2n + 10 \le cn$
  - $(c - 2)\, n \ge 10$
  - $n \ge 10/(c - 2)$
  - Pick $c = 3$ and $n_0 = 10$



Big-Oh Example

- Example: the function n^2 is not $O(n)$
  - $n^2 \le cn$
  - $n \le c$
  - The above inequality cannot be satisfied since c must be a constant



More Big-Oh Examples

- 7n-2

  7n-2 is O(n)

  need c > 0 and $n_0 \ge 1$ such that 7n-2 ≤ c•n for $n \ge n_0$

  this is true for c = 7 and n_0 = 1

- $3n^3 + 20n^2 + 5$

  $3n^3 + 20n^2 + 5$ is $O(n^3)$

  need c > 0 and $n_0 \ge 1$ such that $3n^3 + 20n^2 + 5 \le c\bullet n^3$ for $n \ge n_0$

  this is true for c = 4 and n_0 = 21

- 3 log n + log log n

  3 log n + log log n is O(log n)

  need c > 0 and $n_0 \ge 1$ such that 3 log n + log log n ≤ c•log n for $n \ge n_0$

  this is true for c = 4 and n_0 = 2

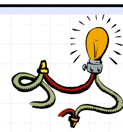
Big-Oh Rules

- If is $f(n)$ a polynomial of degree d, then $f(n)$ is $O(n^d)$, i.e.,
  1. Drop lower-order terms
  2. Drop constant factors
- Use the smallest possible class of functions
  - Say "$2n$ is $O(n)$" instead of "$2n$ is $O(n^2)$"
- Use the simplest expression of the class
  - Say "$3n + 5$ is $O(n)$" instead of "$3n + 5$ is $O(3n)$"

Asymptotic Algorithm Analysis

- asymptotic analysis = determining an algorithms running time in big-Oh notation
- asymptotic analysis steps:
  - We find the worst-case number of primitive operations executed as a function of the input size
  - We express this function with big-Oh notation
- Example:
  - We determine that algorithm ***arrayMax*** executes at most $7n - 2$ primitive operations
  - We say that algorithm ***arrayMax*** "runs in $O(n)$ time" or "runs in order n time"
- Since constant factors and lower-order terms are eventually dropped, we can disregard them when counting primitive operations!

Intuition for Asymptotic Notation



Big-Oh
- $f(n)$ is $O(g(n))$ if $f(n)$ is asymptotically **less than or equal** to $g(n)$

big-Omega
- $f(n)$ is $\Omega(g(n))$ if $f(n)$ is asymptotically **greater than or equal** to $g(n)$

big-Theta
- $f(n)$ is $\Theta(g(n))$ if $f(n)$ is asymptotically **equal** to $g(n)$

little-oh
- $f(n)$ is $o(g(n))$ if $f(n)$ is asymptotically **strictly less** than $g(n)$

little-omega
- $f(n)$ is $\omega(g(n))$ if is asymptotically **strictly greater** than $g(n)$

Relatives of Big-Oh

- **big-Omega**
  - $f(n)$ is $\Omega(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $f(n) \geq c \bullet g(n)$ for $n \geq n_0$
- **big-Theta**
  - $f(n)$ is $\Theta(g(n))$ if there are constants $c' > 0$ and $c'' > 0$ and an integer constant $n_0 \geq 1$ such that $c' \bullet g(n) \leq f(n) \leq c'' \bullet g(n)$ for $n \geq n_0$
- **little-oh**
  - $f(n)$ is $o(g(n))$ if, for any constant $c > 0$, there is an integer constant $n_0 \geq 0$ such that $f(n) \leq c \bullet g(n)$ for $n \geq n_0$
- **little-omega**
  - $f(n)$ is $\omega(g(n))$ if, for any constant $c > 0$, there is an integer constant $n_0 \geq 0$ such that $f(n) \geq c \bullet g(n)$ for $n \geq n_0$

Example Uses of the Relatives of Big-Oh

- **$5n^2$ is $\Omega(n^2)$**

  f(n) is Ω(g(n)) if there is a constant c > 0 and an integer constant $n_0 \geq 1$ such that $f(n) \geq c \cdot g(n)$ for $n \geq n_0$

  let c = 5 and $n_0 = 1$
- **$5n^2$ is $\Omega(n)$**

  f(n) is Ω(g(n)) if there is a constant c > 0 and an integer constant $n_0 \geq 1$ such that $f(n) \geq c \cdot g(n)$ for $n \geq n_0$

  let c = 1 and $n_0 = 1$
- **$5n^2$ is $\omega(n)$**

  f(n) is ω(g(n)) if, for any constant c > 0, there is an integer constant $n_0 \geq 0$ such that $f(n) \geq c \cdot g(n)$ for $n \geq n_0$

  need $5n_0^2 \geq c \cdot n_0 \rightarrow$ given c, the n_0 that satifies this is $n_0 \geq c/5 \geq 0$

More math tools & proofs

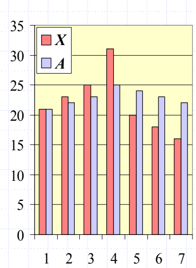
- Correctness of computing average
  - loop invariants and induction
- Recurrence equations
- Strong induction
- Cost of recursive algorithms with recurrence equations.

Computing Prefix Averages

- asymptotic analysis examples: two algorithms for prefix averages
- The i-th prefix average of an array X is average of the first $(i+1)$ elements of X:

$$A[i] = (X[0] + X[1] + \ldots + X[i])/(i+1)$$

- Computing the array A of prefix averages of another array X has applications to financial analysis



Prefix Averages (Quadratic)

- The following algorithm computes prefix averages in quadratic time by applying the definition

```
Algorithm prefixAverages1(X, n)
   Input array X of n integers
   Output array A of prefix averages of X     #operations
A ← new array of n integers
for i ← 0 to n − 1 do
     s ← X[0]
     for j ← 1 to i do
          s ← s + X[j]
     A[i] ← s / (i + 1)
return A
```

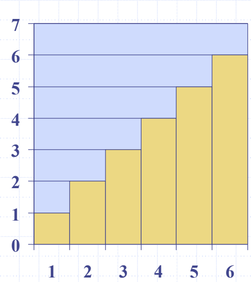

Prefix Averages (Quadratic)

- The following algorithm computes prefix averages in quadratic time by applying the definition

```
Algorithm prefixAverages1(X, n)
   Input array X of n integers
   Output array A of prefix averages of X     #operations
A ← new array of n integers                   n
for i ← 0 to n − 1 do                         n
     s ← X[0]                                 2n
     for j ← 1 to i do                        1 + 2 + …+ (n − 1)
          s ← s + X[j]                        3(1 + 2 + …+ (n − 1))
     A[i] ← s / (i + 1)                       4n
return A                                      1
```


Arithmetic Progression

- The running time of ***prefixAverages1*** is $O(1 + 2 + \ldots + n)$
- The sum of the first n integers is $n(n + 1) / 2$
  - There is a simple visual proof of this fact
- Thus, algorithm ***prefixAverages1*** runs in $O(n^2)$ time



Prefix Averages (Linear, non-recursive)

◆ The following algorithm computes prefix averages in linear time by keeping a running sum

Algorithm *prefixAverages2(X, n)*
  Input array X of n integers
  Output array A of prefix averages of X — #operations

	#operations
$A \leftarrow$ new array of n integers	n
$s \leftarrow 0$	1
for $i \leftarrow 0$ **to** $n - 1$ **do**	n
$\quad s \leftarrow s + X[i]$	n
$\quad A[i] \leftarrow s / (i + 1)$	n
return A	1

◆ Algorithm *prefixAverages2* runs in $O(n)$ time

Prefix Averages (Linear)

◆ The following algorithm computes prefix averages in linear time by computing prefix sums (and averages)

Algorithm *recPrefixSumAndAverage(X, A, n)*
  Input array X of $n \geq 1$ integer.
    Empty array A; A is same size as X.
  Output array $A[0]...A[n\text{-}1]$ changed to hold prefix averages of X.
    returns sum of $X[0], X[1],…,X[n\text{-}1]$

1. **if** n=1
2. $\quad A[0] \leftarrow X[0]$
3. $\quad$ **return** $A[0]$
4. $tot \leftarrow$ recPrefixSumAndAverage(X,A,n-1)
5. $tot \leftarrow tot + X[n\text{-}1]$
6. $A[n\text{-}1] \leftarrow tot / n$
7. **return** tot;

Prefix Averages (Linear)

◆ The following algorithm computes prefix averages in linear time by computing prefix sums (and averages)

Algorithm *recPrefixSumAndAverage(X, A, n)* — T(n) operations
  Input array X of $n \geq 1$ integer.
    Empty array A; A is same size as X.
  Output array $A[0]...A[n\text{-}1]$ changed to hold prefix averages of X.
    returns sum of $X[0], X[1],…,X[n\text{-}1]$

	#operations
if n=1	1
$\quad A[0] \leftarrow X[0]$	3
$\quad$ **return** $A[0]$	2
$tot \leftarrow$ recPrefixSumAndAverage(X,A,n-1)	3+T(n-1)
$tot \leftarrow tot + X[n\text{-}1]$	4
$A[n\text{-}1] \leftarrow tot / n$	4
return tot;	1

Prefix Averages, Linear

- Recurrence equation
  - $T(1) = 6$
  - $T(n) = 13 + T(n-1)$ for $n>1$.
- Solution of recurrence is
  - $T(n) = 13(n-1) + 6$
- $T(n)$ is $O(n)$.