Algorithms, Design and Analysis

Introduction.

Algorithm

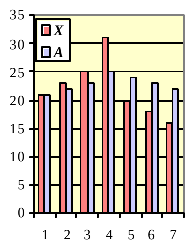
- An *algorithm* is a sequence of unambiguous instructions for solving a problem, i.e., for obtaining a required output for any legitimate input in a finite amount of time.

Computing Prefix Averages

- asymptotic analysis examples: two algorithms for prefix averages
- The ***i***-th prefix average of an array ***X*** is average of the first (i + 1) elements of ***X***:

$A[i] = (X[0] + X[1] + \ldots + X[i])/(i+1)$

- Computing the array ***A*** of prefix averages of another array ***X*** has applications to financial analysis



Prefix Averages (Quadratic)

- The following algorithm computes prefix averages in quadratic time by applying the definition

```
Algorithm prefixAverages1(X, n)
  Input array X of n integers
  Output array A of prefix averages of X
  A ← new array of n integers
  for i ← 0 to n − 1 do
     s ← X[0]
     for j ← 1 to i do
        s ← s + X[j]
     A[i] ← s / (i + 1)
  return A
```

Prefix Averages (Linear, non-recursive)

- The following algorithm computes prefix averages in linear time by keeping a running sum

```
Algorithm prefixAverages2(X, n)
  Input array X of n integers
  Output array A of prefix averages of X
  A ← new array of n integers
  s ← 0
  for i ← 0 to n − 1 do
     s ← s + X[i]
     A[i] ← s / (i + 1)
  return A
```

Prefix Averages (Linear)

- The following algorithm computes prefix averages in linear time by computing prefix sums (and averages)

```
Algorithm recPrefixSumAndAverage(X, A, n)
  Input array X of n ³ 1 integer.
        Empty array A; A is same size as X.
  Output array A[0]…A[n-1] changed to hold prefix averages of X.
        returns sum of X[0], X[1],…,X[n-1]
  if n=1
        A[0] ← X[0]
        return A[0]
  tot ← recPrefixSumAndAverage(X,A,n-1)
  tot ← tot + X[n-1]
  A[n-1] ← tot / n
  return tot;
```

Selection sort

```
Algorithm SelectionSort(A[0..n   1])
//The algorithm sorts a given array by selection sort
//Input: An array A[0..n   1] of orderable elements
//Output: Array A[0..n   1] sorted in ascending order
for i ← 0 to n   2 do
    min ← i
    for j ← i + 1 to n   1 do
        if A[j] < A[min]   min ← j
    swap A[i] and A[min]
```

Insertion sort

```
Algorithm InsertionSort(A[0..n   1])
//Sorts a given array by insertion sort
//Input: An array A[0..n   1] of n orderable elements
//Output: Array A[0..n   1] sorted in nondecreasing order
for i ← 1 to n   1 do
    v ← A[i]
    j ← i   1
    while j ≥ 0 and A[j] > v do
        A[j + 1] ← A[j]
        j ← j   1
    A[j + 1] ← v
```

Mystery algorithm

for *i* := 1 to *n* - 1 do
  max := *i* ;
  for *j* := *i* + 1 to *n* do
    if |A[*j, i*]| > |A[*max, i*]| then *max* := *j* ;
  for *k* := *i* to *n* + 1 do
    swap A[*i, k*] with A[*max, k*];
  for *j* := *i* + 1 to *n* do
    for *k* := *n* + 1 downto *i* do
      A[*j, k*] := A[*j, k*] - A[*i, k*] * A[*j, i*] / A[*i, i*] ;

What is an algorithm?

- Recipe, process, method, technique, procedure, routine,… with following requirements:

1. Finiteness
   - terminates after a finite number of steps
2. Definiteness
   - rigorously and unambiguously specified
3. Input
   - valid inputs are clearly specified
4. Output
   - can be proved to produce the correct output given a valid input

Pseudocode

- Mixture of English, math expressions, and computer code
- Less detailed than a program
- Preferred notation for describing algorithms
- Hides program design issues
- Can write at different levels of detail.

Very High-level pseudocode:

Algorithm ***arrayMax***(***A***, ***n***)
  Input array ***A*** of ***n*** integers
  Output maximum element of ***A***

  currentMax ← ***A***[0]
  Step through each element in A, updating ***currentMax*** when a bigger element is found
  return ***currentMax***

Pseudocode

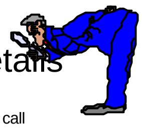
- Mixture of English, math expressions, and computer code
- Less detailed than a program
- Preferred notation for describing algorithms
- Hides program design issues
- Can write at different levels of detail.

Detailed pseudocode

Algorithm ***arrayMax***(***A***, ***n***)
  Input array ***A*** of ***n*** integers
  Output maximum element of ***A***

  currentMax ← ***A***[0]
  for ***i*** ← 1 **to** ***n*** − 1 **do**
    if ***A***[*i*] > ***currentMax*** **then**
      currentMax ← ***A***[***i***]
  return ***currentMax***

Pseudocode Details



- Control flow
  - **if** … **then** … [**else** …]
  - **while** … **do** …
  - **repeat** … **until** …
  - **for** … **do** …
  - Indentation replaces braces
- Method declaration

  Algorithm *method* (*arg* [, *arg* …])

  Input …

  Output …
- Method call

  var.method (*arg* [, *arg* …])
- Return value

  return *expression*
- Expressions

  ← Assignment (like = in Java)

  = Equality testing (like == in Java)

  n^2 Superscripts and other mathematical formatting allowed

Theoretical analysis of time efficiency

Time efficiency is analyzed by determining the number of repetitions of the <u>*basic operation*</u> as a function of <u>*input size*</u>

- <u>*Basic operation*:</u> the operation that contributes most towards the running time of the algorithm

$$T(n) \approx c_{op} C(n)$$

input size

running time

execution time for basic operation

Number of times basic operation is executed

Input size and basic operation examples

Problem	*Input size measure*	*Basic operation*
Search for key in list of *n* items	**Number of items in list *n***	**Key comparison**
Multiply two matrices of floating point numbers	**Dimensions of matrices**	**Floating point multiplication**
Compute a^n	***n***	**Floating point multiplication**
Graph problem	**#vertices and/or edges**	**Visiting a vertex or traversing an edge**

Counting Primitive Operations (§1.1)

- Worst-case primitive operations count, as a function of the input size

Algorithm *arrayMax*(*A*, *n*)	# operations
currentMax ← *A*[0]	2
for *i* ← 1 **to** *n* − 1 **do**	$1 + n$
if *A*[*i*] > *currentMax* **then**	$2(n-1)$
currentMax ← *A*[*i*]	$2(n-1)$
{ increment counter *i* }	$2(n-1)$
return *currentMax*	1
Total	$7n-2$

Counting Primitive Operations (§1.1)

- Best-case primitive operations count, as a function of the input size

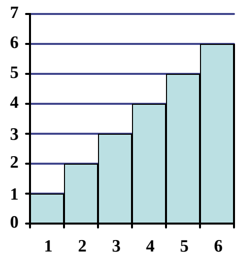
Algorithm *arrayMax*(*A*, *n*)	# operations
currentMax ← *A*[0]	2
for *i* ← 1 **to** *n* − 1 **do**	$1 + n$
if *A*[*i*] > *currentMax* **then**	$2(n-1)$
currentMax ← *A*[*i*]	0
{ increment counter *i* }	$2(n-1)$
return *currentMax*	1
Total	$5n$

Defining Worst [W(n)], Best [B(N)], and Average [A(n)]

- Let I_n = set of all inputs of size n.
- Let t(i) = # of primitive ops by alg on input i.
- W(n) = maximum t(i) taken over all i in I_n
- B(n) = minimum t(i) taken over all i in I_n
- A(n) = $\sum_{i \in I_n} p(i) t(i)$ p(i) = prob. of i occurring.
- We focus on the worst case
  - Easier to analyze
  - Usually want to know how bad can algorithm be
  - average-case requires knowing probability; often difficult to determine

Arithmetic Progression

- The running time of ***prefixAverages1*** is $O(1 + 2 + \ldots + n)$
- The sum of the first n integers is $n(n + 1) / 2$
  - There is a simple visual proof of this fact
- Thus, algorithm ***prefixAverages1*** runs in $O(n^2)$ time



Prefix Averages (Linear)

- The following algorithm computes prefix averages in linear time by computing prefix sums (and averages)

```
Algorithm recPrefixSumAndAverage(X, A, n)          T(n) operations
  Input array X of  n ³  1 integer.
        Empty array A; A is same size as X.
  Output array A[0]…A[n-1] changed to hold prefix averages of X.
        returns sum of X[0], X[1],…,X[n-1]           #operations
  if n=1                                             1
      A[0] ← X[0]                                    3
      return A[0]                                    2
  tot ← recPrefixSumAndAverage(X,A,n-1)              3+T(n-1)
  tot ← tot + X[n-1]                                 4
  A[n-1] ← tot / n                                   4
  return tot;                                        1
```

Prefix Averages, Linear

- Recurrence equation
  - T(1) = 6
  - T(n) = 13 + T(n-1) for n>1.
- Solution of recurrence is
  - T(n) = 13(n-1) + 6
- T(n) is O(n).

Empirical analysis of time efficiency

- Select a specific (typical) sample of inputs
- Use physical unit of time (e.g., milliseconds)

  OR

- Count actual number of basic operations
- Analyze the empirical data

Best-case, average-case, worst-case

For some algorithms efficiency depends on type of input:

- Worst case: W(n) – maximum over inputs of size n
- Best case: B(n) – minimum over inputs of size n
- Average case: A(n) – "average" over inputs of size n
  - Number of times the basic operation will be executed on typical input

Types of formulas for basic operation count

- Exact formula

  e.g., C(n) = n(n-1)/2

- Formula indicating order of growth with specific multiplicative constant

  e.g., C(n) ˜ 0.5 n^2

- Formula indicating order of growth with unknown multiplicative constant

  e.g., C(n) ˜ cn^2

Time efficiency of nonrecursive algorithms

Steps in mathematical analysis of nonrecursive algorithms:

- Decide on parameter *n* indicating *input siz*
- Identify algorithm's *basic operatio*
- Determine *worst*, *average*, and *best* case for input of size *n*
- Set up summation for *C(n)* reflecting algorithm's loop structure
- Simplify summation using standard formulas (see Appendix A)

Example: Sequential search

- *Problem:* Given a list of *n* elements and a search key *K*, find an element equal to *K*, if any.
- *Algorithm:* Scan the list and compare its successive elements with *K* until either a matching element is found (*successful search*) of the list is exhausted (*unsuccessful search*)
- Worst case
- Best case

Prefix Averages (Quadratic)

- The following algorithm computes prefix averages in quadratic time by applying the definition

Algorithm ***prefixAverages1***(X, n)
Input array X of n integers
Output array A of prefix averages of X

		#operations
1.	$A \leftarrow$ new array of n integers	n
2.	**for** $i \leftarrow 0$ **to** $n - 1$ **do**	n
3.	$s \leftarrow X[0]$	$2n$
4.	**for** $j \leftarrow 1$ **to** i **do**	$1 + 2 + \ldots + (n - 1)$
5.	$s \leftarrow s + X[j]$	$3(1 + 2 + \ldots + (n - 1))$
6.	$A[i] \leftarrow s / (i + 1)$	$4n$
7.	**return** A	1

Time efficiency of recursive algorithms

Steps in mathematical analysis of recursive algorithms:

- Decide on parameter *n* indicating *input size*
- Identify algorithm's *basic operation*
- Determine *worst*, *average*, and *best* case for input of size *n*
- Set up a recurrence relation and initial condition(s) for *C*(*n*)-the number of times the basic operation will be executed for an input of size *n* (alternatively count recursive calls).
- Solve the recurrence to obtain a closed form or estimate the order of magnitude of the solution (see Appendix B)