

CENTERS OF MASS

(or center of gravity)

Consider a finite number n of particles of masses m_1, m_2, \dots, m_n placed along the x -axis at distances x_1, x_2, \dots, x_n respectively from the origin (Fig. 7.32). Then

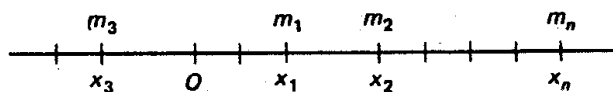


Figure 7.32

the *moment* of this system of n masses about the origin is defined to be

$$m_1x_1 + m_2x_2 + \dots + m_nx_n = \sum_{k=1}^n m_k x_k.$$

The total mass of the system is

$$m = m_1 + m_2 + \dots + m_n = \sum_{k=1}^n m_k.$$

The center of mass of the system of particles is defined to be the point \bar{x} , which has the property that if the total mass were concentrated at \bar{x} then the moment about the origin would be unchanged. Now if the total mass $m = \sum_{k=1}^n m_k$ were concentrated at \bar{x} , the moment would simply be $m\bar{x}$. Therefore

$$m\bar{x} = \sum_{k=1}^n m_k x_k,$$

that is,

$$\bar{x} = \frac{\sum_{k=1}^n m_k x_k}{m} = \frac{\sum_{k=1}^n m_k x_k}{\sum_{k=1}^n m_k},$$

which determines the center of mass of the system.

EXAMPLE

Four masses of 5, 7, 3, and 10 units are placed along the x -axis at the points $(3, 0)$, $(-2, 0)$, $(-5, 0)$, and $(7, 0)$ respectively. Find the center of mass of the system.

SOLUTION

Here $m_1 = 5$, $m_2 = 7$, $m_3 = 3$, and $m_4 = 10$. Also $x_1 = 3$, $x_2 = -2$, $x_3 = -5$, and $x_4 = 7$. Then the center of mass is given by

$$\begin{aligned} \bar{x} &= \frac{\sum_{k=1}^4 m_k x_k}{\sum_{k=1}^4 m_k} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + m_4 x_4}{m_1 + m_2 + m_3 + m_4} \\ &= \frac{5(3) + 7(-2) + 3(-5) + 10(7)}{5 + 7 + 3 + 10} \\ &= \frac{15 - 14 - 15 + 70}{25} = \frac{56}{25}. \end{aligned}$$

from: Arya and Lardner 1979

Now consider a system of n particles of masses m_1, m_2, \dots, m_n located at the points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ respectively in the xy -plane. Then the *moment* of the system about the y -axis is defined to be

$$M_y = m_1x_1 + m_2x_2 + \dots + m_nx_n = \sum_{k=1}^n m_kx_k,$$

and the *moment about the x -axis* is defined to be

$$M_x = m_1y_1 + m_2y_2 + \dots + m_ny_n = \sum_{k=1}^n m_ky_k.$$

The total mass of the system is

$$m = m_1 + m_2 + \dots + m_n = \sum_{k=1}^n m_k.$$

If we imagine the masses to be supported by a weightless tray and assume that each mass occupies exactly one point, then the *center of mass* is the point at which the tray can be supported by a single pinpoint support in such a way as to balance perfectly in a horizontal position (Fig. 7.34). Mathematically, the center of mass (or center of gravity) is the point (\bar{x}, \bar{y}) such that

$$m\bar{x} = M_y, \quad m\bar{y} = M_x.$$

Thus the center of mass (\bar{x}, \bar{y}) of the system of n masses is given by

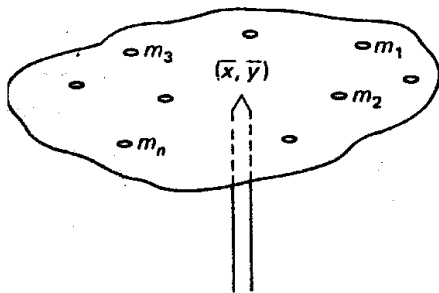


Figure 7.34

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + \dots + m_nx_n}{m_1 + m_2 + \dots + m_n} = \frac{\sum_{k=1}^n m_kx_k}{\sum_{k=1}^n m_k}$$

$$\bar{y} = \frac{m_1y_1 + m_2y_2 + \dots + m_ny_n}{m_1 + m_2 + \dots + m_n} = \frac{\sum_{k=1}^n m_ky_k}{\sum_{k=1}^n m_k}$$

EXAMPLE

Find the center of mass of the system of masses of 2, 3, and 4 located at the points $(1, 2)$, $(3, -7)$, and $(5, 1)$ respectively.

SOLUTION

Here $m_1 = 2$, $m_2 = 3$, and $m_3 = 4$; $(x_1, y_1) = (1, 2)$, $(x_2, y_2) = (3, -7)$, and $(x_3, y_3) = (5, 1)$. Then

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3} = \frac{2(1) + 3(3) + 4(5)}{2 + 3 + 4} = \frac{31}{9}.$$

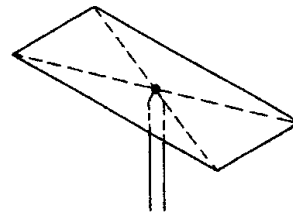
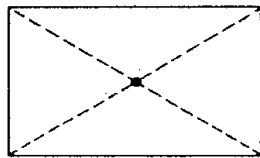
$$\bar{y} = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3} = \frac{2(2) + 3(-7) + 4(1)}{2 + 3 + 4} = -\frac{13}{9}.$$

The center of mass is therefore the point $(\frac{31}{9}, -\frac{13}{9})$.

Let us now consider *homogeneous laminas*. (The word lamina is used to mean a thin sheet or layer of material, for example, a piece of plywood or of sheet-steel.) We say that a lamina is homogeneous if two pieces of it have equal weights whenever their areas are equal. For a homogeneous lamina, we define the density to be the mass per unit area. Therefore the mass of a homogeneous lamina of density D and area A is given by

$$m = DA. \quad (\text{or } m = \rho A)$$

We now wish to define the center of mass of a homogeneous lamina in a way that is to be consistent with our experience with systems of particles. First of all we observe from our common experience that a rectangular sheet of uniform thickness can be balanced at its geometric center, so it is natural for us to define the center of mass of a homogeneous rectangular lamina to be its geometric center (Fig.). In the same

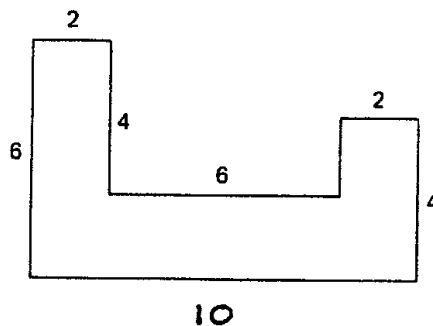


manner, we define the center of mass of a uniform circular region to be its geometric center.

With these definitions of the centers of mass of uniform rectangular and circular laminas, it is possible to find the center of mass of regions that are combinations of any number of rectangles and circles. In doing so, we treat each rectangle or circle as if all of its mass were concentrated at its center. This is illustrated by the following example.

EXAMPLE

A region is made up of a combination of rectangles of uniform density D . The shape and dimensions of the region are illustrated in Fig. . Find the center of mass.



SOLUTION

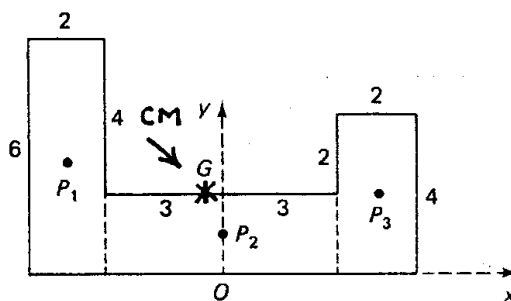
First of all we select the axes of coordinates as shown in the Fig. Then the centers of mass of the three rectangles are the points $P_1, (-4, 3)$; $P_2, (0, 1)$; $P_3, (4, 2)$, and their total masses are $12D, 12D$, and $8D$ respectively. (These are obtained by multiplying the area of each rectangle by the density D .) We may treat the region as a system of three point masses located at the centers of mass, that is, a mass $12D$ at P_1 , a mass $12D$ at P_2 , and a mass $8D$ at P_3 . Then, as before, we have

$$\bar{x} = \frac{12D(-4) + 12D(0) + 8D(4)}{12D + 12D + 8D} = -\frac{16D}{32D} = -\frac{1}{2}$$

and

$$\bar{y} = \frac{12D(3) + 12D(1) + 8D(2)}{12D + 12D + 8D} = \frac{64D}{32D} = 2.$$

The center of mass is therefore the point $(-\frac{1}{2}, 2)$, marked G in the figure.



Procedure:

1. select a coordinate system (most convenient one possible)
2. set up a table for x_i, y_i, A_i and M_{y_i} & M_{x_i} .
3. sum up areas and moments to obtain I_1, I_2 and I_3 .
4. compute coordinates for the center of mass.
5. check answer (use intuition)

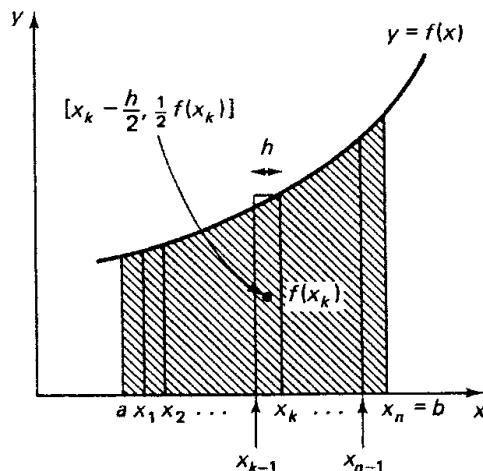
subregion i	\bar{x}_i	\bar{y}_i	A_i	$M_{y_i} = \bar{x}_i \cdot A_i$	$M_{x_i} = \bar{y}_i \cdot A_i$
1	-4	3	12	-48	36
2	0	1	12	0	12
3	4	2	8	32	16
			32	-16	64
			(I_1)	(I_2)	(I_3)

We shall now establish formulas for the coordinates of the center of mass of a plane region that is bounded by the graph of a function $y = f(x)$, the x -axis, and the lines $x = a$, $x = b$. As indicated before, we assume that the density of the region is 1, so that the total mass of the region is its area, given by

$$m = \int_a^b f(x) dx.$$

(since density times thickness cancels)

As when finding the area under the curve, we use the approximation method for defining the moments of such a region about the coordinate axes. Let us divide the interval $a \leq x \leq b$ into n equal parts each of length $h = (b - a)/n$, and let x_k ($k = 0, 1, 2, \dots, n$) be the endpoints of these subintervals. We construct a rectangle on each subinterval and approximate the area under the curve by the area of the corresponding rectangle (Fig. 7). In particular, the k -th rectangle erected on the interval $x_{k-1} \leq x \leq x_k$ has height $f(x_k)$ and width $x_k - x_{k-1} = h$. Its center of mass is its geometric center, which has the coordinates $[x_k - (h/2), \frac{1}{2}f(x_k)]$.



The mass of this rectangle is equal to its area, which is $h \times f(x_k)$. We can imagine the mass of each of the n rectangles to be concentrated at the center of the rectangle, and then the moment about the y -axis of these n rectangles is given by

$$M_y = \sum_{k=1}^n \left[f(x_k) \cdot h \cdot \left(x_k - \frac{h}{2} \right) \right].$$

Similarly the moment about the x -axis of the n rectangles is given by

$$M_x = \sum_{k=1}^n \left[f(x_k) \cdot h \cdot \frac{1}{2} f(x_k) \right].$$

As $n \rightarrow \infty$, the sum of the areas of the rectangles approaches the true area under the curve, and in the same way the moments about the y -axis and x -axis of the rectangles approach the true moments of the area under the curve. We note that as $n \rightarrow \infty$, $h \rightarrow 0$ and $x_k - (h/2) \rightarrow x_k$. Thus for a plane region bounded by $y = f(x)$, the x -axis, and the lines $x = a$, $x = b$, the moments about the y -axis and x -axis are given by

$$M_y = \lim_{n \rightarrow \infty} \sum_{k=1}^n \left[\left(x_k - \frac{h}{2} \right) \cdot f(x_k) \cdot h \right] = \int_a^b x f(x) dx.$$

$$M_x = \lim_{n \rightarrow \infty} \sum_{k=1}^n \left\{ \frac{1}{2} [f(x_k)]^2 \cdot h \right\} = \int_a^b \frac{1}{2} [f(x)]^2 dx.$$

The center of mass (\bar{x}, \bar{y}) is then defined as before by the equations

$$M_y = m\bar{x}, \quad M_x = m\bar{y},$$

so that

$$\bar{x} = \frac{\int_a^b x f(x) dx}{\int_a^b f(x) dx}, \quad \bar{y} = \frac{\frac{1}{2} \int_a^b [f(x)]^2 dx}{\int_a^b f(x) dx}$$

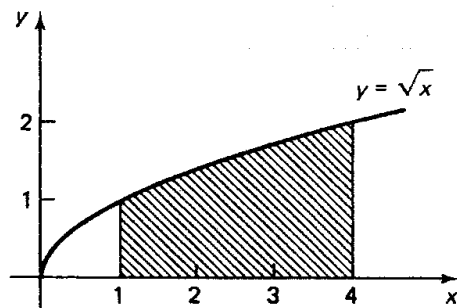
because

$$m = \int_a^b f(x) dx.$$

EXAMPLE

Find the center of mass of the plane region bounded by $y = \sqrt{x}$, the x -axis, and the lines $x = 1$, $x = 4$. **SOLUTION**

Here $f(x) = \sqrt{x}$, $a = 1$, $b = 4$



$$\begin{aligned} I_1: \quad (\text{area}) \quad \int_a^b f(x) dx &= \int_1^4 \sqrt{x} dx = \int_1^4 x^{1/2} dx \\ &= \left[\frac{x^{3/2}}{\frac{3}{2}} \right]_1^4 = \frac{2}{3} [4^{3/2} - 1^{3/2}] \\ &= \frac{2}{3} [2^3 - 1] = \frac{14}{3}. \end{aligned}$$

$$\int_a^b x f(x) dx = \int_1^4 x \cdot \sqrt{x} dx \quad \frac{1}{2} \int_a^b [f(x)]^2 dx = \frac{1}{2} \int_1^4 (\sqrt{x})^2 dx$$

$$I_2: = \int_1^4 x^{3/2} dx \quad I_3: = \frac{1}{2} \int_1^4 x dx$$

$$(M_y) = \left[\frac{x^{5/2}}{5/2} \right]_1^4 = \frac{2}{5} [4^{5/2} - 1^{5/2}] = \frac{2}{5} [2^5 - 1] = \frac{62}{5}$$

$$(M_x) = \frac{1}{2} \left[\frac{x^2}{2} \right]_1^4 = \frac{1}{4} [4^2 - 1^2] = \frac{15}{4}$$

Thus the center of mass (\bar{x}, \bar{y}) is given by

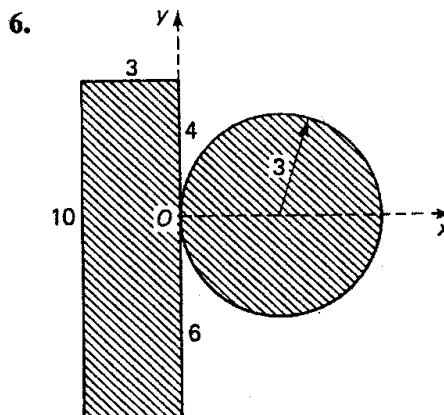
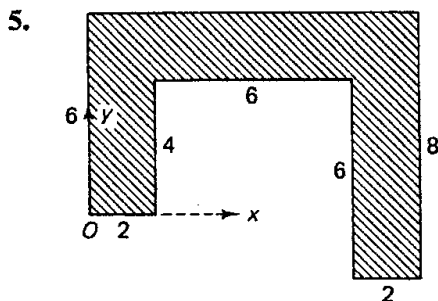
$$\bar{x} = \frac{\int_a^b x f(x) dx}{\int_a^b f(x) dx} = \frac{\frac{62}{5}}{\frac{14}{3}} = \frac{62}{5} \cdot \frac{3}{14} = \frac{93}{35}$$

$$\bar{y} = \frac{\frac{1}{2} \int_a^b [f(x)]^2 dx}{\int_a^b f(x) dx} = \frac{\frac{15}{4}}{\frac{14}{3}} = \frac{15}{4} \cdot \frac{3}{14} = \frac{45}{56}$$

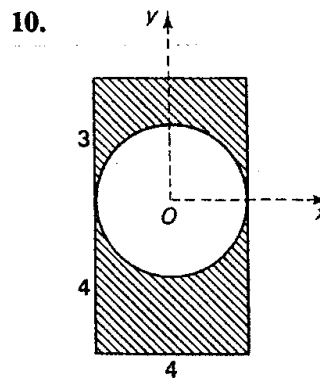
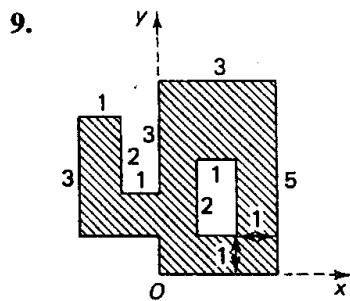
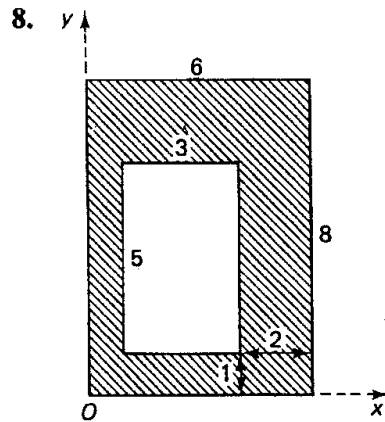
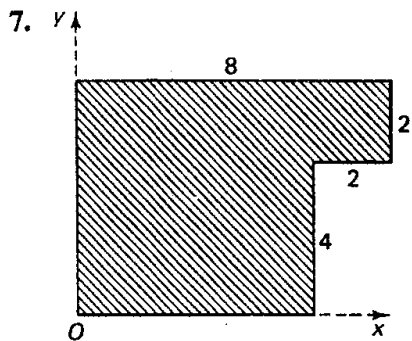
That is, the center of mass is the point $(\frac{93}{35}, \frac{45}{56})$.

EXERCISES

Find the center of mass of the following plane regions of uniform density.



from: **MATHEMATICS FOR THE BIOLOGICAL SCIENCES**
Jagdish C. Arya and Robin W. Lardner (A/L)



In each of the following exercises find the center of mass of the plane region R .
The region R is bounded by:

11. $y = x^2$, $x = 0$, $x = 3$, and $y = 0$ (x -axis).
12. $y = x^2 + 3x + 1$, $x = 1$, $x = 2$, and $y = 0$.
13. $y = x^3$, $x = 0$, $x = 1$, and $y = 0$.
14. $y = \frac{1}{x}$, $x = 1$, $x = 4$, and $y = 0$.
15. $y = \sqrt{a^2 - x^2}$, $x = 0$, $x = a$, and $y = 0$ (quarter-circle).
16. $y = \sqrt{a^2 - x^2}$, $x = -a$, $x = a$, and $y = 0$ (semicircular region).
17. $y = \sin x$, $x = 0$, $x = \pi$, and $y = 0$.
18. $y = \cos x$, $x = 0$, $x = \frac{\pi}{2}$, and $y = 0$.