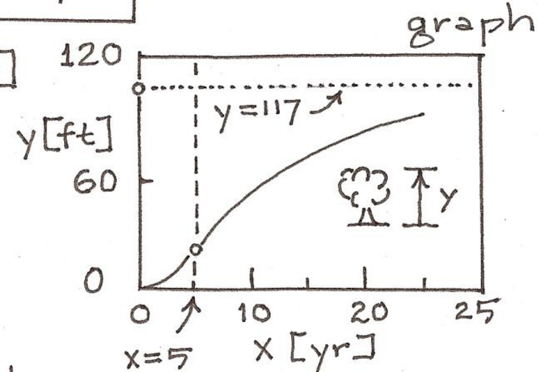
QSci 291 • answers • Hmwk #8

Neu 3rd ed, sec. 5.2, p. 222
Neu 2nd ed, sec. 5.2.3, p. 270

#33) Let y = tree height [ft]
x = time [yr]

then $y = 117e^{-10/x}$ is a particular math. model for tree growth.



Clearly, $y' = \frac{1170}{x^2} e^{-\frac{10}{x}}$,

$y'' = \frac{2340}{x^4}(5-x)e^{-10/x}$ and

$y''' = \frac{2340}{x^6}(3x^2 - 30x + 50)e^{-10/x}$

a.) for all $x > 0$, $y'(x)$ is positive ∴ tree ht. increases (the limiting tree ht., as $x \to \infty$, is 117 [ft])

b.) for $x < 5$, y'' is positive (concave up)
for $x > 5$, y'' is negative (concave down)

c.) see graph

d.) note: $y'' = 0$ and $y''' = -\frac{2340}{5^6}(50)e^{-2}$ (neg. no.)
hence y' has a maximum at $x = 5$, the inflection pt. of $y = 117e^{-10/x}$

ans.

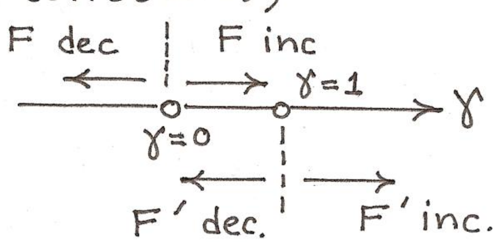
#35) Let F = no. of flowers on a plant and
X = ave. no. of pollinator visits, then

$F = cX^{\gamma}$ (c and γ are constants)

$F' = c\gamma X^{\gamma - 1}$

When $0 < \gamma < 1$,
F increases and
F′ decreases

ans.



QSci 291 • answers • Hmwk #8

Neu. 3rd ed., sec. 5.3, p. 235
Neu. 2nd ed., sec. 5.3, p. 287

#26) Let N = size of a population of living things [#]
t = time [hrs, days, yrs... whatever],

then the Logistic Model (differential eq.) for population growth is: $\frac{dN}{dt} = rN \cdot \left(1 - \frac{N}{K}\right)$

(r & K are constants)

The solution to this differential equation is:

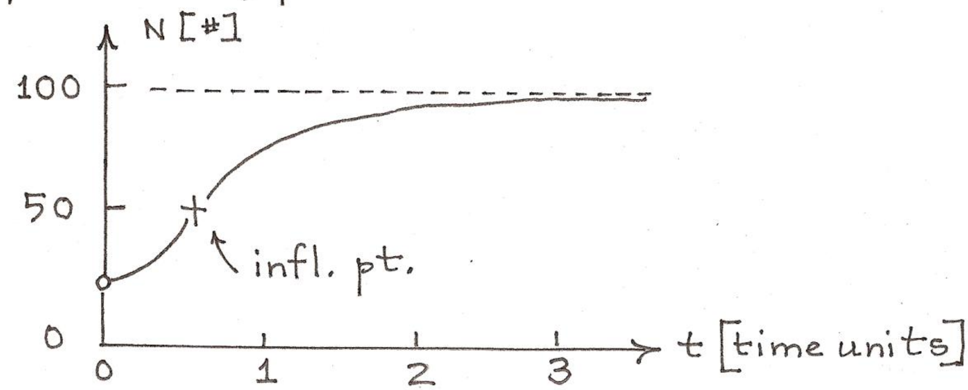
$$N = \frac{K}{\{1 + [(K/N_0) - 1] \cdot e^{-rt}}$$

where N_0 = initial size of the population

If $r = 2$, $K = 100$ and $N_0 = 25$, then

$$N = \frac{100}{1 + 3e^{-2t}}$$

note: the Logistic Model plays a big role in the population dynamics field. Neuhauser, 3rd ed., mentions this often. See example 3, p. 112; problems #28 & 29, p. 113; problem #34, p. 214; problem #27, p. 222; example 6, p. 400; example 1, p. 408 and problem #3, p. 429



QSci 291 • answers • Hmwk #8

#26) cont'd

a) at $t=0$, $N = 100/(1+3)$ or $N = 25$

b) $N' = 600\left[\dfrac{e^{-2t}}{(1+3e^{-2t})^2}\right]$ <u>Q.R.</u>

N' is positive for all $t > 0$ $\therefore$ N increases

c) as $t \to \infty$, $e^{-2t} \to 0$ $\therefore N \to 100/(1+0) = 100$

d) $N'' = -1200\left[\dfrac{(1-3e^{-2t})\cdot e^{-2t}}{(1+3e^{-2t})^3}\right]$ <u>Q.R.</u>

$N'' = 0$ when $1-3e^{-2t} = 0$ or $e^{2t} = 3$

or when $t = \frac{1}{2}\ln(3) = 0.549$

<u>notice</u>: $e^{-2[\frac{1}{2}\ln 3]} = e^{\ln(3^{-1})} = e^{\ln(\frac{1}{3})} = \frac{1}{3}$

Hence, at $t = \frac{1}{2}\ln(3)$, $N = 100/(1+3(1/3)) = 50$

or the inflection pt. occurs when the population is 1/2 the threshold value.

Neu 3rd ed., sec. 5.4, p243
Neu 2nd ed., sec. 5.4.1, p. 296

ans

#4) sketch the study area

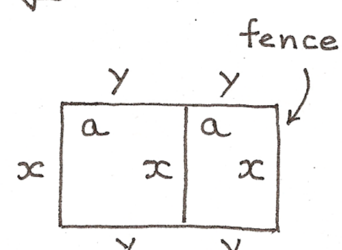
Let: x[ft] = length of division segment

y[ft] = length of other side (see sketch)

L[ft] = total fence length

a[ft²] = area of sub-area

A[ft²] = study area



STUDY AREA (2 sub-areas, rectangles)

QSci 291 • answers • Hmwk # 8

#4) cont'd Note: $L = 3x + 4y$, $a = x \cdot y$ and
$A = 2a = 2xy = 384$

Since $y = \frac{A}{2} \cdot \frac{1}{x}$, then

$L = 3x + 2A\frac{1}{x}$; $L' = 3 - 2A\frac{1}{x^2}$; $L'' = +4A \cdot \frac{1}{x^3}$

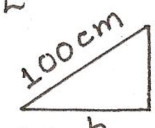
Set $L' = 0$, then $x^2 = \frac{2}{3}A$ or $x = 16$ & $y = 12$

$L''(16) = 4(384)/(16)^3$ (a pos. no.)

Thus, L is a minimum when $x = 16$ ft & $y = 12$ ft

The minimum length of fence required is $\boxed{L = 96\text{ ft}}$ ans

#8) sketch right triangle



Let a[cm] & b[cm] be lengths of triangle sides and A[cm^2] be the area

Note: $A = \frac{1}{2}a \cdot b$ and $a^2 + b^2 = 100^2$ (Pyth$^{\underline{n}}$ Thm)

$\therefore A = \frac{1}{2}a\sqrt{100^2 - a^2}$

(side note: one can procede with $A' = 0$ to find $a = a_0$ and then use a_0 to test for min., max. or infl. pt.)

*Alternative method: use $S = A^2$ to find max., min. or inflection pt. as $S' = 2A \cdot A'$ so if $A' = 0$ then $S' = 0$ at the same crit. pt. Moreover, $S'' = 2(A')^2 + 2A(A'')$ and at a crit. pt. $S''(a_0) = 2A \cdot A''(a_0)$, hence S'' will have the same sign as A''

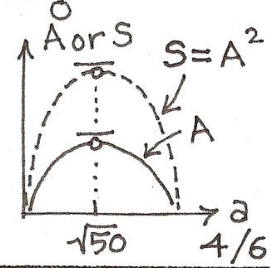
Set $S' = 0$.

$S = \frac{1}{4}a^2(100^2 - a^2)$; $S' = \frac{1}{4}(2a)(100^2 - 2a^2)$; $a^2 = \frac{1}{2}100$

Thus,, $a = \sqrt{50}$ and $b = \sqrt{50}$ and $a = b$

At $a^2 = 49$, $S'(7)$ is pos and $S'(\sqrt{51})$ is neg

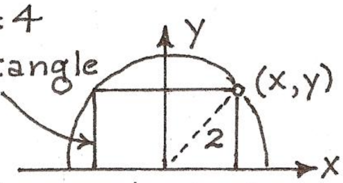
$\therefore$ The area squared will be a maximum at $a = \sqrt{50}$ and $b = \sqrt{50}$



$\boxed{\text{As a consequence, } A \text{ is a maximum when } a = b = \sqrt{50}}$ ans

QSci 291 • answers • Hmwk #8

#10) From the sketh $x^2+y^2=2^2=4$
or $y=\sqrt{4-x^2}$ (eq. of a circle)



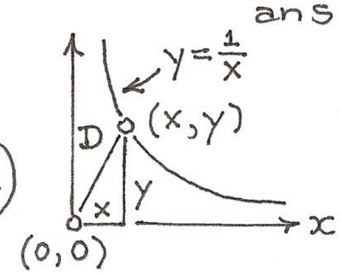
Let A = area of the rectangle,
then $A = 2x \cdot y = 2x \cdot \sqrt{4-x^2}$

* Use square area trick to avoid messy derivative formulas : $S = A^2$

$S' = 4x^2(4-x^2)$; $S' = 4(2x)(2)(2-x^2)$; $S'' = 16(2-3x^2)$

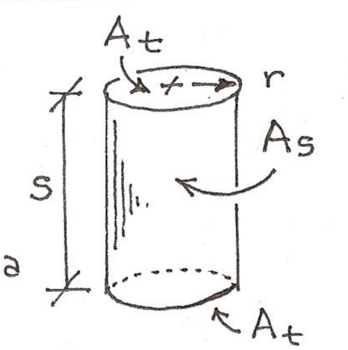
Set $S=0$, $x^2=2$ or $x=+\sqrt{2}$ and $y=\sqrt{2}$
(ignor the crit. pts. $x=0$ & $x=-\sqrt{2}$)
Since $S''(\sqrt{2}) = -64$ (neg.), S is a maximum
at $x=+\sqrt{2}$ & $y=\sqrt{2}$ and the largest area is $\boxed{A=4}$ ans

#13) sketch the distance between the origin and the curve $y=1/x$



Let $S = D^2 = x^2+y^2$, then
$S = x^2+\frac{1}{x^2}$; $S' = 2\left(x-\frac{1}{x^3}\right)$; $S'' = 2\left(1+\frac{3}{x^4}\right)$
Set $S=0$, $x=\frac{1}{x^3}$ or $x^4=1$ and $x=1$
(ignor $x=-1$, $x=i$ and $x=-i$; $i=\sqrt{-1}$)
at $x=1$, $S''(1)=+8$ (pos.), S is a minimum at $x=1$
and the shortest distance is $\boxed{D=\sqrt{2}}$ ans

#17) Let r[cm] = can top radius
s[cm] = can height
A_s [cm²] = area of can side
A_t [cm²] = area of can top (and bottom)
A [cm²] = total can surface area
V [cm³] = can volume



Note: $V = s \cdot A_t = \pi r^2 s = 1000$
$A = A_s + 2A_t = 2\pi r \cdot s + 2\pi r^2$

$A_t = \pi r^2$
$A_s = 2\pi r \cdot s$

Since $s = V/\pi r^2$, $A = 2\pi r\left(\frac{V}{\pi r^2}\right) + 2\pi r^2$

QSci 291 • answers • Hmwk #8

#13) cont'd $A = \frac{2V}{r} + 2\pi r^2$; $A' = 2\left(-\frac{V}{r^2} + \pi r\right)$; $A'' = \frac{4V}{r^3} + 2\pi$

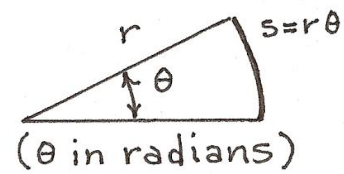
Set $A' = 0$, then $\pi r = \frac{V}{r^2}$ or $r^3 = V/\pi = \frac{1000}{\pi}$

Since $A''(r) = \frac{4V}{r^3} + 2\pi$ is positive

then the can surface area is a minimum

when $\boxed{r = 10/\sqrt[3]{\pi} = 6.28\text{ cm} \ \& \ s = 10/\sqrt[3]{\pi} = 6.28\text{ cm}}$ ans

#19) From the sketch of the circle sector



$p = 2r + s = 2r + r\theta$, perimeter, and

$A = \frac{1}{2}r^2\theta = A_0$ (const.), area

Since $\theta = 2A_0/r^2$, $p = 2\left(r + \frac{A_0}{r}\right)$ &

$p' = 2\left(1 - \frac{A_0}{r^2}\right)$; $p'' = \frac{4A_0}{r^3}$

Set $p' = 0$, then $r = \sqrt{A_0}$ & $p''(\sqrt{A_0})$ is positive

Thus, the perimeter will be smallest when $\boxed{r = \sqrt{A_0}}$ ans

#22) given an age structure model

$r(x) = \frac{1}{x}\left[\ln(\ell(x)) + \ln(m(x))\right]$ with

$\ell(x) = e^{-ax}$ & $m(x) = bx^c$ (a, b and c are constants)

then $r(x)$ can be expressed as (ECE):

$r(x) = \frac{1}{x}\left[-a\cdot x + \ln(b) + c\ln(x)\right] = -a + (\ln b)\frac{1}{x} + c\frac{\ln(x)}{x}$

$\therefore r'(x) = \frac{1}{x^2}\cdot q(x)$ where $q(x) = (c - \ln b) - c\cdot\ln(x)$

& $r''(x) = \frac{-2}{x^3}\left[\frac{c}{2} - q(x)\right]$

Set $r' = 0$, then $q(x_0) = 0$ or $\underline{\ln(x_0) = 1 - (\ln(b))/c}$

Now $x_0 = e^{\ln(x_0)}$, so $\underline{x_0 = e^{1-(\ln(b))/c}}$

note: $q(x_0) = 0$

With $a = 0.1$, $b = 4$ & $c = 0.9$

$x_0 = 0.582$

Since $r''(x_0) = \frac{-2}{x_0^3}\left[\frac{c}{2} - 0\right]$

is negative, then

$\boxed{r(x) \text{ is a maximum at } x_0 = 0.582}$ ans