The trigonometric substitutions are particular examples of the general substitution method. They are often of use when the integrand contains factors of the type $(a^2 - x^2)$, $(a^2 + x^2)$, or $(x^2 - a^2)$, and particularly when such a factor appears underneath a radical sign. Their use is based on the trigonometric square identities (or Pythagorean identities), which are as follows:

$$1 - \sin^2\theta = \cos^2\theta$$

$$1 + \tan^2\theta = \sec^2\theta \quad \text{or} \quad \sec^2\theta - 1 = \tan^2\theta.$$

The substitutions themselves consist of writing either $x = a\sin\theta$ or $x = a\tan\theta$ or $x = a\sec\theta$, where θ is the new variable of integration and a is a certain constant determined by the given integrand. The choice of substitution depends on which of the three factors appears in the integrand, and is summarized in the following table.

If the Integral Involves		The Appropriate Substitution Is
(i)	$a^2 - x^2$	$x = a\sin\theta$
(ii)	$a^2 + x^2$	$x = a\tan\theta$
(iii)	$x^2 - a^2$	$x = a\sec\theta$

The purpose of these substitutions is to change the factor $a^2 - x^2$, $a^2 + x^2$, or $x^2 - a^2$ that appears in the integrand to a single squared trigonometric function. For example, if we put $x = a\sin\theta$ in $a^2 - x^2$, we have:

$$a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2(1 - \sin^2\theta) = a^2\cos^2\theta.$$

Thus the factor $(a^2 - x^2)$ becomes a simple function of the new variable θ. If this factor appears under a square root sign an even simpler function of θ is obtained since $\sqrt{a^2 - x^2} = a\cos\theta$.

taken from: Arya & Lardner. 1972
Math. for the Biol. Sci.
Prentice-Hall

EXAMPLE Evaluate

$$\int \frac{x}{\sqrt{x^2 - 9}}\, dx.$$

SOLUTION In this example we have a choice of methods. First of all we can write

$$\int \frac{x}{\sqrt{x^2 - 9}}\, dx = \frac{1}{2} \int \frac{2x\, dx}{\sqrt{x^2 - 9}}.$$

Since in this case the differential of $x^2 - 9$, which is $2x\, dx$, appears, we put

$$x^2 - 9 = y \quad \text{and then} \quad 2x\, dx = dy.$$

Therefore

$$\begin{aligned}
\int \frac{x}{\sqrt{x^2 - 9}}\, dx &= \frac{1}{2} \int \frac{2x\, dx}{\sqrt{x^2 - 9}} \\
&= \frac{1}{2} \int \frac{dy}{\sqrt{y}} \\
&= \tfrac{1}{2} \int y^{-1/2}\, dy \\
&= \frac{1}{2} \cdot \frac{y^{-1/2+1}}{(-\frac{1}{2} + 1)} \\
&= \sqrt{y} + C \\
&= \sqrt{x^2 - 9} + C
\end{aligned}$$

because $y = x^2 - 9$.

ALTERNATIVELY Because the integrand involves $\sqrt{x^2 - a^2} = \sqrt{x^2 - 9}$ with $a = 3$, we can make use of the trigonometric substitution

$$x = a \sec \theta \quad \text{or} \quad x \quad 3 \sec \theta.$$

Then

$$dx = 3 \sec \theta \tan \theta\, d\theta.$$

Therefore

$$\begin{aligned}
\int \frac{x}{\sqrt{x^2 - 9}}\, dx \quad & \int \frac{3 \sec \theta}{\sqrt{9 \sec^2 \theta - 9}} \cdot 3 \sec \theta \tan \theta\, d\theta \\
& \int \frac{9 \sec^2 \theta \tan \theta}{3\sqrt{\sec^2 \theta - 1}}\, d\theta \\
&= 3 \int \frac{\sec^2 \theta \tan \theta}{\tan \theta}\, d\theta \\
&= 3 \int \sec^2 \theta\, d\theta \\
&= 3 \tan \theta + C.
\end{aligned}$$

The final step is to express this result in terms of the original variable x. We have

$$\tan \theta = \sqrt{\sec^2 \theta - 1} = \sqrt{(x/3)^2 - 1} = \tfrac{1}{3}\sqrt{x^2 - 9}.$$

Therefore the given integral becomes

$$\int \frac{x}{\sqrt{x^2 - 9}}\, dx = \sqrt{x^2 - 9} + C,$$

the same answer as found by the earlier method.

EXAMPLE Evaluate

$$\int \frac{1}{(a^2 + x^2)^{3/2}}\, dx.$$

SOLUTION Since the integrand involves $(a^2 + x^2)$ we put $x = a \tan \theta$. Then $dx = a \sec^2 \theta \, d\theta$ and so

$$\begin{aligned}
\int \frac{1}{(a^2 + x^2)^{3/2}}\, dx &= \int \frac{1}{(a^2 + a^2 \tan^2 \theta)^{3/2}} \cdot a \sec^2 \theta \, d\theta \\
&= \int \frac{1}{(a^2 \sec^2 \theta)^{3/2}} \cdot a \sec^2 \theta \, d\theta \\
&= \int \frac{1}{a^3 \sec^3 \theta} \cdot a \sec^2 \theta \, d\theta \\
&= \frac{1}{a^2} \int \frac{1}{\sec \theta}\, d\theta \\
&= \frac{1}{a^2} \int \cos \theta \, d\theta \\
&= \frac{1}{a^2} \sin \theta + C.
\end{aligned}$$

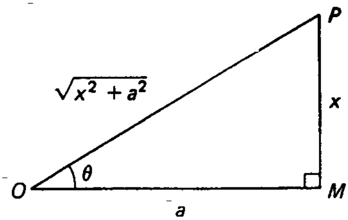

Figure 6.2

We must express $\sin \theta$ in terms of x when we have $\tan \theta = x/a$. From the right-angled triangle OMP (Fig. 6.2), in which θ is the angle at O, we have:

$$\tan \theta = \frac{MP}{OM} = \frac{x}{a} \quad \text{(given).}$$

Therefore we can take $MP = x$ and $OM = a$. Then

$$OP^2 = MP^2 + OM^2 = x^2 + a^2 \quad \text{or} \quad OP = \sqrt{a^2 + x^2}.$$

$$\sin \theta = \frac{MP}{OP} = \frac{x}{\sqrt{a^2 + x^2}}$$

$$\int \frac{1}{(a^2 + x^2)^{3/2}}\, dx = \frac{1}{a^2} \cdot \frac{x}{\sqrt{a^2 + x^2}} + C.$$

EXAMPLE Evaluate

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx, \quad a > 0.$$

SOLUTION Here the substitution $a^2 - x^2 = u$ cannot work because its differential $-2x\,dx$ does not appear in the integral nor can it be obtained by multiplying by any constant. Instead we put

$$x = a \sin\theta, \quad \text{so that}$$

$$dx = a\cos\theta\,d\theta, \quad \theta = \operatorname{Sin}^{-1}\frac{x}{a}$$

Then

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx \quad \int \frac{1}{\sqrt{a^2 - a^2\sin^2\theta}} \cdot a\cos\theta\,d\theta$$

$$\int \frac{1}{a\sqrt{1 - \sin^2\theta}} \cdot a\cos\theta\,d\theta$$

$$= \int \frac{1}{a\cos\theta} \cdot a\cos\theta\,d\theta$$

$$= \int d\theta \quad \theta + C$$

$$= \operatorname{Sin}^{-1}\frac{x}{a} + C.$$

NOTE. When we take $\sqrt{1 - \sin^2\theta} = \cos\theta$, we are being consistent with the principal value of $\theta = \operatorname{Sin}^{-1}(x/a)$ for which $-\pi/2 < \theta < \pi/2$ since $\cos\theta$ is positive for $-\pi/2 < \theta < \pi/2$.

EXERCISES 6.5

Make use of trigonometric substitutions to evaluate the following integrals:

1. $\displaystyle\int \frac{1}{a^2 + x^2}\,dx.$

2. $\displaystyle\int \frac{dt}{4 + t^2}$

3. $\displaystyle\int \frac{dy}{\sqrt{9 - y^2}}.$

4. $\displaystyle\int \frac{d\theta}{16 - \theta^2}.$

5. $\displaystyle\int \frac{dx}{x\sqrt{x^2 - 16}}, \quad (x > 4).$

6. $\displaystyle\int \frac{dy}{y\sqrt{y^2 - 25}}, \quad (y > 5).$

7. $\displaystyle\int \frac{dt}{(t^2 + 9)^{3/2}}.$

8. $\displaystyle\int \frac{dx}{(25 - x^2)^{3/2}}.$

9. $\displaystyle\int \frac{dx}{(x^2 - a^2)^{3/2}}.$

10. $\displaystyle\int \frac{x}{\sqrt{9 - x^2}}\,dx.$

11. $\displaystyle\int \frac{x}{(a^2 - x^2)^{3/2}}\,dx.$

12. $\displaystyle\int \frac{\theta\,d\theta}{\theta^2 + a^2}.$

13. $\displaystyle\int \frac{\theta\,d\theta}{\theta^2 + 3}.$

14. $\displaystyle\int \frac{x\,dx}{x^2 + 5}.$